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THE LYAPUNOV-SCHMIDT METHOD FOR TWO-POINT BOUNDARY VALUE PROBLEMS

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Abstract. We apply the Lyapunov-Schmidt method as a numerical method for some boundary value problems. The advantage of this method consists of the important reduction of the dimension of the nonlinear system to be solved. Our MATLAB package based on this method behaves better than other known packages such as bvp4c or sbvp. Test problems are also presented.

Key Words and Phrases: Lyapunov-Schmidt method, boundary value problems, MATLAB package.

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1. Introduction

The *Lyapunov-Schmidt* (LS) method is an efficient method, which can be used for different types of boundary value problems. Elaborated in the years 1906-1908 and reformulated in a modern language by L. Cesari after 1963 [4], this method applies to some nonlinear equations of the type

$$Lu = Nu, \tag{1}$$

for instance $u''(x) = f(x, u(x), u'(x))$, in the presence of some boundary conditions, considered on the domain of the linear operator L.

Let X and Y be real Banach spaces and let F be an application

$$F : X \times \mathbb{R} \to \mathrm{Y}$$

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satisfying

$$F(0,\lambda)=0,\quad \forall\lambda\in\mathbb{R} \text{ and } \mathrm{F}\in \mathrm{C}^2$$

We are looking for nontrivial solutions of the equation $F(u,\lambda)=0$.

The value λ_0 is a *bifurcation value* (or $(0,\lambda_0)$ is a *bifurcation point*) for the above equation if every neighborhood of $(0,\lambda_0)$ in $X\times\mathbb{R}$ contains nontrivial solutions of it. The following important result holds.

THEOREM. *If the point $(0,\lambda_0)$ is a bifurcation point for the equation $F(u,\lambda)=0$ then the Fréchet derivative $F_u(0,\lambda_0)$ cannot be a linear homeomorphism of X to Y.*

In the sequel we will consider so-called Fredholm operators. A linear operator $L:X\to Y$ is called a *Fredholm operator* if the kernel of L, $\ker L$ is finite dimensional, the range of L, $\mathrm{im}L$, is closed in Y and the co-kernel of L, $\mathrm{coker}L$, is also finite dimensional. Concerning $F_u(0,\lambda_0)$ we have:

THEOREM. *Let $F_u(0,\lambda_0)$ be a Fredholm operator with kernel V and co-kernel Z. Then there exists a closed subspace W of X and a closed subspace T of Y such that*

$$X=V\oplus W,\quad Y=Z\oplus T$$

The operator $F_u(0,\lambda_0)|_W:W\to T$ is bijective and has a continuous inverse, hence it is a linear homeomorphism of W onto T.

We may decompose now every $u\in X$ and $F:X\to Y$ uniquely

$$u=u_1+u_2,\quad u_1\in V,\quad u_2\in W$$
$$F=F_1+F_2,\quad F_1:X\to Z,\quad F_2:X\to T$$

hence the equation $F(u,\lambda)=0$ is equivalent to the system of equations

$$F_1(u_1,u_2,\lambda)=0$$
$$F_2(u_1,u_2,\lambda)=0$$

If we denote by $L=F_u(0,\lambda_0)$, using a Taylor expansion we have

$$F(u,\lambda)=F(0,\lambda_0)+Lu+N(u,\lambda)$$

and, consequently, the considered equation becomes

$$Lu+N(u,\lambda)=0$$

or

$$Lu_2+N(u_1+u_2,\lambda)=0$$

Let now $Q : Y \to Z$ and $I - Q : Y \to T$ be projections determined by the decomposition. Then, the above equation leads to

$$QN(u, \lambda) = 0 \tag{2}$$

$$u_2 + L^{-1}(I - Q)N(u_1 + u_2, \lambda) = 0 \tag{3}$$

The equation (3) is a fixed point equation. If u_2 can be determined as a function of u_1 and λ, the equation (2) becomes an equation in a finite dimensional space for the finite dimensional u_1.

2. The Lyapunov-Schmidt method as a numerical method

Being used for a long time only for the theoretical demonstration of the existence of the solutions of such a problem, including the branching of Navier-Stokes solutions for example, the above LS method (or the *alternative method*, following Cesari) is also very useful for the effective approximation of these solutions. We will shortly present a constructive variant of the LS method, following [7].

Let S be a real, separable Hilbert space, $L : D(L) \subset S \to S$ a linear operator and $N : D(N) \subset S \to S$ a nonlinear operator. We impose the following assumptions:

a) L is a closed operator (i.e. $x_n \to x$ and $Lx_n \to y$ imply $x \in D(L)$ and $Lx = y$), self-adjoint, $D(L)$ is dense in S and the $\dim Ker\,(L) = p > 0$ is finite,

b) L has the eigenvalues $\lambda_1 = ... = \lambda_p = 0, \lambda_{p+1} > 0, ...$ such that $\lambda_{i+1} \geq \lambda_i$ and $\lambda_i \to \infty$ when $i \to \infty$; the corresponding eigenfunctions $\Phi_1, \Phi_2, ...$ determine an orthonormal complete system in S,

c) there is a subspace S' of S which is complete with regard to a norm μ and $D(L) \subset S'$; for every $x \in D(L)$ its Fourier series $\sum_{k=1}^{\infty} (x, \Phi_k)\, \Phi_k$ converges in the norm μ too and $\{\mu(\Phi_k)/\lambda_k\}_{k>p} \in l^2$. Additionally we admit that there is an $\alpha > 0$ such that for every $x \in S'$ we have $\|x\| \leq \alpha \mu(x)$,

d) $D(N) \cap D(L) \neq \emptyset, D(N) \subset S', D(N)$ is closed vs. the norm μ,

e) for every $R > 0$ there is $\beta_R > 0$ and $b_R > 0$ such that for all $x, y \in D(N)$ with $\mu(x) \leq R, \mu(y) \leq R$ we have $\mu(Nx - Ny) \leq \beta_R \mu(x - y)$ and $\mu(Nx) \leq b_R$.

Our purpose is the study of the existence of the solutions of the equation (1) in $D(L)\cap D(N)$, their numerical approximation and the evaluation of the errors.

Let $m \geq p$ be and

$$S_m = sp\{\Phi_1, ..., \Phi_m\}, S_0 = \{0\}$$

Obviously, $S_m \subset D(L)$. We define the operators $P_m : S \to S_m$ and $H_m : S \to S$ by the following:

If

$$u \in S, \qquad u = \sum_{k=1}^{\infty} (u, \Phi_k)\, \Phi_k$$

then

$$P_m u = \sum_{k=1}^{m} (u, \Phi_k)\, \Phi_k, \qquad H_m u = \sum_{k=m+1}^{\infty} (u, \Phi_k)\, \Phi_k$$

It can be proved that H_m is well defined and for all $u \in S$ we have $H_m u \in D(L)$ while $H_m = (L\Big|_{S_m^{\perp}})^{-1}$. More, from $\|H_m\| = 1/\lambda_{m+1}$ and $\mu(H_m) \leq \alpha\sigma(m)$, where

$$\sigma(m) = \left[\sum_{k=m+1}^{\infty} \left(\frac{\mu(\Phi_k)}{\lambda_k}\right)^2\right]^{1/2}$$

and $\mu(H_m) = \sup_{\mu(u)=1} \mu(H_m u)$, we have $\lim_{m\to\infty} \mu(H_m) = 0$. At the same time we have $H_m L u = (I - P_m)u$.

Let us suppose, additionally, that

f) $R(H_m) \subset D(N), S_m \subset D(N)$ and $D(N)$ is a subspace of S'.

Let now $\overline{u} \in D(L) \cap D(N)$ be a solution of the equation $Lu = Nu$. By applying the operators H_m and P_m to this equation, we find

$$\overline{u} = P_m \overline{u} + H_m N \overline{u}, \tag{4}$$

called the *auxiliary equation* of the problem, and

$$P_m(L\overline{u} - N\overline{u}) = 0, \tag{5}$$

called the *bifurcation equation* of the problem. Conversely, every solution of the system of (4) and (5), belonging to $D(L)\cap D(N)$, is also a solution of (1).

The auxiliary equation is, in fact, a fixed point problem. For its study, let $a > 0, b > 0$ and an "approximating" solution u_0 of the equation $Lu = Nu$ be.

Let $u^* \in S_m$, $u^* = \sum_{k=1}^m c_k \Phi_k$ be so that $\mu(u^* - u_0) \leq a$. We denote

$$S_{u^*}^b = \{u \in D(N) \,|P_m u = u^*, \mu(I - P_m)u \leq b\}$$

We also define the operator $T_{u^*}^b : S_{u^*}^b \to S$ by

$$T_{u^*}^b(u) = u^* + H_m N u$$

for all $u \in S_{u^*}^b$. It can show that for a sufficiently large m, $T_{u^*}^b$ becomes a contraction with respect to the metric space $S_{u^*}^b$ so, according to the Banach fixed point theorem, the operator $T_{u^*}^b$ admits an unique fixed point $y(u^*)$, called the *associate element for* u^* and which can be obtained by the method of successive approximations. So, we define another operator, $\tau_m^b : S_m \to D(L) \cap S_m^\perp$ by $\tau_m^b u^* = H_m N(u^* + \tau_m^b u^*)$. Consequently, for every $u^* \in S_m$, the associate element, $y(u^*) = u^* + \tau_m^b u^*$ i.e. it fulfils the auxiliary equation.

This element also satisfies the bifurcation equation if

$$P_m(Ly(u^*) - Ny(u^*)) = 0$$

i.e. if u^* fulfils the system

$$(\lambda_k u^* - N(u^* + \tau_m^b u^*), \Phi_k) = 0, \quad k = 1, 2, ..., m \tag{6}$$

called the *system of determining equations*. This is a system on $\mathbf{R}^m$ for the coefficients $c_k, k = 1, ..., m$ of u^*. So, we have the theorem:

THEOREM. *If b, m are sufficiently large then the equation (1) admits a solution $\overline{u}$ if and only if the system of determining equations (6) admits a solution u^* and then $\overline{u} = u^* + \tau_m^b u^*$.*

Consequently, the study of the existence of the solutions of the equation $Lu = Nu$ can be reduced to the study of the existence of the solutions of the determining equations and, more, their approximation into $\mathbf{R}^m$ leads to the approximation of the solutions of the equation $Lu = Nu$ into S'. Summarizing, the approximating algorithm is:

a) We are looking for an approximate solution of the equation $Lu = Nu$ of the form

$$u = \sum_{k=1}^{m} c_k \Phi_k + \sum_{k=m+1}^{N} c_k \Phi_k$$

where $0 \leq m \leq N$.

b) By fixing $u^* = \sum_{k=1}^{m} c_k \Phi_k$, we generate the associate function $y(u^*)$ performing the iterations

$$y^0 = u^*,\ y^{s+1} = u^* + H_m N y^s = u^* + \sum_{k=m+1}^{N} C_k^s \Phi_k,\ s = 0, 1, ..., S$$

c) With $y = y^{S+1}$ as an approximation of the associated function, we can write the system $Lu^* = P_m N y$ of the determining equations, with the unknowns $c_1, ..., c_m$. This system of the form $F(c_1, ..., c_m) = 0$ is then numerically solved, by a suitable method, for instance by the Newton's method. Every evaluation of the function F means the reiteration of the b) step. Finally, thus determined u^* generates, also by the b) iterations, an approximation of the solution of the equation $Lu = Nu$.

We remark that in the case of *Galerkin's method,* the approximating solutions are being looked for under the form $u^* = \sum_{k=1}^{N} c_k \Phi_k$, where the coefficients $c_k, k = 1, ..., N$ are determined from the equations $(Lu^* - Nu^*, \Phi_k) = 0, k = 1, ..., N$ i.e.

$$(\lambda_k u^* - Nu^*, \Phi_k) = 0, k = 1, ..., N$$

These equations are got from the determining equations for $m = N$. If $m = 0$ the system of the determining equations disappears. The associate function to a certain u^* verifies the equation $y = L^{-1} N y$, so the algorithm reduces, in this case, to the transformation of the equation $Lu = Nu$ into a fixed point problem. Obviously, this case arises only when there exists the inverse L^{-1} and $L^{-1}N$ is a contraction.

3. The algorithm

There are some methods suitable for nonlinear boundary value problems, for example `SBVP` or `bvp4c` which is now part of Matlab. These methods are based on spline functions and collocation approximations.

Another method is Method of Weighted Residual. In this case, we are looking for an approximate solution of the equation $Lu = Nu$ of the form

$$u = \sum_{i=1}^{N} c_i \varphi_i$$

which leads to the following system for the unknowns c_i

$$\sum_{i=1}^{N} c_i L\varphi_i = N\left(\sum_{i=1}^{N} c_i\varphi_i\right).$$

Well known methods are Galerkin, collocation or eigenfunction expansion where the eigenfunctions of L could be efficiently calculated using the packages SLEDGE or SLEIGN2.

We obtain the equation

$$\sum_{i=1}^{m} c_i\lambda_i\varphi_i + \sum_{i=m+1}^{N} c_i\lambda_i\varphi_i = N\left(\sum_{i=1}^{N} c_i\varphi_i\right)$$

where m is a positive integer, less than N. In this case, the projection H_m is of the form

$$H_m u = \sum_{i=m+1}^{N} \frac{c_i}{\lambda_i}\varphi_i$$

so that we have finally

$$\sum_{i=m+1}^{N} c_i\varphi_i = H_m N\left(\sum_{i=1}^{N} c_i\varphi_i\right) = \sum_{i=m+1}^{N} C_i\varphi_i.$$

By applying H_m to our equations, we are led to this form of the auxiliary equation

$$c_i = C_i(c_1, ..., c_N), i = m+1, ..., N$$

from where we calculate $c_{m+1}, ..., c_N$ as functions of c_1 $...c_m$, again a fixed point problem. The determining equation becomes now of the form

$$\sum_{i=1}^{m} c_i\lambda_i\varphi_i = Q_m N\left(\sum_{i=1}^{N} c_i\varphi_i\right)$$

by applying the projection Q_m and this is a small finite dimensional system for $c_1, ..., c_m$.

In fact, in LS methods, the true unknowns are $c_1, ..., c_m$; the other coefficients $c_{m+1}, ..., c_N$ are calculated as coefficients of the associated fixed point.

4. Matlab implementation

The first version of our package applies only to the Sturm-Liouville case for the linear operator L, in the form

$$Lu = \frac{1}{x^\alpha}\frac{1}{\nu(x)}\frac{d}{dx}\left(x^\alpha p(x)\frac{du}{dx}\right) + q(x)\frac{du}{dx} + g(x)u$$

with the boundary conditions

$$au'(0) + bu(0) + a_1u'(1) + b_1u(1) = 0$$
$$cu'(1) + du(1) + c_0u'(0) + d_0u(0) = 0.$$

There exists a Matlab package `MATSLISE` of V. Ledoux (2004), based on the works of L. Ixaru which uses the so called CP methods to calculate the eigenfunctions of Sturm-Liouville or Schrodinger operators but this package works slowly. A more interesting package is `MWRtools` of R. A. Adomaitis (1998-2001) [2] which uses spectral methods to calculate the eigenfunctions of the Sturm-Liouville operator in order to solve some *linear* boundary value problems.

This package uses plenary the capabilities of Matlab to perform matrix operations. For example, if we compute

$$u^N(x) = \sum_{i=1}^{N} a_i\varphi_i(x), \qquad x \in [0,1]$$

where the eigenfunctions φ_i are represented as column vectors of values corresponding to a Gauss-Radau-Lobatto grid and a_i are the coefficients, we have simply, in Matlab, $u^N = \varphi \cdot a$, where $\varphi(\mathbf{x}) = [\varphi_1(\mathbf{x}), ..., \varphi_N(\mathbf{x})]$ and $a = [a_1, ..., a_N]^T$.

The quadrature is simply

$$\int_0^1 f(x)x^\alpha dx = \mathbf{w}^T \cdot \mathbf{f}$$

i.e. the product of the vector of weights by the vector of function values at the same Gauss-Lobatto-Radau grid. Similarly, the first and the second derivative of f are calculated by these matrix times vector products

$$\frac{df}{dx} = \mathbf{A} \cdot \mathbf{f}, \qquad \frac{d^2f}{dx^2} = \mathbf{B} \cdot \mathbf{f}$$

where A and B are the corresponding differentiation matrices, obtaining the vectors of the values of the derivatives at that grid.

The projection of the function f onto the eigenfunctions is of the form $\mathbf{a} = \varphi^T \cdot \mathbf{W} \cdot \mathbf{f}$, where W is the weight of the inner product. Here a is the vector of the coefficients of the expansion of f as an eigenfunction series.

The Matlab subroutines we use are

(1) `pd.m` to calculate the Gauss-Lobatto-Radau grid and the differentiation matrices
(2) `sl.m` to calculate the eigenfunction and the eigenvalues of the Sturm-Liouville operator with the given boundary conditions
(3) `wip.m` to calculate the coefficients of the expansion
(4) `newton.m` to solve the (small) nonlinear system for the true unknowns $a_1, ..., a_m$
(5) `asoc.m` to perform the successive iteration to calculate the other coefficients $a_{m+1}, ..., a_N$.

5. Examples

The package is under construction. We present here some test problems, using 50 grid points and 21 eigenfunctions. The m-files of the package could be obtained by e-mail from the author (`dtrif@math.ubbcluj.ro`).

5.1. **Example 1.** Solve the boundary value problem

$$x'' + x = 0.5t - 0.5x^3$$
$$x(0) = 0$$
$$x(1) + x'(1) = 0$$

Here the eigenvalues of the linear part are known,

$$\lambda_i = 1 - l_i^2, \text{ where } \sin l_i = -l_i \cos l_i$$
$$\varphi_i = c_i \sin(l_i t).$$

The errors between our solution and the solution calculated by `bvp4c` of Matlab is less than 14×10^{-6}, see figure 1.

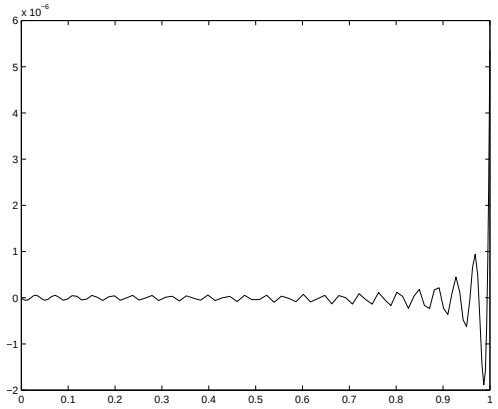

FIGURE 1

5.2. **Example 2.** The next example is the Emden equation (with a singularity)

$$y'' + \frac{2}{x}y' + y^5 = 0$$
$$y'(0) = 0$$
$$y(1) = \frac{\sqrt{3}}{2}.$$

The general Sturm-Liouville operator has now the form

$$\frac{1}{x^\alpha}\frac{1}{x}\left(x^\alpha x u'\right)' = \lambda u \text{ with } \alpha = 1, \quad \nu(x) = x.$$

The known solution is

$$y(x) = \left(1 + \frac{x^2}{3}\right)^{-\frac{1}{2}}$$

and figure 2 shows the error.

5.3. **Example 3.** This example comes from the combustion theory [6],

$$-u'' + Mu' = DY_oY_f e^{-\frac{\theta}{T_0 + u}}$$
$$Mu(0) - u'(0) = 0$$
$$u(1) = 0$$

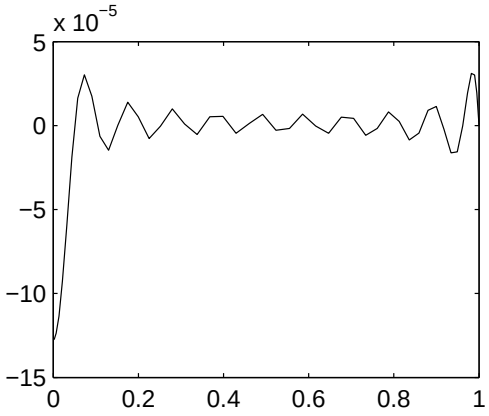

FIGURE 2

where

$$
\begin{aligned}
Y_o &= -u + Y_{o1} e^{M(x-1)} \\
Y_f &= -u + 1 - e^{M\,(x-1)} \\
T_0 &= 0.118, \quad Y_{o1} = 0.21, \quad \theta = 2.6.
\end{aligned}
$$

In this case we have two parameters, M and D. Again we know the eigenvalues and the eigenfunctions of the linear part,

$$
\begin{aligned}
\lambda_n &= \frac{M^2}{4} + \mu_n, \text{ where } \tan \mu_n + 2\frac{\mu_n}{M} = 0 \\
\varphi_n &= e^{\dfrac{Mx}{2}} \sin \mu_n (1-x).
\end{aligned}
$$

The computed solution for $M = 6$ and $D = 5.5 \times 10^8$ (with a high gradient at the burning front) agrees well with those of [6] for these values of parameters. Figure 3 shows the coefficients of the eigenfunction expansion of the solution. We can see that the first five coefficients carry, practically, the whole quantitative information about the solution.

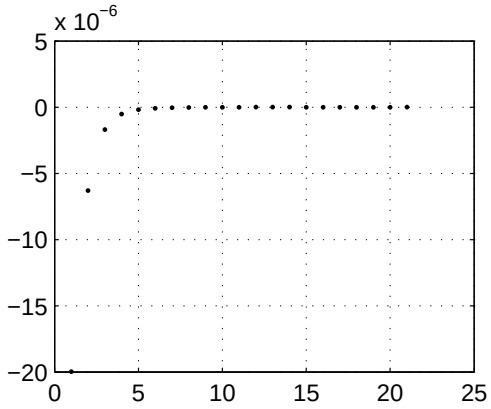

FIGURE 3

5.4. **Example 4.** The next example is the Burgers equation which exhibits a near shock [5],

$$u_t = \upsilon u_{xx} - u\,u_x, \quad \upsilon = \frac{0.01}{\pi}$$
$$u(x,0) = -\sin(\pi x)$$
$$u(0,t) = u(1,t) = 0.$$

Now we search the solution in the form

$$u(x,t) = \sum_k u_k(t)\varphi_k(x)$$

where the coefficients u_k are now functions of t. The auxiliary and the determining equations become differential equations

$$u_k' = \lambda_k u_k - C_k, \qquad k = 1, ..., m$$
$$\frac{1}{\lambda_k}u_k' = u_k - \frac{1}{\lambda_k}C_k, \qquad k = m+1, ..., N.$$

Using a numerical method, say backward Euler for simplicity, we obtain the following iterations for the fixed point coefficients

$$c_k^{n+1} = \frac{1+\upsilon\lambda_k\dfrac{\Delta t}{2}}{1-\upsilon\lambda_k\dfrac{\Delta t}{2}}c_k^n - \frac{\dfrac{\Delta t}{2}}{1-\upsilon\lambda_k\dfrac{\Delta t}{2}}\left[C_k^{n+1} + C_k^N\right]$$

where $k = m + 1, ..., N$ and the time step size $\Delta t = \dfrac{1}{300\pi}$.

The magnitude of the eigenvalues is increasing with k, so we can choose m such that this operator becomes a contraction with a small contraction factor. The maximal magnitude of the derivative of the numerical solution is $u'_{\max} = 151.6771$ (the analytical value is 152.00516) at the moment $\pi \cdot t_{\max} = 1.6033$ (the analytical value is 1.6037), see [5]. Figure 4 shows the solution (with the near shock) at $t_{\max}$.

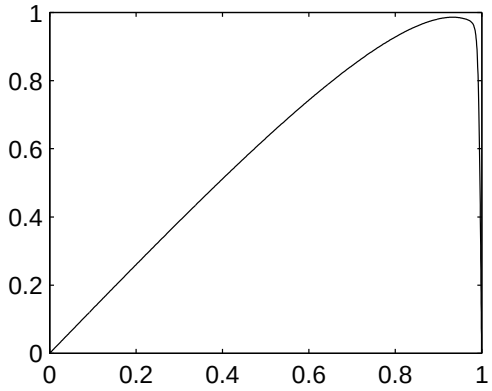

FIGURE 4

6. COMPARISON

The comparison between LS method and `SBVP 1.0` of Auzinger [1] and `bvp4c` of Matlab (see Matlab help) shows a computing time for LS about 3 to 5 times larger. The Matlab profile reports show that about 75% of the computing time was spent on computation of the eigenfunctions and only about 6% on the effective calculations of the numerical solution. But we have good reasons to use LS method.

(1) We can build a database with known eigenfunctions
(2) In the problems with parameters, where we have (for example) bifurcations, or in evolution problems, we can use repeatedly the same eigenfunctions
(3) The eigenfunctions carry physical information, so that our LS solution has a better structure for studies

(4) LS method could be easily extended to 2D or 3D
(5) In all the cases, we have to solve finally a very small nonlinear system (usually $m = 0, 1, 2$), values which also carry information about bifurcation behaviour.

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