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PERIODIC PROBLEMS WITH ϕ-LAPLACIAN INVOLVING NON-ORDERED LOWER AND UPPER FUNCTIONS

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Abstract. Existence principles for the BVP $(\phi(u'))' = f(t,u,u')$, $u(0) = u(T)$, $u'(0) = u'(T)$ are presented. They are based on the method of lower/upper functions and on the Leray-Schauder topological degree. In contrast to the results known up to now, we need not assume that they are well-ordered.

Key Words and Phrases: ϕ-Laplacian, lower/upper functions, periodic solutions, topological degree.

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1. Formulation of the problem and the fixed point operator

This paper is devoted to the periodic problem with a one-dimensional ϕ-Laplacian

$$(\phi(u'))' = f(t,u,u'), \tag{1.1}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{1.2}$$

where

$$\begin{cases} 0 < T < \infty, \quad f \text{ is an } \mathbb{L}_1\text{-Carathéodory function on } [0,T]\times\mathbb{R}^2, \\ \phi : \mathbb{R} \mapsto \mathbb{R} \text{ is an increasing homeomorphism such that } \phi(\mathbb{R}) = \mathbb{R}. \end{cases} \tag{1.3}$$

A typical example of a function ϕ is the p-Laplacian $\phi_p(y) = |y|^{p-2}\,y$, $p > 1$.

Recently, the problem (1.1), (1.2) in special cases, when ϕ is a p-Laplacian or the right hand side does not depend on u', have been investigated by several authors. Existence and multiplicity results for the non resonance case have been presented e.g. by M. Del Pino, R. Manásevich and A. Murúa [4], J. Mawhin and Manásevich [9], Liu Bing [8] and Yan Ping [17], while the resonance case has been considered e.g. by C. Fabry and D. Fayyad [5] or Liu Bin [7]. A nice survey of the subject with an exhaustive bibliography has been provided by J. Mawhin in [10].

The papers which are devoted to the lower/upper functions method for the problem (1.1), (1.2) mostly assume well-ordered σ_1/σ_2, i.e. $\sigma_1 \le \sigma_2$ on $[0,T]$. We can refer to the papers by A. Cabada and R. Pouso [1], by M. Cherpion, C. De Coster and P. Habets [3] and by S. Staněk [15]. While [1] and [3] concerned the problem (1.1), (1.2) under the Nagumo type two-sided growth conditions, in [15] only one-sided growth conditions of the Nagumo type were needed. Making use of the lower/upper functions method P. Jebelean and J. Mawhin in [6] and S. Staněk in [16] obtained the first existence results also for singular periodic problems with a ϕ-Laplacian. The paper [2] by A. Cabada, P. Habets and R. Pouso is, to our knowledge, the only one presenting the lower/upper functions method for the problem $(\phi(u'))' = f(t,u)$, (1.2) under the assumption that $\sigma_1 \ge \sigma_2$ on $[0,T]$, i.e. lower/upper functions are in the reverse order. If $\phi = \phi_p$ the authors get the solvability for $1 < p \le 2$, only.

Our aim is to offer existence principles for the problem (1.1), (1.2) in terms of lower/upper functions which need not be well-ordered (see Theorem 3.2). Moreover, we will not impose any additional restrictions on ϕ.

As usual, we denote by $\mathbb{C}$ the set of functions continuous on $[0,T]$, $\mathbb{C}^1$ is the set of functions $u \in \mathbb{C}$ with the first derivative continuous on $[0,T]$, $\mathbb{L}_1$ is the set of functions Lebesgue integrable on $[0,T]$ and $\mathbb{AC}$ is the set of functions absolutely continuous on $[0,T]$. For $x \in \mathbb{L}_1$, we put

$$\|x\|_\infty = \sup_{t\in[0,T]} \text{ess}\, |x(t)|, \quad \|x\|_1 = \int_0^T |x(t)|\,\mathrm{d}t \quad \text{and} \quad \overline{x} = \frac{1}{T}\int_0^T x(s)\,\mathrm{d}s.$$

It is well known that C^1 becomes a Banach space when equipped with the norm $\|x\|_{\mathbb{C}^1} = \|x\|_\infty + \|x'\|_\infty$. For $R \in (0,\infty)$ we define $\mathcal{B}(R) = \{u \in \mathbb{C}^1 :$

$\|u\|_{\mathbb{C}^1} < R\}$. The set of $\mathbb{L}_1$-Carathéodory functions on $[0,T]\times\mathbb{R}^2$ is denoted by Car, i.e. Car is the set of functions $f:\ [0,T]\times\mathbb{R}^2\mapsto\mathbb{R}$ having the following properties: (i) for each $x\in\mathbb{R}$ and $y\in\mathbb{R}$ the function $f(\cdot,x,y)$ is measurable on $[0,T]$; (ii) for almost every $t\in[0,T]$ the function $f(t,\cdot,\cdot)$ is continuous on $\mathbb{R}^2$; (iii) for each compact set $K\subset\mathbb{R}^2$ there is a function $m_K\in\mathbb{L}_1$ such that $|f(t,x,y)|\le m_K(t)$ holds for a.e. $t\in[0,T]$ and all $(x,y)\in K$.

If Ω is an open bounded subset of a Banach space $\mathbb{X}$ and the operator $\mathcal{F}:$ $cl(\Omega)\mapsto\mathbb{X}$ is completely continuous and such that $\mathcal{F}(u)\ne u$ for all $u\in\partial\,\Omega$, then we can define the *Leray-Schauder topological degree* $\deg(\mathcal{I}-\mathcal{F},\Omega)$. Here $\mathcal{I}$ is the identity operator on $\mathbb{X}$ and $\mathrm{cl}(\Omega)$ and $\partial\Omega$ denote the closure and the boundary of Ω, respectively.

For a homeomorphism $\psi:\mathbb{R}\mapsto\mathbb{R}$ and $q\in\mathbb{R}$ we denote

$$\{q\}_\psi := \max\{-\psi(-q),\psi(q)\}. \tag{1.4}$$

Definition 1.1. A *solution* of the problem (1.1), (1.2) is a function $u\in\mathbb{C}^1$ such that $\phi(u')\in\mathbb{AC}$, $(\phi(u'(t)))'=f(t,u(t),u'(t))$ for a.e. $t\in[0,T]$ and (1.2) is satisfied.

Remark 1.2. Notice that the condition (1.2) is equivalent to the condition $u(0)=u(T)=u(0)+u'(0)-u'(T)$.

Definition 1.3. A function $\sigma\in\mathbb{C}^1$ is an *upper function* of (1.1), (1.2) if $\phi(\sigma')\in\mathbb{AC}$,

$$(\phi(\sigma'(t)))'\le f(t,\sigma(t),\sigma'(t)) \qquad \text{for a.e. } t\in[0,T], \tag{1.5}$$

$$\sigma(0)=\sigma(T),\ \sigma'(0)\le\sigma'(T). \tag{1.6}$$

If the inequalities in (1.5)–(1.6) are reversed, σ is called an *upper function.*

To transform the problem (1.1), (1.2) into a fixed point problem in $\mathbb{C}^1$, we will make use of some ideas from [9] (see also e.g. [1], [10], [17] or [13]). Having in mind Remark 1.2, let us consider the quasilinear Dirichlet problem

$$(\phi(x'))'=h(t) \quad \text{a.e. on } [0,T], \quad x(0)=x(T)=d, \tag{1.7}$$

with $h\in\mathbb{L}_1$ and $d\in\mathbb{R}$. A function $x\in\mathbb{C}^1$ is a solution of (1.7) if and only if

$$x(t)=d+\int_0^t\phi^{-1}\Big(\phi(x'(0))+\int_0^s h(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s \quad \text{for } t\in[0,T]$$

and

$$\int_0^T \phi^{-1}\Big(\phi(x'(0)) + \int_0^s h(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s = 0.$$

Since ϕ is increasing on $\mathbb{R}$ and $\phi(\mathbb{R}) = \mathbb{R}$, for each fixed $\ell \in \mathbb{C}$ the equation

$$\int_0^T \phi^{-1}\Big(a + \ell(t)\Big)\,\mathrm{d}t = 0$$

has exactly one solution $a = a(\ell)$ in $\mathbb{R}$ for each $\ell \in \mathbb{C}$. So, we can define an operator $\mathcal{K} : \mathbb{L}_1 \mapsto \mathbb{C}^1$ by

$$(\mathcal{K}(h))(t) = \int_0^t \phi^{-1}\Big(a\big(\int_0^\tau h(s)\,\mathrm{d}s\big) + \int_0^\tau h(s)\,\mathrm{d}s\Big)\,\mathrm{d}\tau \text{ for a.e. } t \in [0,T]. \quad (1.8)$$

Furthermore, let $\mathcal{N} : \mathbb{C}^1 \mapsto \mathbb{L}_1$ and $\mathcal{F} : \mathbb{C}^1 \mapsto \mathbb{C}^1$ have the form

$$(\mathcal{N}(u))(t) = f(t, u(t), u'(t)) \quad \text{for a.e. } t \in [0,T]$$

and

$$(\mathcal{F}(u))(t) = u(0) + u'(0) - u'(T) + (\mathcal{K}(\mathcal{N}(u)))(t) \quad \text{for } t \in [0,T]. \quad (1.9)$$

It follows from the definition of $\mathcal{K}$ that $x \in \mathbb{C}^1$ is a solution to (1.7) if and only if $x = d + \mathcal{K}(h)$. Consequently, $u \in \mathbb{C}^1$ is a solution to (1.1), (1.2) if and only if $\mathcal{F}(u) = u$. Taking into account [9, Proposition 2.2], we can summarize:

Lemma 1.4. *Let* $\mathcal{F} : \mathbb{C}^1 \mapsto \mathbb{C}^1$ *be defined by* (1.9). *Then* $\mathcal{F}$ *is completely continuous and* $u \in \mathbb{C}^1$ *is a solution to* (1.1), (1.2) *if and only if* $\mathcal{F}(u) = u$.

2. Well-ordered case

The main result of this section is Theorem 2.1 which determines the Leray-Schauder degree of the operator representing the problem (1.1), (1.2) in the case that it has a couple of well-ordered lower/upper functions.

Beside (1.3), we will work with the following assumptions:

$$\sigma_1 \text{ and } \sigma_2 \text{ are respectively lower and upper functions of (1.1), (1.2),} \quad (2.1)$$

$$\sigma_1 < \sigma_2 \text{ on } [0,T] \quad (2.2)$$

and with the following class of auxiliary problems:

$$(\phi(v'))' = \eta(v')\, f(t, v, v'), \quad v(0) = v(T), \quad v'(0) = v'(T), \quad (2.3)$$

where η may be an arbitrary continuous function mapping $\mathbb{R}$ into $[0,1]$.

For $\rho > 0$ we define

$$\Omega_\rho = \big\{u \in \mathbb{C}^1 : \sigma_1 < u < \sigma_2 \text{ on } [0,T] \quad \text{and} \quad \|u'\|_\infty < \rho\big\}. \tag{2.4}$$

Theorem 2.1. *Assume that* (1.3), (2.1) *and* (2.2) *hold. Furthermore, suppose that there exists* $r^* \in (0,\infty)$ *such that*

$$\begin{cases} \|v'\|_\infty < r^* \quad \text{for each continuous } \eta : \mathbb{R} \mapsto [0,1] \text{ and for} \\ \text{each solution } \ v \ \text{ of } \ (2.3) \text{ such that } \ \sigma_1 \le v \le \sigma_2 \text{ on } [0,T]. \end{cases} \tag{2.5}$$

Finally, let $\mathcal{F} : \mathbb{C}^1 \mapsto \mathbb{C}^1$ *and* Ω_ρ *be defined by* (1.9) *and* (2.4), *respectively. Then*

$$\deg(\mathcal{I} - \mathcal{F}, \Omega_\rho) = 1 \quad \text{for each } \ \rho \ge r^* \text{ such that } \mathcal{F}(u) \ne u \text{ on } \partial\Omega_\rho.$$

Proof. We will start with the following "maximum principle" assertion:
CLAIM. *Assume* (1.3), (2.1), (2.2) *and let* $\widetilde{f} \in \text{Car}$ *and* $d \in \mathbb{R}$ *be such that*

$$\begin{cases} \widetilde{f}(t,x,y) < \ f(t,\sigma_1(t),\sigma_1'(t)) \text{ for a.e. } t \in [0,T], \text{ all } \ x \in (-\infty,\sigma_1(t)) \\ \qquad\qquad \text{and all } \ y \in \mathbb{R} \text{ such that } \ |y - \sigma_1'(t)| \le \dfrac{\sigma_1(t) - x}{\sigma_1(t) - x + 1}, \\ \widetilde{f}(t,x,y) > \ f(t,\sigma_2(t),\sigma_2'(t)) \text{ for a.e. } t \in [0,T], \text{ all } \ x \in (\sigma_2(t),\infty) \\ \qquad\qquad \text{and all } \ y \in \mathbb{R} \text{ such that } \ |y - \sigma_2'(t)| \le \dfrac{x - \sigma_2(t)}{x - \sigma_2(t) + 1} \end{cases} \tag{2.6}$$

and

$$\sigma_1(0) \le d \le \sigma_2(0). \tag{2.7}$$

Then any solution u *of the problem*

$$\begin{cases} (\phi(u'(t)))' = \widetilde{f}(t,u(t),u'(t)) \quad \text{a.e. on } \ [0,T], \\ u(0) = u(T) = d \end{cases} \tag{2.8}$$

satisfies the estimate

$$\sigma_1 \le u \le \sigma_2 \text{ on } [0,T]. \tag{2.9}$$

Proof of CLAIM. Let u be a solution of (2.8) and $v = u - \sigma_2$ on $[0,T]$. Assume that

$$\sup_{t\in[0,T]} v(t) > 0. \tag{2.10}$$

Due to (1.6) and (2.7), we have $v(0) = v(T) \le 0$ and hence there are $\alpha \in (0,T)$ and $\beta \in (\alpha, T]$ such that $v(\alpha) = \sup_{t\in[0,T]} v(t)$, $v'(\alpha) = 0$, and

$$v(t) > 0 \quad \text{and} \quad |v'(t)| < \frac{v(t)}{v(t)+1} \ \text{ for } \ t \in [\alpha, \beta]. \tag{2.11}$$

Using (1.5) and (2.6), we obtain

$$(\phi(u'(t)))' - (\phi(\sigma_2'(t)))' = \widetilde{f}(t, u(t), u'(t)) - (\phi(\sigma_2'(t)))'$$

$$> f(t, \sigma_2(t), \sigma_2'(t)) - (\phi(\sigma_2'(t)))' \ge 0$$

for a.e. $t \in [\alpha, \beta]$. Hence

$$0 < \int_\alpha^t \Big(\phi(u'(s)))' - (\phi(\sigma_2'(s))' \Big)\, \mathrm{d}s = \phi(u'(t)) - \phi(\sigma_2'(t))$$

for all $t \in (\alpha, \beta]$. Therefore $v'(t) = u'(t) - \sigma_2'(t) > 0$ for all $t \in (\alpha, \beta]$. This contradicts the fact that v has a maximum at α, i.e. $u \le \sigma_2$ on $[0,T]$.

If we put $v = \sigma_1 - u$ on $[0,T]$ and use the properties of σ_1 instead of σ_2, we prove that $u \ge \sigma_1$ on $[0,T]$ by a similar argument and so we complete the proof of Claim.

Let r^* be such that (2.5) is true and let

$$\Omega = \big\{ u \in \mathbb{C}^1 : \sigma_1 < u < \sigma_2 \text{ on } [0,T] \text{ and } \ \|u'\|_\infty < r^* \big\} \tag{2.12}$$

and

$$\mathcal{F}(x) \ne x \quad \text{for each} \ \ x \in \partial\Omega. \tag{2.13}$$

Furthermore, let

$$R^* = r^* + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty, \tag{2.14}$$

$$\eta(y) = \begin{cases} 1 & \text{for } \ |y| \le R^*, \\ 2 - \dfrac{|y|}{R^*} & \text{for } \ R^* < |y| < 2\,R^*, \\ 0 & \text{for } \ |y| \ge 2\,R^* \end{cases} \tag{2.15}$$

and

$$g(t,x,y) = \eta(y)\, f(t,x,y) \text{ for a.e. } t \in [0,T] \text{ and all } \ (x,y) \in \mathbb{R}^2. \tag{2.16}$$

Then σ_1/σ_2 are lower/upper functions for the modified problem

$$(\phi(u'))' = g(t, u(t), u'(t)), \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{2.17}$$

and there is an $m \in \mathbb{L}_1$ such that $|g(t,x,y)| \le m(t)$ for a.e. $t \in [0,T]$ and all $(x,y) \in [\sigma_1(t), \sigma_2(t)] \times \mathbb{R}^2$. Put

$$\omega_i(t,\zeta) = \sup_{z\in\mathbb{R},\, |\sigma_i'(t)-z|\le\zeta} |f(t,\sigma_i(t),\sigma_i'(t)) - f(t,\sigma_i(t),z)|$$

for $\quad i = 1,2 \quad$ and $\quad (t,\zeta) \in [0,T] \times [0,\infty) \quad$ and

$$\widetilde{f}(t,x,y) = \begin{cases} g(t,\sigma_1(t),y) - \omega_1(t, \dfrac{\sigma_1(t)-x}{\sigma_1(t)-x+1}) & \text{if } x < \sigma_1(t), \\ g(t,x,y) & \text{if } x \in [\sigma_1(t),\sigma_2(t)], \\ g(t,\sigma_2(t),y) + \omega_2(t, \dfrac{x-\sigma_2(t)}{x-\sigma_2(t)+1}) & \text{if } x > \sigma_2(t). \end{cases}$$

Then $\widetilde{f} \in \mathrm{Car}$,

$$\widetilde{f}(t,x,y) = g(t,x,y) \text{ for a.e. } t \in [0,T] \text{ and all } (x,y) \in [\sigma_1(t),\sigma_2(t)] \times \mathbb{R}, \tag{2.18}$$

there is $\widetilde{m} \in \mathbb{L}_1$ such that

$$|\widetilde{f}(t,x,y)| \le \widetilde{m}(t) \text{ for a.e. } t \in [0,T] \text{ and all } (x,y) \in \mathbb{R}^2, \tag{2.19}$$

and the relations (2.6) are valid if f is replaced by g. Define

$$\alpha(x) = \begin{cases} \sigma_1(0) & \text{for } x < \sigma_1(0), \\ x & \text{for } \sigma_1(0) \le x \le \sigma_2(0), \\ \sigma_2(0) & \text{for } x > \sigma_2(0). \end{cases} \tag{2.20}$$

Consider an auxiliary problem

$$(\phi(u'))' = \widetilde{f}(t,u,u'), \quad u(0) = u(T) = \alpha(u(0) + u'(0) - u'(T)). \tag{2.21}$$

By Lemma 1.4, (2.21) is equivalent to the operator equation $\widetilde{\mathcal{F}}(u) = u$, where $\widetilde{\mathcal{F}} : \mathbb{C}^1 \mapsto \mathbb{C}^1$ is a completely continuous operator defined by

$$\widetilde{\mathcal{F}}(u) = \alpha(u(0) + u'(0) - u'(T)) + \mathcal{K}(\widetilde{\mathcal{N}}(u)) \tag{2.22}$$

with $\mathcal{K} : \mathbb{L}_1 \mapsto \mathbb{C}^1$ given by (1.8) and $\widetilde{\mathcal{N}} : \mathbb{C}^1 \mapsto \mathbb{L}_1$ of the form

$$(\widetilde{\mathcal{N}}(u))(t) = \widetilde{f}(t,u(t),u'(t)) \quad \text{for a.e. } t \in [0,T]. \tag{2.23}$$

Due to (2.19), (2.22) and (2.23), we can find $R_0 \in (0,\infty)$ such that

$$\Omega \subset \mathcal{B}(R_0) \quad \text{and} \quad \widetilde{\mathcal{F}}(u) \in \mathcal{B}(R_0) \quad \text{for all } u \in \mathbb{C}^1.$$

In particular, $\|u\|_{\mathbb{C}^1} < R_0$ for each $\lambda \in [0,1]$ and each $u \in \mathbb{C}^1$ such that $u = \lambda\,\widetilde{\mathcal{F}}(u)$. So, the operator $\mathcal{I} - \lambda\,\widetilde{\mathcal{F}}$ is a homotopy on $\mathrm{cl}(\mathcal{B}(R_0)) \times [0,1]$. Hence,

$$\deg(\mathcal{I} - \widetilde{\mathcal{F}}, \mathcal{B}(R_0)) = \deg(\mathcal{I}, \mathcal{B}(R_0)) = 1. \tag{2.24}$$

Let

$$\widetilde{\Omega} = \big\{u \in \Omega : \sigma_1(0) < u(0) + u'(0) - u'(T) < \sigma_2(0)\big\}. \tag{2.25}$$

Then

$$\widetilde{\mathcal{F}} = \mathcal{F} \text{ on } \mathrm{cl}(\widetilde{\Omega}) \tag{2.26}$$

and

$$\Big(\mathcal{F}(u) = u \quad \text{and} \quad u \in \Omega\Big) \implies u \in \widetilde{\Omega}. \tag{2.27}$$

We will prove that the following complementary implication is true as well:

$$\widetilde{\mathcal{F}}(u) = u \implies u \in \widetilde{\Omega}. \tag{2.28}$$

Indeed, let $u \in \mathbb{C}^1$ be a fixed point of $\widetilde{\mathcal{F}}$. In particular, we have

$$\sigma_1(0) \le u(0) = u(T) = \alpha(u(0) + u'(0) - u'(T)) \le \sigma_2(0). \tag{2.29}$$

By Claim, this implies that

$$\sigma_1 \le u \le \sigma_2 \text{ on } [0,T]. \tag{2.30}$$

In accordance with (2.18), we have

$$\widetilde{f}(t, u(t), u'(t)) = g(t, u(t), u'(t)) \text{ for a.e. } t \in [0,T] \tag{2.31}$$

Now, we will show that u satisfies also the second condition from (1.2), i.e. that $u'(0) = u'(T)$ holds. Obviously, this is true whenever

$$\sigma_1(0) \le u(0) + u'(0) - u'(T) \le \sigma_2(0). \tag{2.32}$$

If

$$u(0) + u'(0) - u'(T) > \sigma_2(0), \tag{2.33}$$

then, by (1.6), (2.20) and (2.29), $u(0) = u(T) = \sigma_2(0) = \sigma_2(T)$ and $u'(0) > u'(T)$. This together with (2.30) and (2.33) may hold only if $\sigma_2'(0) \ge u'(0) > u'(T) \ge \sigma_2'(T)$, which contradicts (1.6). Similarly we would prove that the relation $u(0) + u'(0) - u'(T) \ge \sigma_1(0)$ is true as well. This means that (2.32) and hence also $u'(0) = u'(T)$ hold. To summarize, u satisfies (2.21), (2.31) and (1.2) and, consequently, it is a solution to (2.17). Now, by (2.30), (2.15), (2.16) and (2.5), the relation $\|u'\|_\infty < r^* \le R^*$ follows. Therefore $u \in \mathrm{cl}(\Omega)$ and, due to (2.26) and (2.13), we obtain

$$u \in \Omega \quad \text{and} \quad \mathcal{F}(u) = u.$$

Finally, by (2.27), we conclude that $u \in \widetilde{\Omega}$, which completes the proof of (2.28).

Making use of (2.27), (2.28) and (2.24) and taking into account the excision property of the degree, we get

$$\deg(\mathcal{I} - \mathcal{F}, \Omega) = \deg(\mathcal{I} - \mathcal{F}, \widetilde{\Omega}) = \deg(\mathcal{I} - \widetilde{\mathcal{F}}, \widetilde{\Omega}) = \deg(\mathcal{I} - \widetilde{\mathcal{F}}, \mathcal{B}(R_0)) = 1.$$

Finally, since according to (2.5) all the fixed points u of $\mathcal{F}$ such that $\sigma_1 < u < \sigma_2$ on $[0,T]$ belong to Ω, we can conclude that

$$\deg(\mathcal{I} - \mathcal{F}, \Omega_\rho) = \deg(\mathcal{I} - \mathcal{F}, \Omega) = 1$$

holds for each $\rho \ge r^*$ such that $\mathcal{F}(x) \ne x$ on $\partial\Omega_\rho$. □

Remark 2.2. Let us emphasize that up to now for the problem (1.1), (1.2) no result giving the degree of the related operator with respect to the set determined by the associated lower and upper functions like Theorem 2.1 is known to us. Implementing an arbitrary condition ensuring the existence of $r^* \in (0,\infty)$ with the property (2.5) and making use of the standard approximation technique, we could complete alternate proofs of already known existence results (see e.g. [1, Theorem 3.1] or [15, Theorem 1]) valid when the existence of a pair σ_1/σ_2 of lower/upper functions associated with the given problem and such that $\sigma_1 \le \sigma_2$ on $[0,T]$ is supposed. Usually, conditions of the Nagumo type are used to this aim. To our knowledge, the most general version of such conditions which covers also the case with ϕ-Laplacian is provided by [15, Lemma 2.1] due to S. Staněk. For the purpose of Section 3 the following simple a priori estimate will be sufficient.

Lemma 2.3. *Let $\phi : \mathbb{R} \mapsto \mathbb{R}$ be an increasing homeomorphism such that $\phi(\mathbb{R}) = \mathbb{R}$ and let $h \in \mathbb{L}_1$. Then there is $r^* \in (0,\infty)$ such that $\|u'\|_\infty < r^*$ holds for each $u \in \mathbb{C}^1$ fulfilling* (1.2) *and such that $\phi(u') \in \mathbb{AC}$ and*

$$(\phi(u'(t)))' > h(t) \quad (\mathit{or} \ \ (\phi(u'(t)))' < h(t)) \ \mathit{for\ a.e.}\ t \in [0,T].$$

Proof. Let $h \in \mathbb{L}_1$. By the proof of [11, Lemma 1.1], we can see that $\|v\|_\infty < \|h\|_1$ holds for each $v \in \mathbb{AC}$ such that $v(0) = v(T)$, $v(t_v) = 0$ for some $t_v \in (0,T)$ and $v'(t) > h(t)$ (or $v'(t) < h(t)$ for a.e. $t \in [0,T]$. The assertion of the lemma follows by setting $v=\phi(u')-\phi(0)$ and $r^*=\{|\phi(0)|+\|h\|_1\}_{\phi^{-1}}$, where we are using the notation introduced in (1.4)).

□

3. Non-ordered case

Our main result is Theorem 3.2 which is the first known existence principle for periodic problems with a general ϕ-Laplacian and non-ordered lower/upper functions. We shall start this section with the following auxiliary assertion which will be helpful for its proof.

Lemma 3.1. *Let σ_1, $\sigma_2 \in \mathbb{C}$ and $u \in \mathbb{C}$ be such that*

$$u(t_u) < \sigma_1(t_u) \quad \text{and} \quad u(s_u) > \sigma_2(s_u) \quad \text{for some} \;\; t_u, s_u \in [0,T]. \tag{3.1}$$

Then there exists $\tau_u \in [0,T]$ such that

$$\min\{\sigma_1(\tau_u),\sigma_2(\tau_u)\} \le u(\tau_u) \le \max\{\sigma_1(\tau_u),\sigma_2(\tau_u)\}. \tag{3.2}$$

Proof. If $u(0) < \min\{\sigma_1(0),\sigma_2(0)\}$ while (3.2) does not hold, then necessarily $u(t) < \min\{\sigma_1(t),\sigma_2(t)\}$ on $[0,T]$ which contradicts (3.1). If $u(0) > \max\{\sigma_1(0),\sigma_2(0)\}$ while (3.2) does not hold, then $u(t) > \max\{\sigma_1(t),\sigma_2(t)\}$ on $[0,T]$ which again contradicts (3.1). □

Theorem 3.2. *Assume* (1.3), (2.1), *and*

$$\sigma_1(\tau) > \sigma_2(\tau) \quad \text{for some} \;\; \tau \in [0,T]. \tag{3.3}$$

Furthermore, let $h \in \mathbb{L}_1$ be such that

$$f(t,x,y) > h(t) \quad (\text{or } f(t,x,y) < h(t)) \quad \text{for a.e. } t \in [0,T] \text{ and all } \; x,y \in \mathbb{R}.$$

Then the problem (1.1), (1.2) *has a solution u satisfying* (3.2) *for some $\tau_u \in [0,T]$.*

Proof. Assume e.g. that

$$f(t,x,y) > h(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } \; x,y \in \mathbb{R} \tag{3.4}$$

is the case and put $\widetilde{h}(t) = -(|h(t)|+2)$ for a.e. $t \in [0,T]$. By Lemma 2.3, there is $r^* \in (0,\infty)$ such that

$$\begin{cases} \|u'\|_\infty < r^* \quad \text{for each} \;\; u \in \mathbb{C}^1 \;\; \text{fulfilling} \;\; (1.2), \;\; \phi(u') \in \mathbb{AC} \;\; \text{and} \\ (\phi(u'(t))' > \widetilde{h}(t) \quad \text{for a.e. } t \in [0,T]. \end{cases} \tag{3.5}$$

Furthermore, put

$$c^* = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + T\, r^* \tag{3.6}$$

and define

$$\widetilde{f}(t,x,y) = \begin{cases} -(|h(t)|+1) & \text{if } \ x \le -(c^*+1), \\ f(t,x,y) + (x+c^*)\,(|h(t)|+1+f(t,x,y)) & \\ & \text{if } \ -(c^*+1) < x < -c^*, \\ f(t,x,y) & \text{if } \ -c^* \le x \le c^*, \\ f(t,x,y) + (x-c^*)\,|h(t)| & \text{if } \ c^* < x < c^*+1, \\ f(t,x,y) + |h(t)| & \text{if } \ x \ge c^*+1. \end{cases} \tag{3.7}$$

Let us consider an auxiliary problem

$$(\phi(u'))' = \widetilde{f}(t,u,u'), \quad u(0) = u(T), \quad u'(0) = u'(T). \tag{3.8}$$

The functions σ_1 and σ_2 are respectively a lower and an upper function of (3.8). Furthermore,

$$\widetilde{f}(t,x,y) > \widetilde{h}(t) \quad \text{ for a.e. } t \in [0,T] \text{ and all } \ x,y \in \mathbb{R}. \tag{3.9}$$

Moreover,

$$\widetilde{f}(t,x,y) < 0 \quad \text{ for a.e. } t \in [0,T] \text{ and all } \ x \in (-\infty, -c^*-1], y \in \mathbb{R}, \tag{3.10}$$

$$\widetilde{f}(t,x,y) > 0 \quad \text{ for a.e. } t \in [0,T] \text{ and all } \ \text{ and all } \ x \in [c^*+1, \infty), y \in \mathbb{R}. \tag{3.11}$$

In particular, $\sigma_3(t) \equiv -c^* - 2$ and $\sigma_4(t) \equiv c^* + 2$ are respectively a lower and an upper function for (3.8). Let us denote

$$\Omega_0 = \{u \in \mathbb{C}^1 : \ \sigma_3 < u < \sigma_4 \text{ on } [0,T], \ \|u'\|_\infty < r^*\},$$

$$\Omega_1 = \{u \in \Omega_0 : \ \sigma_3 < u < \sigma_2 \text{ on } [0,T]\},$$

$$\Omega_2 = \{u \in \Omega_0 : \ \sigma_1 < u < \sigma_4 \text{ on } [0,T]\}$$

and

$$\Omega = \Omega_0 \setminus \mathrm{cl}(\Omega_1 \cup \Omega_2).$$

Clearly, Ω is the set of all $u \in \Omega_0$ for which the relations $\|u'\|_\infty < r^*$ and (3.1) are satisfied.

By Lemma 1.4, the problem (3.8) is equivalent to the operator equation $\widetilde{\mathcal{F}}(u) = u$ in $\mathbb{C}^1$, where

$$\widetilde{\mathcal{F}}(u) = u(0) + u'(0) - u'(T) + \mathcal{K}(\widetilde{\mathcal{N}})(u) \tag{3.12}$$

and $\mathcal{K} : \mathbb{L}_1 \mapsto \mathbb{C}^1$ and $\widetilde{\mathcal{N}} : \mathbb{C}^1 \mapsto \mathbb{L}_1$ are given respectively by (1.8) and (2.23) (with $\widetilde{f}$ defined now by (3.7)). Next we will prove the following two assertions:
Claim 1. *If* $\widetilde{\mathcal{F}}(u) = u$ *and* $u \in \mathrm{cl}(\Omega_0)$, *then* $u \in \Omega_0$.
Proof of Claim 1. Let $\widetilde{\mathcal{F}}(u) = u$ and $u \in \partial\Omega_0$. Since, by (3.5) and (3.9), we have $\|u'\|_\infty < r^*$, this can happen only if

$$u(\alpha) = \max_{t\in[0,T]} u(t) = c^* + 2 \quad \text{or} \quad u(\alpha) = \min_{t\in[0,T]} u(t) = -(c^* + 2) \tag{3.13}$$

for some $\alpha \in [0, T)$. In the former case, we have $u'(\alpha) = 0$ and $u(t) > c^* + 1$ on $[\alpha, \beta]$ for some $\beta \in (\alpha, T]$. Due to (3.11), we have also $(\phi(u'(t)))' = \widetilde{f}(t, u(t), u'(t)) > 0$ for a.e. $t \in [\alpha, \beta]$, i.e. $u'(t) > 0$ on $(\alpha, \beta]$, a contradiction. Similarly we can prove that the latter case in (3.13) is impossible.
Claim 2. *If* $\widetilde{\mathcal{F}}(u) = u$ *and* $u \in \mathrm{cl}(\Omega)$, *then* $\|u\|_\infty < c^*$.
Proof of Claim 2. Let $\widetilde{\mathcal{F}}(u) = u$ and $u \in \mathrm{cl}(\Omega)$. By (3.5) and (3.9) we have $\|u'\|_\infty < r^*$ and, by Lemma 3.1 and Claim 1, $\|u\|_\infty < \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + T\, r^* = c^*$.

Now, there are two cases to consider: either $\widetilde{\mathcal{F}}(u) = u$ for some $u \in \partial\Omega$ or $\widetilde{\mathcal{F}}(u) \neq u$ on $\partial\Omega$. If $\widetilde{\mathcal{F}}(u) = u$ for some $u \in \partial\Omega$, then, by Claim 2, $\|u\|_\infty < c^*$ must hold, i.e. $\mathcal{F}(u) = \widetilde{\mathcal{F}}(u) = u$. Hence, by Lemma 1.4, u is a solution to (1.1), (1.2). If $\widetilde{\mathcal{F}}(u) \neq u$ on $\partial\Omega$, then, as $\partial\Omega = \partial\Omega_0 \cup \partial\Omega_1 \cup \partial\Omega_2$, we can apply Theorem 2.1 to get

$$\deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_0) = \deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_1) = \deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_2) = 1.$$

Furthermore, by (3.3) we have $\Omega_1 \cap \Omega_2 = \emptyset$. Therefore

$$\deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega) = \deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_0) - \deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_1) - \deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_2) = -1$$

and there exists $u \in \Omega$ such that $\widetilde{\mathcal{F}}(u) = u$. By Claim 2, we have $\|u\|_\infty < c^*$ which, by virtue of (3.7) and (3.12), yields $\widetilde{\mathcal{F}}(u) = \mathcal{F}(u) = u$, i.e. u is a solution to (1.1), (1.2). □

Remark 3.3. Theorem 3.2 combined with techniques introduced e.g. in [14] enables one to prove some existence results to certain problems with singularities. In particular, we can derive results similar to those by P. Jebelean and

J. Mawhin [6], but without restricting ourselves to p-Laplacians. This will be the content of some of our forthcoming papers. For an extension to impulsive periodic problems with ϕ-Laplacian, see our preprints [13].

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