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WEAK-HICKS CONTRACTIONS

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Abstract. A class of probabilistic contractions, larger than the class of Hicks C-contractions is presented and a fixed point theorem is proved.

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This paper deals with a type of probabilistic contractions, obtained by weakening Hicks' contractivity condition [4].

The outline of the paper is as follows: In Section 1 (Preliminaries), some basic definitions and concepts from the theory of probabilistic metric spaces that are used in our work are given. In Section 2, the notion of weak-Hicks probabilistic contraction is presented. It is shown that the class of these contractions is larger than that of Hicks contractions and a fixed point theorem in complete Menger spaces under g-convergent t-norms is proved.

1. Preliminaries

We begin by briefly recalling some definitions and notions from probabilistic metric spaces theory that we will use in the sequel. For more details we refer the reader to [1], [3], [13].

A *triangular norm* T (shorter *t-norm*) is a binary operation on the unit interval $[0, 1]$, which is associative, commutative, nondecreasing at both places

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and $T(x,1) = x$ for every $x \in [0,1]$ that is, $([0,1], T, \geq)$ is an ordered commutative semigroup with 1 as identity element. As examples we mention the t-norms T_L (*Łukasiewicz* t-norm), T_P and T_M, defined by $T_L(a,b) = \max\{a+b-1, 0\}$, $T_P(a,b) = ab$ and $T_M(a,b) = \min\{a,b\}$.

If T is a t-norm and $(x_{1,}\ x_2, ..., x_n) \in [0,1]^n$ $(n \in N)$, then one can define recurrently $T_{i=1}^n x_i$ by 1, if $n = 0$, and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_{i.}, x_n)$ for all $n \geq 1$. Moreover, we can extend T to a countable infinitary operation defining $T_{i=1}^\infty x_i$, for any sequence $(x_i)_{i \in N}$, as $\lim_{n\to\infty} T_{i=1}^n x_i$ (this limit always exists).

We say ([2]) that T is *geometrically convergent* (*g-convergent*) if, for all $q \in (0,1)$,

$$\lim_{n\to\infty} T_{i=n}^\infty (1 - q^i) = 1.$$

T_L and the t norms of *Hadžić type* [3] are examples of g-convergent t-norms. Other examples can be found in [3]. As it is proved in [2], if $\lim_{n\to\infty} T_{i=n}^\infty (1 - q_0^i) = 1$ for some $q_0 \in (0,1)$, then T is g-convergent. Also, note that if T is g-convergent then $\sup\limits_{t<1} T(t,t) = 1$.

A *distance distribution function* is a mapping $F : [0,\infty) \to [0,1]$ which is increasing, left continuous on $(0,\infty)$ and $F(0) = 0$. The class of all distribution functions is denoted by Δ_+. D_+ is the subset of Δ_+ containing all functions F which also satisfy the condition $\lim\limits_{t\to\infty} F(t) = 1$.

If X is a nonempty set, a mapping F from $X \times X$ to Δ_+ is called *a probabilistic distance on* X and $F(x,y)$ is denoted by F_{xy}. The triple (X, F, T), where X is a nonempty set, F is a probabilistic distance on X and T is a t-norm is said to be *a Menger space* (in the *sense of Schweizer* & *Sklar*) if the following conditions hold:

$$(PM0) \ : \ F_{xy}(t) = 1 \text{ for every } t > 0 \text{ if and only if } x = y$$

$$(PM1) \ : \ F_{xy} = F_{yx} \ \forall x, y \in X$$

$$(PM2_M) \ : \ F_{xy}(t+s) \geq T(F_{xz}(t), F_{zy}(s)), \ \forall x, y, z \in X, \ \forall t, s > 0.$$

If (X, F, T) is a Menger space with $\sup\limits_{t<1} T(t,t) = 1$, then the family

$$\{U_{\varepsilon,\lambda}\}_{\varepsilon>0, \lambda\in(0,1)}, \text{ where } U_{\varepsilon,\lambda} = \{(x,y) \in X \times X : \ F_{xy}(\varepsilon) > 1 - \lambda\}$$

is a base for a metrizable uniformity on X, named the *F-uniformity* and denoted by $\mathcal{U}_F$. The F-uniformity is also generated by the family $\{V_\delta\}_{\delta>0}$ where $V_\delta := U_{\delta,\delta}$.

$\mathcal{U}_F$ naturally determines a topology $\mathcal{T}_F$ on X, named *the F-topology:*

$$O \in \mathcal{T}_F \text{ iff } \forall x \in O\ \exists \varepsilon > 0,\ \exists \lambda \in (0,1) \text{ such that } U_{\varepsilon,\lambda}(x) \subset O.$$

In what follows the topological notions refer to the F-uniformity.

The following definitions for *Picard* and *weakly Picard operators* will be also used:

Definition 1.1. ([12]) Let (X, d) be a metric space. The mapping $f : X \to X$ is said to be a *weakly Picard operator* if the sequence $(f^n(x))_{n\in N}$ converges for every $x \in X$ and the limit is a fixed point of f. f is called a *Picard operator* if it is a weakly Picard operator and the set of its fixed points contains a unique element.

2. Weak Hicks-contractions

The following type of probabilistic contractions, intensively studied in the fixed point theory in probabilistic metric spaces (see e.g. [1], [3]), has been introduced by *T. L. Hicks.*

Definition 2.1. ([4]) Let F be a probabilistic metric on S. The mapping $f : S \to S$ is called *Hicks C-contraction* (or *C-contraction*) if there exists $k \in (0,1)$ such that the following implication holds for every $p, q \in S$:

$$(H):\quad t \in (0,\infty),\ F_{pq}(t) > 1 - t \Rightarrow F_{f(p)f(q)}(kt) > 1 - kt.$$

Radu [9] showed that every C-contraction on a Menger space (S, F, T) with Range$(F) \subset D_+$ and $T \geq T_L$ is actually a Banach contraction in the metric space $(S, \mathbf{K})$, where $\mathbf{K}(p, q) = \sup\{t \mid t \leq 1 - F_{pq}(t)\}$ and he also proved a more general result:

Theorem 2.1. ([9]) *Let (S, F, T) be a complete Menger space such that Range$(F) \subset D_+$ and $\sup_{a<1} T(a, a) = 1$. Then every C-contraction f on S has a unique fixed point which is the limit of the sequence $(f^n(p))_{n\in\mathbf{N}}$ for every $p \in S$.*

It should be noted that in the proofs of the above mentioned theorems the condition Range$(F) \subset D_+$ on the probabilistic metric is not necessary. As a matter of fact, the proof is based on the following "boundedness property" of the probabilistic metric:

$$\delta > 1 \Rightarrow F_{pq}(\delta) > 1 - \delta.$$

From this property one obtains that

$$F_{f^n(p)f^n(q)}(k^n(1+\varepsilon)) > 1 - k^n(1+\varepsilon) \ \ \forall n \in N, n \geq 1, \ \forall p, q \in S$$

and now it is clear that the existence and the uniqueness of the fixed point follow even in the absence of the condition Range$(F) \subset D_+$.

By weakening the condition (H), we propose the following

Definition 2.2. ([6]) Let S be a nonempty set and F be a probabilistic distance on S. A mapping $f : S \to S$ is said to be a *weak-Hicks contraction* (*w-H contraction* for short) if there exists $k \in (0,1)$ such that, for all $p, q \in S$,

$$(w\text{-}H): \quad t \in (0,1), \ F_{pq}(t) > 1 - t \Rightarrow F_{f(p)f(q)}(kt) > 1 - kt.$$

As we will see, the weakness of contraction condition $(w\text{-}H)$ can be counterbalanced by a more convenient condition on the probabilistic metric, namely $F_{xf(x)}(1) > 0$ for some x.

Example 2.1. Let $X = [0, \infty)$ and

$$F_{xy}(t) = \frac{\min(x,y)}{\max(x,y)}, \forall t \in (0,\infty), \forall x, y \in X, \ x \neq y.$$

It is known ([10], [11]) that (X, F, T) is a complete Menger space under the triangular norm $T = T_P \geq T_L$.

Also, it can easily be seen that the mapping $g : X \to X$, $g(x) = \begin{cases} 0, \text{ if } x = 0 \\ 1, \text{ if } x > 0 \end{cases}$ is a w-H contraction for every $k \in (0,1)$.

Indeed, if $F_{g(x)g(y)}(kt) \leq 1 - kt$ then (exactly) one of the numbers x and y is 0, which implies $F_{xy}(t) = 0 < 1 - t$.

Note that this mapping has two fixed points: $x = 0$ and $x = 1$. Therefore, unlike C-contractions, a weak-Hicks contraction is not necessarily a Picard operator on the metric space $(X, \mathbf{K})$.

Example 2.2. Let $X = \{a, b, c\}$ and define $F_{pq}(t) = 0$ for all $p, q \in X$, $p \neq q$ and $t > 0$. It is easy to see that (X, F, T_L) is a complete Menger space.

Consider the mapping $f : X \to X$, defined by $f(a) = a$, $f(b) = c$, $f(c) = b$.

Since $t \in (0, 1)$, $F_{pq}(t) > 1 - t \Rightarrow p = q \Rightarrow F_{f(p)f(q)}(kt) = 1$, f is a weak-Hicks contraction. Note that f is not a weakly Picard operator in $(X, \mathbf{K})$ (and hence it is not a C-contraction, too), for the sequence $(f^n(b))$ does not converge.

Thus we just have proved:

Proposition 2.1. *The class of weak-Hicks contractions strictly contains the class of C-contractions.*

In the proof of Theorem 2.2 we need the following continuity lemma:

Lemma 2.1. *Every w-H contraction is (uniformly) continuous.*

Indeed, if $\varepsilon > 0$ and $\lambda \in (0, 1)$ are given, we choose $\delta \in (0, 1)$ such that $0 < k\delta < \text{Min}\{\varepsilon, \lambda\}$. Then

$$(p, q) \in V_\delta \Longrightarrow F_{pq}(\delta) > 1 - \delta \Longrightarrow F_{f(p)f(q)}(k\delta) > 1 - k\delta \Longrightarrow$$
$$F_{f(p)f(q)}(k\delta) > 1 - \lambda \Longrightarrow (f(p), f(q)) \in U_{\varepsilon,\lambda}.$$

Theorem 2.2. *Let (X, F, T) be a complete Menger space under the g-convergent t-norm T and f be a weak-Hicks contraction on X. Then f has a fixed point iff there exists $x \in X$ such that $F_{xf(x)}(1) > 0$.*

Proof. The direct implication is obvious, for $F_{uu}(t) = 1$, $\forall\ t > 0$.

For the converse implication, first we show that there exists $\delta \in (0, 1)$ such that $F_{xf(x)}\ (\delta) > 1 - \delta$. Indeed, if we suppose that $F_{xf(x)}\ (\delta) \leq 1 - \delta$ for all $\delta \in (0, 1)$, then $F_{xf(x)}\ (1) \leq 0$ (for $F_{xf(x)}$ is left continuous), which is a contradiction.

Now, using (w-H), we can prove by induction that

$$F_{f^n(x)f^{n+1}(x)}(k^n\delta) > 1 - k^n\delta,\ \forall n \in N.$$

Next we will show that the sequence $(f^n(x))_{n\in N}$ is a Cauchy sequence, that is

$$\forall\varepsilon > 0, \forall\lambda \in (0, 1)\exists n_0 (= n_0(\varepsilon, \lambda)) : F_{f^n(x)f^{n+m}(x)}(\varepsilon) > 1 - \lambda, \forall n \geq n_0, \forall m \in N.$$

Let $\varepsilon > 0$ and $\lambda \in (0, 1)$ be given. Since the series $\sum_{n=1}^{\infty} k^n\delta$ is convergent, there exists $n_1 (= n_1(\varepsilon))$ such that $\sum_{n=n_1}^{\infty} k^n\delta < \varepsilon$.

Then, for all $n \geq n_1$ and $m \in N$ we have

$$F_{f^n(x)f^{n+m}(x)}(\varepsilon) \geq F_{f^n(x)f^{n+m}(x)}(\sum_{i=n_1}^{\infty} k^i\delta) \geq$$

$$F_{f^n(x)f^{n+m}(x)}(\sum_{i=n}^{n+m-1} k^i\delta) \geq T_{i=1}^{m} x_{n+i-1}$$

where $x_j := F_{f^j(x)f^{j+1}(x)}(k^j\delta)$ satisfies $x_j \geq 1-k^j\delta, \forall j \geq n$. Let n_2 be such that $T_{i=1}^{\infty}(1-k^{n_2+i-1}) > 1-\lambda$ (such a number n_2 does exist, for T is g-convergent).

Then, for all $n \geq n_0 := Max\{n_1, n_2\}$ and $m \in N$,

$$F_{f^n(x)f^{n+m}(x)}(\varepsilon) \geq T_{i=1}^{m}(1-k^{n+i-1}\delta) \geq T_{i=1}^{m}(1-k^{n+i-1}) \geq$$

$$T_{i=1}^{m}(1-k^{n_2+i-1}) \geq T_{i=1}^{\infty}(1-k^{n_2+i-1}) > 1-\lambda.$$

Thus, $(f^n(x))_{n\in N}$ is a Cauchy sequence. By the completeness of (X, F, T) and the continuity of f it follows that $(f^n(x))_{n\in N}$ converges to a fixed point of f. This competes the proof.

Corollary 2.1. ([6, 7]) *Let (S, F, T) be a complete Menger space where either $T \geq T_L$ or T is of Hadžić type and $f : S \to S$ be a weak-Hicks contraction. If $F_{pf(p)}(1) > 0$ for some $p \in S$, then f has a fixed point.*

In the final part of the paper, using the method of Hicks [5], we will obtain a fixed point theorem in metric spaces.

A probabilistic semi-metric space (S, F) is said to be *of class $\mathcal{H}$* ([5]) if there exists a metric d on S such that the following "compatibility relation" holds:

$$t > 0,\ d(p,q) < t \Leftrightarrow F_{pq}(t) > 1-t.$$

It is known ([8]) that (S, F) is of class $\mathcal{H}$ iff the mapping $\mathbf{K}$, defined on $S \times S$ by $\mathbf{K}(p,q) = \sup\{t \geq 0 \mid t \leq 1 - F_{pq}(t)\}$ is a metric on S (Ky Fan metric). It is also known ([5]) that if (S, F, T) is a Menger space under the t-norm $T \geq T_L$, then (S, F) is of class $\mathcal{H}$.

The relation $d(p,q) < t \Leftrightarrow F_{pq}(t) > 1-t$ allows us to transpose some results from probabilistic metric spaces to metric spaces:

Suppose (S, F, T) be a complete Menger space with $T \geq T_L$. Since (S, F) is of class $\mathcal{H}$, $\mathbf{K}$ is a metric on S and $\mathbf{K}(p,q) < t \Longleftrightarrow F_{pq}(t) > 1-t$. Thus

(S, F) is complete iff $(S, \mathbf{K})$ is complete. Also, note that $\mathbf{K}(x, y) < 1$ iff there exists $t \in (0,1)$ such that $F_{xy}(t) > 0$ (this is equivalent to $F_{xy}(1) > 0$) and that f is a w-H contraction on S iff $\mathbf{K}(f(p), f(r)) \leq k\mathbf{K}(p, r)$ for all p, r such that $\mathbf{K}(p, r) < 1$. Indeed, if $\mathbf{K}(p, r) < \varepsilon < 1$ then $F_{pr}(\varepsilon) > 1 - \varepsilon$, so by ($w$-$H$), $F_{f(p)f(r)}(k\varepsilon) > 1 - k\varepsilon$, which implies $\mathbf{K}(f(p), f(r)) < k\varepsilon$ and therefore $\mathbf{K}(f(p), f(r)) \leq k\mathbf{K}(p, r)$. Conversely, suppose $\mathbf{K}(f(p), f(q)) \leq k\mathbf{K}(p, q)$. Then $F_{pq}(t) > 1 - t \Rightarrow \mathbf{K}(p, q) < t \Rightarrow \mathbf{K}(f(p), f(q)) < kt \Rightarrow F_{f(p)f(q)}(kt) > 1 - kt$.

From Theorem 2.2 it follows

Proposition 2.2. *If $(X,\ d)$ is a complete metric space and $f : X \rightarrow X$ is a mapping with the property : there are $a > 0$ and $k \in (0,1)$ such that $d(f(x), f(y)) \leq kd(x, y)$ for all x, y with $d(x, y) < a$, then f has a fixed point provided $d(x, f(x)) < a$ for some $x \in X$.*

Note that in the case when (S, F, T) is a complete Menger space with $T \geq T_L$ a direct proof of Theorem 2.2 may be obtained from the observation that $d(x, f(x)) < 1$ implies

$$d\left(f^n(x), f^{n+1}(x)\right) \leq k^n d(x, f(x)), \forall n \in N$$

and therefore

$$\sum_n d\left(f^n(x), f^{n+1}(x)\right) < \sum_n k^n < \infty$$

where d is the compatible metric.

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