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A STABILITY RESULT IN FIXED POINT THEORY

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Abstract. Let K be a compact convex subset of a Banach space. We consider a complete metric space of all the continuous self-mappings of K and show that a typical element of this space (in the sense of Baire category) has a fixed point which is stable under small perturbations of the mapping.

Key Words and Phrases: Baire category, Banach space, compact convex set, continuous mapping, fixed point, generic property

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1. Introduction

Let $K \subset X$ be a nonempty, compact and convex subset of a Banach space $(X, ||\cdot||)$. Denote by $\mathcal{A}$ the set of all continuous mappings $A : K \to K$. For each $A, B \in \mathcal{A}$ set

$$d(A, B) = \sup\{||Ax - Bx|| : \ x \in K\}.$$

It is clear that $(\mathcal{A}, d)$ is a complete metric space. By Schauder's fixed point theorem [3], for each $A \in \mathcal{A}$ there exists $x_* \in K$ such that $Ax_* = x_*$. In this note we are interested in the possible generic stability of such fixed points. For various other generic aspects of fixed point theory, see [1], [2], [4], and the references therein.

We begin with the following simple result.

Proposition 1.1. Let $A \in \mathcal{A}$, $\Omega = \{x \in K : \ Ax = x\}$, and let $\epsilon > 0$. Then there exists a positive number δ such that for each $B \in \mathcal{A}$ satisfying $d(A, B) \le \delta$ and each $x \in K$ satisfying $Bx = x$, there exists $y \in \Omega$ such that $||x - y|| \le \epsilon$.

Proof. If this result were not true, there would exist a sequence $\{B_n\}_{n=1}^{\infty} \subset \mathcal{A}$ satisfying

$$d(A, B_n) \le 1/n \text{ for all integers } n \ge 1, \tag{1.1}$$

and a sequence $\{x_n\}_{n=1}^{\infty} \subset K$ such that for each integer $n \ge 1$,

$$B_n x_n = x_n \text{ and } \inf\{||x_n - y|| : \ y \in \Omega\} > \epsilon. \tag{1.2}$$

Since K is a compact, we may assume without loss of generality that there exists

$$x_* = \lim_{n\to\infty} x_n. \tag{1.3}$$

It follows from (1.3), (1.2), (1.1), and the continuity of A that

$$||Ax_* - x_*|| \le ||Ax_* - Ax_n|| + ||B_n x_n - Ax_n|| + ||B_n x_n - x_n||$$

$$+||x_n - x_*|| \le ||Ax_* - Ax_n|| + 1/n + ||x_n - x_*|| \to 0 \text{ as } n \to \infty.$$

Thus $Ax_* = x_*$, $x_* \in \Omega$, and (1.3) contradicts (1.2). The contradiction we have reached proves Proposition 1.1.

In view of this result, it is natural to ask if, given $A \in \mathcal{A}$, there is a fixed point $x_* \in K$ of A with the following property:

For each $\epsilon > 0$, there exists $\delta > 0$ such that for each $B \in \mathcal{A}$ satisfying $d(A, B) \le \delta$, there exists $y \in K$ such that $By = y$ and $||y - x_*|| \le \epsilon$.

Consider first the following simple example.

Example. Let $X = R^1$, $K = [0, 1]$, and let $Ax = x$ for all $x \in K$. Clearly, the fixed point set of A is the interval $[0, 1]$. For each integer $n \ge 1$, define

$$A_n x = (1 - 1/n)x \ \text{ and } B_n x = \min\{x + 1/n, 1\} \text{ for all } x \in [0, 1].$$

Clearly, $B_n, A_n \to A$ as $n \to \infty$. It is easy to see that for each $n \ge 1$, the fixed point set of A_n is the singleton $\{0\}$, while the fixed point set of B_n is the interval $[1 - 1/n, 1]$.

This example shows that in general the answer to our question is negative. Nevertheless, we show in this paper that for a typical $A \in \mathcal{A}$ (in the sense of Baire category) the answer is positive.

2. The main result

Let $K \subset X$ be a nonempty, closed and convex subset of a Banach space $(X, ||\cdot||)$. Denote by $\mathcal{A}$ the family of all continuous mappings $A : K \to K$ such that the closure of $A(K)$ is a compact set in the norm topology. It is well known [3] that for each $A \in \mathcal{A}$, there is $x_A \in K$ such that $Ax_A = x_A$.

For each $A, B \in \mathcal{A}$ set

$$d(A,B) = \sup\{||Ax - Bx|| : \ x \in K\}. \tag{2.1}$$

It is not difficult to see that $(\mathcal{A}, d)$ is a complete metric space. We now state the main result of our paper.

Theorem 2.1. There exists a subset $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open everywhere dense subsets of $(\mathcal{A}, d)$ so that for each $A \in \mathcal{F}$, there exists $x_* \in K$ such that

(i) $Ax_* = x_*$;

(ii) for each $\epsilon > 0$, there exists $\delta > 0$ such that if $B \in \mathcal{A}$ satisfies $d(A,B) \leq \delta$, then there is $z \in K$ which satisfies $Bz = z$ and $||z - x_*|| \leq \epsilon$.

3. Proof of Theorem 2.1

We precede the proof of Theorem 2.1 with two auxiliary propositions.

Proposition 3.1. Let $A \in \mathcal{A}$, $\epsilon > 0$, and let $x_* \in K$ satisfy $Ax_* = x_*$. Then there exist $B \in \mathcal{A}$ and $\delta > 0$ such that $d(B,A) \leq \epsilon$ and $Bz = x_*$ for each $z \in K$ satisfying $||z - x_*|| \leq \delta$.

Proof. Since A is continuous, there exists $\delta > 0$ such that for each $z \in K$ satisfying $||z - x_*|| \leq 4\delta$, the following inequality holds:

$$||Az - x_*|| \leq \epsilon/4. \tag{3.1}$$

By Urysohn's theorem, there exists a continuous function $\lambda : X \to [0,1]$ such that

$$\lambda(z) = 1 \text{ for each } z \in X \text{ satisfying } ||z - x_*|| \leq \delta \tag{3.2}$$

and

$$\lambda(z) = 0 \text{ for each } z \in X \text{ satisfying } ||z - x_*|| \geq 2\delta. \tag{3.3}$$

Define

$$Bz = \lambda(z)x_* + (1 - \lambda(z))Az \tag{3.4}$$

for all $z \in K$.

Clearly, $B : K \to K$ is continuous, $B(K)$ is contained in a compact subset of X, and

$$Bx_* = x_*. \tag{3.5}$$

By (3.4), (3.2) and (3.3), for each $z \in K$ satisfying $||z - x_*|| \leq \delta$, we have

$$Bz = x_*, \tag{3.6}$$

and for each $z \in K$ satisfying $||z - x_*|| \geq 2\delta$,

$$Bz = Az. \tag{3.7}$$

It follows from (3.4) and the choice of δ (see (3.1)) that for each $z \in K$ satisfying $||z - x_*|| \leq 2\delta$,

$$||Bz - Az|| = ||\lambda(z)x_* + (1 - \lambda(z))Az - Az|| \leq$$

$$\leq ||x_* - Az|| \leq \epsilon/4.$$

This completes the proof of Proposition 3.1.

Proposition 3.2. Let $A \in \mathcal{A}$, $\epsilon > 0$, let $x_* \in K$ be a fixed point of A, and let $B \in \mathcal{A}$ and $\delta > 0$ be as guaranteed by Proposition 3.1. Then for each $C \in \mathcal{A}$ satisfying $d(C, B) \leq \delta$, there is $y \in K$ such that

$$Cy = y \text{ and } ||y - x_*|| \leq d(C, B).$$

Proof. By Proposition 3.1,

$$d(A, B) \leq \epsilon \tag{3.8}$$

and

$$Bz = x_* \text{ for each } z \in K \text{ satisfying } ||z - x_*|| \leq \delta. \tag{3.9}$$

Assume that $C \in \mathcal{A}$ satisfies

$$d(C, B) \leq \delta. \tag{3.10}$$

Set

$$\Omega = \{z \in K : \ ||z - x_*|| \leq d(C, B)\}. \tag{3.11}$$

Clearly, Ω is a closed convex set. It follows from (3.11), (3.10) and (3.9) that for each $z \in \Omega$,

$$||x_* - Cz|| \leq ||x_* - Bz|| + ||Bz - Cz|| = ||Bz - Cz|| \leq d(C, B)$$

and $Cz \in \Omega$. Thus $C(\Omega) \subset \Omega$. Clearly $C(\Omega) \subset C(X)$ is contained in a compact subset of X. By Schauder's fixed point theorem, there is $y \in \Omega$ such that $Cy = y$. Proposition 3.2 is proved.

Completion of the proof of Theorem 2.1.

Let $A \in \mathcal{A}$ and $\epsilon \in (0,1)$. By Propositions 3.1 and 3.2, there exist

$$A_\epsilon \in \mathcal{A},\ x_{A,\epsilon} \in K,\ \text{and}\ \delta_{A,\epsilon} \in (0,1)$$

such that

$$d(A, A_\epsilon) \leq \epsilon, \tag{3.12}$$

$$A_\epsilon z = x_{A,\epsilon} \text{ for each } z \in K \text{ satisfying } ||z - x_{A,\epsilon}|| \leq \delta_{A,\epsilon}, \tag{3.13}$$

and the following property holds:

(P) For each $C \in \mathcal{A}$ satisfying $d(C, A_\epsilon) \leq \delta_{A,\epsilon}$ there is $y \in K$ such that

$$Cy = y \text{ and } ||y - x_{A,\epsilon}|| \leq d(C, A_\epsilon).$$

For each integer $i \geq 1$, set

$$\mathcal{U}(A, \epsilon, i) = \{C \in \mathcal{A} :\ d(C, A_\epsilon) < \delta_{A,\epsilon}/i\}. \tag{3.14}$$

Define

$$\mathcal{F} = \cap_{i=1}^{\infty} \cup \{\mathcal{U}(A, \epsilon, i) :\ A \in \mathcal{A},\ \epsilon \in (0,1)\}. \tag{3.15}$$

Clearly, $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $(\mathcal{A}, d)$.

Let $B \in \mathcal{F}$. For each integer $i \geq 1$, there are $A_i \in \mathcal{A}$ and $\epsilon_i \in (0,1)$ such that

$$B \in \mathcal{U}(A_i, \epsilon_i, i). \tag{3.16}$$

It follows from (3.16), (3.14), and property (P) that for each integer $i \geq 1$, there exists $y_i \in K$ such that

$$By_i = y_i \tag{3.17}$$

and

$$||y_i - x_{A_i,\epsilon_i}|| \leq d(A, (A_i)_{\epsilon_i}) \leq \delta_{A_i,\epsilon_i}/i. \tag{3.18}$$

Since $\{y_i\}_{i=1}^{\infty} \subset B(K)$, there is a subsequence $\{y_{i_k}\}_{k=1}^{\infty}$ which converges to $x_* \in K$. It is obvious that $Bx_* = x_*$.

Let $\epsilon > 0$. There exists a natural number k such that

$$i_k^{-1} < 8^{-1}\epsilon \text{ and } ||y_{i_k} - x_*|| \leq \epsilon/8. \tag{3.19}$$

It follows from (3.18) and (3.19) that

$$||y_{i_k} - x_{A_{i_k},\epsilon_{i_k}}|| \le 1/i_k < \epsilon/8. \tag{3.20}$$

Inequalities (3.19) and (3.20) imply that

$$||x_* - x_{A_{i_k},\epsilon_{i_k}}|| \le ||x_* - y_{i_k}|| + ||y_{i_k} - x_{A_{i_k},\epsilon_{i_k}}|| \le \epsilon/4. \tag{3.21}$$

Let

$$C \in \mathcal{U}(A_{i_k}, \epsilon_{i_k}, i_k). \tag{3.22}$$

It follows from (3.22), (3.14), (3.19), and property (P) that there exists $z \in K$ such that

$$Cz = z \text{ and } ||z - x_{A_{i_k},\epsilon_{i_k}}|| \le d(C, (A_{i_k})_{\epsilon_{i_k}}) \le 1/i_k \le \epsilon/8.$$

When combined with (3.21), this implies that

$$||z - x_*|| \le ||z - x_{A_{i_k},\epsilon_{i_k}}|| + ||x_{A_{i_k},\epsilon_{i_k}} - x_*|| \le \epsilon/2.$$

This concludes the proof of Theorem 2.1.

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