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ASPECTS IN HYPERBOLIC A-PROPERNESS

DAN PASCALI

Courant Institute, New York University
251, Mercer St, New York, NY
10012-1185, SUA
Faculty of Mathematics, "Ovidius" University
B-dul Mamaia 124, 900527 Constanţa, Romania
E-mail: dp39@nyu.edu

Abstract. The hyperbolic A-properness on Hilbert spaces was introduced in connection with the approximation-solvability of the semilinear equation Lu+Su=f where the linear part has an infinite-dimensional kernel. Its extension to Banach spaces requires an approximation scheme Γ, called the Fredholm factorization associated with L.

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The talk is devoted to the study the approximation-solvability of semilinear equations of the form

$$Tu = Lu + N(u) = f, \tag{1}$$

between Banach spaces X and Y, when the nonlinear perturbation N satisfies a condition of A-properness with respect to the linear part. The idea of approximation solvability is to replace the above equation by a sequence of approximate equations

$$T_n u_n = L_n u_n + N_n(u_n) = f_n, \tag{2}$$

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between sequences spaces $\{X_n\}$ and $\{Y_n\}$, to which more classical methods of resolution can be applied, and to extract a solution of the original problem from a sequence of solutions of approximate equations (2).

W.V. Petryshyn introduced in the sixties a comprehensive class of mappings, beyond the compact ones, which permit to pass from the equations (2) to the equation (1). This is so-called *approximation-proper*(A-proper) operators. In other words, the approximation-solvability of the initial equation (1) is equivalent to its finite-dimensional solvability (2) whenever T is A-proper. As usually in studying of semilinear equations, the size of nonlinearity interacts topological properties and spectral of the linear part.

W.V. Petryshyn and others studied intensively the elliptic case when $\dim KerL < \infty$. We introduced [5] an A-proper technique for almost self-adjoint operators in a real Hilbert space H, which covers also the case when $KerL$ is infinite-dimensional. This is called the *hyperbolic A-properness* because L inherits the properties of the generalized d'Alembertian in the study of non-trivial periodic solutions of one dimensional space semilinear wave of the form

$$\begin{cases} u_{tt} - u_{xx} + g(t,x,u) = 0 \ , \ (t,x) \in R \times (0,\pi) \ , \\ u(t,0) = u(t,\pi) = 0 \ , \ t \in R \ , \\ u(t+2\pi, x) = u(t,x) \ , \ (t,x) \in \Omega \ , \end{cases} \tag{3}$$

where $\Omega = (0,2\pi) \times (0,\pi)$ and g: $\Omega \times R \longmapsto R$. We assume that the function $g(t,x,r)$ is 2π-periodic in t, measurable in $(t,x) \in \Omega$ for each $r \in R$ and continuous in r for almost all $(t,x) \in \Omega$. Let $\widetilde{C}_0^2$ be the space of twice continuous differentiable functions v:$\overline{\Omega} \longmapsto R$ such that $v(t,0) = v(t,\pi) = 0$ and $v(t,x)$ is 2π-periodic in t. As $C_0^\infty \subset \widetilde{C}_0^2$ and C_0^∞ is dense in $H = L^2(\Omega)$ with respect to the norm $\|v\| = (v,v)^{1/2}$ where $(v,w) = \int\limits_\Omega vw$, it is natural to consider the problem (3) in $L^2(\Omega)$.

Moreover, the eigenfunctions of the d'Alembertian operator $\Box = \partial_{tt}^2 - \partial_{xx}^2$ with periodic conditions (3) can be written as

$$\Psi_{nk}(x,t) = \begin{cases} \frac{\sqrt{2}}{\pi}\sin(nx)\sin(kt) \ , \ (n,t) \in N \times N \ , \\ \frac{1}{\pi}\sin(nx) \ , \ n \in N \ , \ k = 0 \\ \frac{\sqrt{2}}{\pi}\sin(nx)\cos(kt) \ , \ n \in N \ , \ -k \in N \ , \end{cases}$$

A computation yields $\Box\Psi_{nk} = \left(n^2 - k^2\right)\Psi_{nk}$. The set $\{\Psi_{nk}\}_{(n,k)\in N\times Z}$ forms an orthonormal basis in H and the wave operator admits abstract realization $L : D(L) \subset H \longmapsto H$ defined by

$$Lu = \sum_{n=1}^{\infty}\sum_{k=-\infty}^{\infty}\left(n^2 - k^2\right)(u, \Psi_{nk})\,\Psi_{nk}$$

with

$$D(L) = \left\{u \in H \mid \sum\sum\left(n^2 - k^2\right)^2\|(u, \Psi_{nk})\|^2 < +\infty\right\}.$$

The periodic boundary conditions in (3) are incorporated in $D(L)$.The unbounded selfadjoint operator L is densely defined in H with $\dim KerL = \infty$. Therefore, there exists an orthogonal topological decomposition

$$H = KerL \oplus R(L),$$

where $KerL$ is the nullspace and $R(L)$ is the range of L. Moreover, L has a pure point spectrum of eigenvalues

$$\sigma(L) = \left\{\lambda_{nk} = n^2 - k^2 \mid (n,k) \in N \times Z\right\}$$

with corresponding eigenfunction Ψ_{nk}. Clearly, $\sigma(L)$ is unbounded from above and below, as well as any nonzero eigenvalue has a finite multiplicity. Finally, the restriction L_0 of L to $D(L) \cap R(L)$ is onto and so it has an inverse $K = L_0^{-1} : R(L) \longmapsto D(L) \cap R(L)$ with the discrete spectrum ([2])

$$\sigma(K) = \left\{\tfrac{1}{\lambda} \mid \lambda \in \sigma(L)\diagdown\{0\}\right\}.$$

In particular, for $f \in L^2(\Omega)$ the solution of the linear equation $Lu = f$ has the form

$$Kf = \sum_{k,nk\neq n}\frac{(f, \Psi_{nk})}{k^2 - n^2}\Psi_{nk}$$

In addition, it is easy to show that K is a compact operator as uniform limit of finite rank.

By the above hypotheses g: $\Omega \times R \longmapsto R$ is a Caratheodory function. In addition, we assume that it has at most linear growth, i.e., there is a constant $c > 0$ and a function $g_0 \in L^2(\Omega)$ such that

$$|g(t,x,u)| \leq c\,|u| + g_0(t,x)$$

for almost all $(t, x) \in \Omega$ and $u \in R$. The Nemytskii or superposition operator

$$(Nu)(t, x) = g(t, x, u(t, x)),$$

generated by g, is bounded and continuous from $L^2(\Omega)$ into itself.

Consequently, a *generalized solution* of (3) is a function $u \in L^2(\Omega)$ such that

$$(u, v_{tt} - v_{xx}) + (N(u), v) = 0 \quad \forall v \in \widetilde{C}_0^2 \ .$$

From here on, we shall write this equation in the equivalent operator form

$$Lu + N(u) = 0, \tag{4}$$

where L has the properties mentioned before.

More general even, we denote by Λ the set of all closed operators, densely defined linear operators $L : H \to H$ and such that $R(L) = (KerL)^{\perp}$. In other words,

$$H = KerL \oplus R(L),$$

and let Q be the orthogonal projection onto $R(L)$. Consider a sequence of increasing finite-dimensional subspaces $\{X_n\}$ of $N(L)$ such that $\bigcup X_n$ is dense in $KerL$ and let P_n be the orthogonal projection onto X_n and consider now the sequence $\{H_n\}$ of subspaces in H, finite-dimensional in the first component, $H_n = X_n + R(L)$ and the associated orthogonal projections $J_n = P_n + Q : H \to H_n$.

The double sequence $\gamma_0 = \{H_n, J_n\}$ is an approximation scheme and a sequence $\{u_n\}$ is said to be *L-convergent* and write $u_n \xrightarrow{L} u$ if

$$(I - Q)u_n \rightharpoonup (I - Q)\, u \text{ and } Qu_n \to Q\, u$$

With respect to this convergence due to J. Mawhin [3],[7], we introduced [4] the following

Definition 1. A mapping $T : H \to H$ is *A_L-proper* with respect to γ_0 if the restrictions $T_n : H_n \to H_n$ are continuous for every n and whenever $\{n_k\}$ is a sequence of positive integers and $\{u_{n_k}\}$ is a corresponding bounded sequence such that $T_{n_k} u_{n_k} \to g$ for some $g \in H$, it follows that $\{u_{n_k}\}$ is L-convergent to $u \in H$, at least on a subsequence, and $Tu = g$.

We can prove

Theorem 1. *Let $L \in \Lambda$ and $N : H \to H$ be a demicontinuous perturbation. Suppose that there is an bounded neighborhood $\Omega \subset H$ of the origin such* *that* $N(\overline{\Omega})$ *is bounded and* KQN *is compact on* $\overline{\Omega}$. *If the finite-dimensional equations (2) are solvable and* $L+N$ *is* A_L*-proper, then the equation (1) is* L*-approximation-solvable.*

Later on, we consider the restrictions $N_n = J_nN$ and $M_n = P_n - P_nN - KQN$ and $\Omega_n = \Omega \bigcap H_n$. We remark that $u \in \Omega_n$ is a solution of (4) iff $u = M_nu$.

We have the following continuation variant of the above theorem:

Theorem 2. *Assume that the assumptions in Theorem 1 hold. If for every* $n \in \mathbb{N}$,

$$Lu + (1-\lambda)J_nu + \lambda N_nu \neq 0,$$

for each $(u, \lambda) \in D(L) \bigcap \partial\Omega_n \times (0, 1)$, *then the initial equation (4) has at least a solution.*

This definition is a link between Petryshyn's A-properness and the hyperbolic degree introduced by J. Berkovitz and V. Mustonen [1]. L-approximation solvability of the equation (1), involves actually two steps:

a) Finite-dimensional solvability of (2), i.e., there exists a solution $u_n \in D(L) \cap H_n$ of the equation $Lu_n + N(u_n) = 0$ for n large enough;

b) Prove that $L + N$ is an A_L-proper mapping.

We introduce now a large class of nonlinear perturbations for which b) holds.

Definition 2. A mapping $N : \overline{\Omega} \to H$ is said to be *L-pseudomonotone* if for every sequence $\{u_n\} \subset \overline{\Omega}$ such that

$$u_n \xrightarrow{L} u \quad \text{and} \quad \limsup(N(u_n), u_n - u) \leq 0$$

it follows that $(N(u_n), u_n) \to (N(u), u)$ and $N(u_n) \rightharpoonup N(u)$ in H.

Moreover, M_n is completely continuous on H_n if N is L-pseudomonotone with respect to $L \in \Lambda$.

W. Krawcewicz [2] extended the above definition when $L, N : X \to Y$ are mappings between two real Banach spaces X, Y and $D(L)$ is a dense subspace of X and assume that $L : D(L) \to Y$ is a linear operator satisfying the following hypotheses:

(L_1) L is closed operator;

(L_2) $KerL$ and $R(L)$ are closed and there are two closed subspaces $X_0 \subset X$ and $Y_0 \subset Y$ such that $X = KerL + X_0$ and $Y = Y_0 + R(L)$;

(L_3) $\dim KerL = \operatorname{codim} R(L) = \infty$.

Let $P_n : X \to X_0$ and $Q_n : Y \to R(L)$ be the linear projections associated with the direct splittings in (L_2). Denote $\{Y_n'\}$ a *filtration* of Y_0, that is, for every $n \in N$, Y_n' is a finite-dimensional subspace of Y_0 such that $Y_n' \subset Y_{n+1}'$ and $\bigcup Y_n'$ is dense in Y_0 . Consider also the filtration $\{X_n'\}$ of $KerL$ such that $\dim X_n' = \dim Y_n'$, which is always possible by virtue of (L_3). We may suppose that $\bigcup X_n'$ is dense in $KerL$ and there are $P_n' : KerL \to X_n'$ and $Q_n' : Y_0 \to Y_n'$ such that

$$\begin{gathered} \forall x \in KerL \quad P_n'x \to x \quad \text{as } n \to \infty \\ \forall y \in Y_0 \quad Q_n'y \to y \quad \text{as } n \to \infty. \end{gathered}$$

For each n, put $X_n = X_n' + X_0$ and $Y_n = Y_n' + R(L)$ and define

$$\begin{aligned} P_n &= P + P_n'(I - P) : X \to X_n, \\ Q_n &= Q + Q_n'(I - Q) : Y \to Y_n, \end{aligned}$$

the linear projections over X_n and Y_n, respectively, and note that

$$\begin{gathered} P_n x \to x \quad \text{for all } x \in X, \\ Q_n y \to y \quad \text{for all } y \in Y. \end{gathered}$$

Let us consider now $L_n : X_n \cap D(L) \to Y_n$, the restriction $L \mid_{X \cap D(L)}$ to X_n and we can easily show that L_n is a Fredholm operator of index zero. This is the reason why the special approximation scheme $\Gamma = (\{X_n\}, \{P_n\}; \{Y_n\}, \{Q_n\})$ with the components designed above is called the *Fredholm factorization* associated with L.

Let Ω be an open set in X and $N : \overline{\Omega} \to Y$ a (nonlinear) mapping. Denote $\Omega_n = \Omega \cap X_n$ and $N_n : \overline{\Omega}_n \to Y_n$ where $N_n x = Q_n N x$ for all $x \in \Omega_n$,and suppose that $\Omega_n \neq \emptyset$ for all $n \in N$.

Defintion 3. The mapping $N : \overline{\Omega} \to Y$ is called *A_L-proper* if the restrictions N_n are continuous for every n, and whenever $\{u_n\}$,with $u_n \in \overline{\Omega}_n \bigcap D(L)$, is a bounded subsequence such that

$$L_n u_n + N(u_n) = f_n \to f \quad \text{as } n \to \infty,$$

it follows that $\{u_n\}$ is L-convergent to $u \in \overline{\Omega} \bigcap D(L)$, at least for a subsequence, and $Lu + N(u) = f$.

The restriction $L\,|_{X_0}\colon X_0 \cap D(L) \to R(L)$ being injective, we may consider its inverse $K : R(L) \to X_0$, which is a bounded linear operator by virtue of the closed graph theorem.

Theorem 3. *Assume that the space X is reflexive and the linear operator L satisfies the above hypotheses. Let $N : X \to X^*$ be a bounded mapping for which KQN is compact. Then any sequence $\{u_n\}$, $u_n \in X_n \cap D(L)$ with the properties $u_n \to u$ and $Lu_n + P^*N(u_n) \to f$ contains a subsequence $\{u_{n'}\}$ such that* $u_{n'} \xrightarrow{L} u$ and $(N(u_{n'}), u_{n'} - u) \to 0.$

In the case of Hilbertian case, we have $P_n = P_n^*$ and the hypothesis concerning the compactness of KQN can reduce to a similar (weaker) condition for type (S) with respect to L.

Proposition 4. *Let H be a Hilbert space, $L : D(L) \subseteq H \to H$ a selfadjoint operator and $N : H \to H$ a bounded mapping. For any sequence $\{u_n\}$, $u_n \in H_n \cap D(L)$, with $u_n \rightharpoonup u$, $Lu_n + P_nNu_n \to f$ and* $\liminf\,(Nu_n - Nu, Lu_n - Lu) \leq 0$ *there exists a subsequence $\{u_{n'}\} \subseteq \{u_n\}$ such that $u_{n'} \xrightarrow{L} u$ and $(N(u_{n'}), u_{n'} - u) \to 0$ as $n' \to \infty$.*

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