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NONAUTONOMOUS PERIODIC SYSTEMS WITH NONSMOOTH POTENTIAL

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Abstract. We consider nonautonomous periodic systems with nonsmooth potential. Using a variational approach based on the nonsmooth generalized Mountain Pass Theorem we establish the existence of a nonconstant solution.

Key Words and Phrases: Locally Lipschitz function, generalized subdifferential, nonsmooth critical point theory, generalized Mountain Pass Theorem, nonsmooth Palais-Smale condition, nonconstant solution.

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1. Introduction

In this paper we study nonlinear periodic systems driven by the one-dimensional p-Laplacian and having a nonsmooth potential.

Recently nonlinear boundary value problems involving the p-Laplacian, have been studied by several authors. We mention the works of Dang-Oppenheimer [4], Del Pino-Manasevich-Murua [5], Fabry-Fayyad [6], Guo [7], Manasevich-Mawhin [11] and Papageorgiou-Yannakakis [13]. The above works with the exception of Manasevich-Mawhin and Papageorgiou-Yannakakis deal with the

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scalar problem (i.e. $N = 1$) and their method is based on degree theory or on the theory of nonlinear operators of monotone type.

Here we deal with systems involving the one-dimensional p-Laplacian and with a potential function which is nonsmooth. So the resulting problem is multivalued (differential inclusion). More precisely the problem under consideration is the following:

$$\left\{ \begin{array}{l} (\|x'(t)\|^{p-2}x'(t))' \in \partial j(t,x(t)) \text{ a.e on } T=[0,b] \\ x(0)=x(b),\ x'(0)=x'(b),\ 2 \leq p < \infty. \end{array} \right\} \tag{1.1}$$

The potential function $j(t,x)$ is locally Lipschitz in $x \in \mathbb{R}^{\mathbb{N}}$ and $\partial j(t,x)$ denotes the generalized subdifferential in the sense of Clarke [3]. Our approach is based on the nonsmooth critical point theory of Chang [2] (for extensions see Kourogenis-Papageorgiou [10]. Problems with nonsmooth locally Lipschitz potential, are known as hemivariational inequalities and arise in mechanics and engineering (see Naniewicz-Panagiotopoulos [12]).

Our proof uses the following nonsmooth extension of the so-called Linking Theorem of Rabinowitz [14], p.28.

Theorem 1.1. *If X is a reflexive Banach space, $X = Y \oplus V$ with $dimY < +\infty$, $\varphi : X \to \mathbb{R}$ is locally Lipschitz, satisfies the nonsmooth PS-condition and the following conditions hold*

(i) *there exist $r,\ \alpha > 0$ such that for all $v \in V$ with $\|v\| = r$ we have $\varphi(v) \geq \alpha$;*

(ii) *there exist $e \in \partial B_1 \cap V$ ($B_1 = \{x \in X : \|x\| < 1\}$) and $R > r$ such that if $Q = \{v \in V : \|v\| \leq R\} \oplus \{\lambda e : 0 < \lambda < R\}$, then $\varphi|_{\partial Q} \leq 0$ with ∂Q being the boundary of Q in $Y \oplus \mathcal{R}e$,*

then $c = \inf\limits_{\gamma\in\Gamma} \max\limits_{u \in Q} \varphi(\gamma(u))$ where $\Gamma = \{\gamma \in C(\overline{Q}, X) : \gamma|_{\partial Q} = identity\}$ is a critical value of φ with critical point $x \in X$ and $c \geq \alpha$. Moreover, if $c = \alpha$, then $x \in V$.

Proof. Using Proposition 5.9 (p.29) of Rabinowitz (which is still valid in the present nonsmooth setting), for every $\gamma \in \Gamma$ we have

$$\begin{aligned} &\gamma(Q) \cap \partial B_r \cap V \neq \emptyset \ \ (\text{where } B_r = \{x \in X : \|x\| < r\}) \\ \Rightarrow &c \geq \alpha. \end{aligned}$$

Next we show that c is a critical value of φ. Suppose that this is not true. Using the nonsmooth deformation theorem of Chang [2] and Kourogenis-Papageorgiou [10], we can find $\varepsilon \in (0, \frac{\alpha}{2})$ and $\xi : X \to X$ a homeomorphism such that

$$\xi(x) = x \text{ for all } x \notin \{u \in X : |\varphi(u) - c| < \frac{\alpha}{2}\} \tag{1.2}$$

$$\text{and } \varphi(\xi(x)) \leq c - \varepsilon \text{ for all } x \in X \text{ with } \varphi(x) \leq c + \varepsilon. \tag{1.3}$$

From the minimax definition of c, we can find $\gamma \in \Gamma$ such that

$$\max\,[\varphi(\gamma(u)) : u \in \overline{Q}] \leq c + \varepsilon. \tag{1.4}$$

Let $g = \xi \circ \gamma$. Evidently from hypothesis *(ii)* and (1.2), we obtain that $g \in \Gamma$. Then from (1.3) and (1.4) we obtain that $c \leq c - \varepsilon$, a contradiction. Finally if $c = \alpha$, then clearly $x \in V$. □

2. Existence Theorem

In this section we deal with problem (1.1). Our hypotheses on the nonsmooth potential are the following:

H(j): $j : T \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a function such that $j(\cdot, 0) \in L^{\infty}(T)$

(i) for every $x \in \mathbb{R}^{\mathbb{N}}$, $t \to j(t,x)$ is measurable;

(ii) for almost all $t \in T$, $x \to j(t,x)$ is locally Lipschitz;

(iii) for every $r > 0$, there exists $\theta_r \in L^q(T)$ $(\frac{1}{p} + \frac{1}{q} = 1)$ such that for almost all $t \in T$, all $x \in \mathbb{R}^{\mathbb{N}}$ with $\|x\| \leq r$ and all $u \in \partial j(t,x)$ we have $\|u\| \leq \theta_r(t)$;

(iv) there exist $\eta > p$ and $M > 0$ such that for almost all $t \in T$ and all $x \in \mathbb{R}^{\mathbb{N}}$ with $\|x\| \geq M$ we have $j^0(t,x;x) \leq \eta j(t,x) < 0$;

(v) for almost all $t \in T$ and all $x \in \mathbb{R}^{\mathbb{N}}$ with $\|x\| \leq 1$ we have

$$-\frac{1}{pb^p} \leq j(t,x).$$

We consider the energy functional $\varphi : W^{1,p}_{per}(T, \mathbb{R}^{\mathbb{N}}) \to \mathbb{R}$ defined by

$$\varphi(x) = \frac{1}{p}\|x'\|_p^p + \int_0^b j(t, x(t))dt.$$

It is easy to see that φ is locally Lipschitz (see also Chang [2] or Hu-Papageorgiou [9]).

Proposition 2.1. If hypotheses $H(j)$ hold, then φ satisfies the nonsmooth PS-condition.

Proof. First we show that for almost all $t \in T$ and all $x \in \mathbb{R}^{\mathbb{N}}$, we have

$$j(t,x) \leq \alpha_1(t) - \alpha_2(t)\|x\|^{\eta} \text{ with } \alpha_1, \alpha_2 \in L^q(T)_+. \tag{2.1}$$

To this end let N_0 be the Lebesgue-null set outside of which hypotheses $H(j)(ii) \to (v)$ hold and let $t \in T \backslash N_0$, $x \in \mathbb{R}^{\mathbb{N}}$, $\|x\| > M$. We set $\psi(t,\lambda) = j(t,\lambda x)$, $\lambda \geq 1$. Evidently $\psi(t,\cdot)$ is locally Lipschitz. Moreover, from Theorem 2.3.10, p.45, of Clarke [3], we have that

$$\begin{aligned}
&\partial\psi(t,\lambda) \subseteq (\partial_x j(t,\lambda x), x)_{\mathbb{R}^{\mathbb{N}}} \\
\Rightarrow &\lambda\partial\psi(t,\lambda) \subseteq (\partial_x j(t,\lambda x), \lambda x)_{\mathbb{R}^{\mathbb{N}}} \\
\Rightarrow &\lambda\psi'(t,\lambda) \leq \eta\psi(t,\lambda) \text{ for almost all } \lambda \geq 1.
\end{aligned}$$

Here we have used the fact that $\psi(t,\cdot)$ being locally Lipschitz it is differentiable for almost all $\lambda \geq 1$ and also we have used hypothesis $H(j)(iv)$. Because $\psi(t,\lambda) < 0$ (see hypothesis $H(j)(iv)$), we obtain

$$\frac{\eta}{\lambda} \leq \frac{\psi'(t,\lambda)}{\psi(t,\lambda)} \text{ for almost all } \lambda \geq 1.$$

Integrating this inequality from $\lambda = 1$ to $\lambda = \lambda_0 > 1$, we obtain

$$\begin{aligned}
&\ln\lambda_0^{\eta} \leq \ln\frac{\psi(t,\lambda_0)}{\psi(t,1)} \\
\Rightarrow &\psi(t,\lambda_0) \leq \lambda_0^{\eta}\psi(t,1) \text{ (since } \psi(t,\lambda) < 0).
\end{aligned}$$

So we have proved that for $t \in T\backslash N_0$, $\|x\| \geq M$ and $\lambda \geq 1$, we have

$$j(t,\lambda x) \leq \lambda^{\eta} j(t,x).$$

Using this fact, we obtain

$$j(t,x) = j\left(t, \frac{\|x\|}{M}\frac{Mx}{\|x\|}\right) \leq \left(\frac{\|x\|}{M}\right)^{\eta} j\left(t, \frac{Mx}{\|x\|}\right).$$

From the Lebourg Mean Value Theorem (see Clarke [3], p.41) we have that for almost all $t \in T$ and all $\|z\| = M$ we have

$$\begin{aligned}
&|j(t,z) - j(t,0)| = |(u(t),z)_{\mathbb{R}^{\mathbb{N}}}| \text{ with } u(t) \in \partial j(t,\mu(t)z),\ \mu(t) \in (0,1) \\
\Rightarrow &|j(t,z)| \leq |j(t,0)| + \alpha_M(t)M = \beta(t) \text{ with } \beta \in L^q(T).
\end{aligned}$$

So if we set $\beta_1(t) = \max\,[j(t,z) : \|z\| = M]$, evidently $\beta_1 \leq 0$ (see hypothesis $H(j)(iv)$) and $\beta_1 \in L^q(T)$. For all $t \in T\backslash N_0$ and all $\|x\| \geq M$ we have

$$j(t,x) \leq \left(\frac{\|x\|}{M}\right)^{\eta} \beta_1(t). \tag{2.2}$$

Also for $t \in T\backslash N_0$, $j(t,\cdot)$ is bounded on $\overline{B}_M = \{y \in \mathbb{R}^{\mathbb{N}} : \|y\| \leq M\}$. Then from (2.2) and since $\beta_1 \leq 0$ we obtain (2.1).

Now let $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{per}((T,\mathbb{R}^{\mathbb{N}})$ be a sequence such that

$$|\varphi(x_n)| \leq M_1 \text{ for all } n \geq 1 \text{ with } M_1 > 0 \text{ and } m(x_n) \to 0.$$

For every $n \geq 1$, let $x_n^* \in \partial\varphi(x_n)$ such that $m(x_n) = \|x_n^*\|$. The existence of such elements follows from the fact that $\partial\varphi(x_n) \subseteq W^{1,p}_{per}(T,\mathbb{R}^{\mathbb{N}})^*$ is weakly compact and the norm functional is weakly lower semicontinuous. Also let $A : W^{1,p}_{per}(T,\mathbb{R}^{\mathbb{N}}) \to W^{1,p}_{per}(T,\mathbb{R}^{\mathbb{N}})^*$ be the nonlinear operator defined by

$$\langle A(x), y\rangle = \int_0^b \|x'(t)\|^{p-2}(x'(t), y'(t))_{\mathbb{R}^{\mathbb{N}}} dt.$$

It is easy to check (see also Hu-Papageorgiou [9]) that A is demicontinuous, monotone, hence it is maximal monotone. For every $n \geq 1$, we have

$$x_n^* = A(x_n) + u_n$$

with $u_n \in L^q(T,\mathbb{R}^{\mathbb{N}})$, $u_n(t) \in \partial j(t, x_n(t))$ a.e. on T (see Clarke [3], Corollary p.47). From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{per}(T,\mathbb{R}^{\mathbb{N}})$ we have that

$$\frac{\eta}{p}\|x_n'\|_p^p + \int_0^b \eta j(t, x_n(t))dt \leq M_1\eta \tag{2.3}$$

$$\text{and } |\langle x_n^*, x_n\rangle| \leq \varepsilon_n\|x_n\| \text{ with } \varepsilon_n \downarrow 0. \tag{2.4}$$

Inequality (2.4) becomes

$$\begin{aligned} &-\langle A(x_n), x_n\rangle - (u_n, x_n)_{pq} \leq \varepsilon_n\|x_n\| \\ \Rightarrow &-\|x_n'\|_p^p - \int_0^b (u_n(t), x_n(t))_{\mathbb{R}^{\mathbb{N}}} dt \leq \varepsilon_n\|x_n\|. \end{aligned} \tag{2.5}$$

Adding (2.3) and (2.4), we obtain

$$(\frac{\eta}{p}-1)\|x_n'\|_p^p+\int_0^b(\eta j(t,x_n(t))-(u_n(t),x_n(t))_{\mathbb{R}^{\mathbb{N}}})\,dt\leq M_1\eta+\varepsilon_n\|x_n\|$$
$$\Rightarrow(\frac{\eta}{p}-1)\|x_n'\|_p^p+\int_0^b\big(\eta j(t,x_n(t))-j^0(t,x_n(t);x_n(t))\big)\,dt\leq M_1\eta+\varepsilon_n\|x_n\|. \tag{2.6}$$

We have

$$\begin{aligned}&\int_0^b\big(\eta j(t,x_n(t))-j^0(t,x_n(t);x_n(t))\big)\,dt\\ =&\int_{\{\|x_n(t)\|<M\}}\big(\eta j(t,x_n(t))-j^0(t,x_n(t);x_n(t))\big)\,dt+\\ &\int_{\{\|x_n(t)\|\geq M\}}\big(\eta j(t,x_n(t))-j^0(t,x_n(t);x_n(t))\big)\,dt.\end{aligned} \tag{2.7}$$

As before by virtue of the Lebourg Mean Value Theorem it follows that there exists $\xi>0$ such that

$$-\xi_1\leq\int_{\{\|x_n(t)\|<M\}}\big(\eta j(t,x_n(t))-j^0(t,x_n(t);x_n(t))\big)\,dt. \tag{2.8}$$

Also from hypothesis $H(j)(iv)$ it follows that

$$0\leq\int_{\{\|x_n(t)\|\geq M\}}\big(\eta j(t,x_n(t))-j^0(t,x_n(t);x_n(t))\big)\,dt. \tag{2.9}$$

Using (2.8) and (2.9) in (2.7), we obtain

$$-\xi_1\leq\int_0^b\big(\eta j(t,x_n(t))-j^0(t,x_n(t);x_n(t))\big)\,dt. \tag{2.10}$$

Using (2.10) in (2.6), we obtain

$$\left(\frac{\eta}{p}-1\right)\|x_n'\|_p^p\leq\xi_2(1+\|x_n\|)\ \text{ for some }\ \xi_2>0\text{ and all }n\geq 1. \tag{2.11}$$

From (2.11) we shall infer that $\{x_n\}_{n\geq 1}\subseteq W^{1,p}_{per}(T,\mathbb{R}^{\mathbb{N}})$ is bounded. To this end we argue by a contradiction. So suppose that $\{x_n\}_{n\geq 1}\subseteq W^{1,p}_{per}(T,\mathbb{R}^{\mathbb{N}})$ is not bounded. By passing to a subsequence if necessary, we may assume that $\|x_n\|\to+\infty$. Set $y_n=\dfrac{x_n}{\|x_n\|}$, $n\geq 1$. Due to the reflexivity of $W^{1,p}_{per}(T,\mathbb{R}^{\mathbb{N}})$ and the compact embedding of $W^{1,p}_{per}(T,\mathbb{R}^{\mathbb{N}})$ into $C(T,\mathbb{R}^{\mathbb{N}})$, by passing to a suitable subsequence if necessary, we may assume that $y_n \xrightarrow{w} y$ in $W^{1,p}_{per}(T,\mathbb{R}^{\mathbb{N}})$ and $y_n \to y$ in $C(T,\mathbb{R}^{\mathbb{N}})$. Divide (2.11) by $\|x_n\|^p$. We obtain

$$\begin{aligned}
&\left(\frac{\eta}{p}-1\right)\|y_n'\|_p^p \le \xi_2\left(\frac{1}{\|x_n\|^p}+\frac{1}{\|x_n\|^{p-1}}\right)\\
\Rightarrow&\left(\frac{\eta}{p}-1\right)\|y'\|_p^p \le 0 \ \ (\text{recall that } y_n' \xrightarrow{w} y' \text{ in } L^p(T,\mathbb{R}^{\mathbb{N}}))\\
\Rightarrow& y'=0 \ \ (\text{since } \eta > p)\\
\Rightarrow& y=c\in\mathbb{R}^{\mathbb{N}} \ \text{ and } \ y_n \to c \ \text{ in } \ W^{1,p}_{per}(T,\mathbb{R}^{\mathbb{N}}).
\end{aligned}$$

Consider the direct sum decomposition $W^{1,p}_{per}(T,\mathbb{R}^{\mathbb{N}}) = \mathbb{R}^{\mathbb{N}} \oplus V$ where $V = \{v \in W^{1,p}_{per}(T,\mathbb{R}^{\mathbb{N}}) : \int_0^b v(t)dt = 0\}$. So if $z \in W^{1,p}_{per}(T,\mathbb{R}^{\mathbb{N}})$, we have $z = \overline{z} + \widehat{z}$ with $\overline{z} \in \mathbb{R}^{\mathbb{N}}$ and $\widehat{z} \in V$. We have

$$x_n = \overline{x}_n + \widehat{x}_n \ \text{ and } \ y_n = \overline{y}_n + \widehat{y}_n \ \text{ with } \ \overline{y}_n = \frac{\overline{x}_n}{\|x_n\|} \ \text{ and } \ \widehat{y}_n = \frac{\widehat{x}_n}{\|x_n\|}, \ n \ge 1.$$

From the previous considerations, we have

$$\widehat{y}_n = \frac{\widehat{x}_n}{\|x_n\|} \to 0 \ \text{ in } \ W^{1,p}_{per}(T,\mathbb{R}^{\mathbb{N}}) \ \text{ and } \ \overline{y}_n = \frac{\overline{x}_n}{\|x_n\|} \to c \ \text{ in } \ \mathbb{R}^{\mathbb{N}}.$$

Suppose that $c = 0$. Then $y_n \to 0$ in $W^{1,p}_{per}(T,\mathbb{R}^{\mathbb{N}})$, a contradiction to the fact that $\|y_n\| = 1$ for all $n \ge 1$. So $c \neq 0$. This means that for all $t \in T$, $\|x_n(t)\| \to +\infty$. In fact we shall show that this convergence is uniform in $t \in T$, i.e. that $\min\limits_{t\in T}\|x_n(t)\| \to +\infty$. For this purpose, recall that $y_n \to c$ in $W^{1,p}_{per}(T,\mathbb{R}^{\mathbb{N}})$. So given $0 < \varepsilon < \|c\|$, we can find $n_0 \ge 1$ such that for all $n \ge n_0$ and all $t \in T$ we have

$$\begin{aligned}
&\|y_n(t) - c\| < \varepsilon\\
\Rightarrow& 0 < \|c\| - \varepsilon < \|y_n(t)\|.
\end{aligned}$$

Since by hypothesis $\|x_n\| \to +\infty$, given $\mu > 0$, we can find $n_1 \ge 1$ such that for all $n \ge n_1$ we have

$$\|x_n\| \ge \mu > 0.$$

Therefore for $n \ge n_2 = \max\{n_0, n_1\}$ and for $t \in T$, we have

$$\begin{aligned}
&\frac{\|x_n(t)\|}{\mu} \ge \frac{\|x_n(t)\|}{\|x_n\|} = \|y_n(t)\| > \|c\| - \varepsilon = \xi_3 > 0\\
\Rightarrow& \|x_n(t)\| > \mu\xi_3 > 0.
\end{aligned}$$

Since $\mu > 0$ was arbitrary and $\xi_3 > 0$ it follows that $\min_{t \in T} \|x_n(t)\| \to +\infty$ as $n \to +\infty$. So without any loss of generality we may assume that $\|x_n(t)\| > 0$ for all $t \in T$ and all $n \geq 1$. From the choice of the sequence $\{x_n\}_{n \geq 1} \subseteq W^{1,p}_{per}(T, \mathbb{R}^{\mathbb{N}})$, we have

$$\begin{aligned}
&-\frac{pM_1}{\|x_n\|^p} \leq \|y_n'\|_p^p + \int_0^b \frac{pj(t, x_n(t))}{\|x_n\|^p} dt \\
\Rightarrow &-\frac{pM_1}{\|x_n\|^p} \leq \|y_n'\|_p^p + \int_0^b \frac{pj(t, x_n(t))}{\|x_n(t)\|^p} \|y_n(t)\|^p dt \\
\Rightarrow &-\frac{pM_1}{\|x_n\|^p} \leq 1 + \int_0^b \frac{p\alpha_1(t) - p\alpha_2(t)\|x_n(t)\|^\eta}{\|x_n(t)\|^p} \|y_n(t)\|^p dt \ \ \text{(see (2.1))}.
\end{aligned}$$

Since $\eta > p$ and $\min_{t \in T} \|x_n(t)\| \to +\infty$, by passing to the limit as $n \to \infty$ we reach a contradiction. This proves that $\{x_n\}_{n \geq 1} \subseteq W^{1,p}_{per}(T, \mathbb{R}^{\mathbb{N}})$ is bounded. So we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}_{per}(T, \mathbb{R}^{\mathbb{N}})$ and $x_n \to x$ in $C(T, \mathbb{R}^{\mathbb{N}})$. From the choice of the sequence $\{x_n\}_{n \geq 1} \subseteq W^{1,p}_{per}(T, \mathbb{R}^{\mathbb{N}})$, we have

$$\begin{aligned}
&|\langle x_n^*, x_n - x\rangle| \leq \varepsilon_n \|x_n - x\| \ \text{ with } \ \varepsilon_n \downarrow 0 \\
\Rightarrow &\langle A(x_n), x_n - x\rangle + (u_n, x_n - x)_{pq} \leq \varepsilon_n \|x_n - x\|.
\end{aligned}$$

Since $\{x_n\}_{n \geq 1} \subseteq W^{1,p}_{per}(T, \mathbb{R}^{\mathbb{N}})$ is bounded (hence it is bounded in $C(T, \mathbb{R}^{\mathbb{N}})$), by virtue of hypothesis $H(j)(iii)$, we have that $\{u_n\}_{n \geq 1} \subseteq L^q(T, \mathbb{R}^{\mathbb{N}})$ is bounded. So $(u_n, x_n - x)_{pq} \to 0$. Therefore

$$\limsup_{n \to \infty} \langle A(x_n), x_n - x\rangle \leq 0.$$

Because A is maximal monotone, it is generalized pseudomonotone and so $\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle \Rightarrow \|x_n'\|_p \to \|x'\|_p$. Because $x_n' \xrightarrow{w} x'$ in $L^p(T, \mathbb{R}^{\mathbb{N}})$ and the latter is unoformly convex, from the Kadec-Klee property it follows that $x_n' \to x'$ in $L^p(T, \mathbb{R}^{\mathbb{N}})$ and so finally we have $x_n \to x$ in $W^{1,p}_{per}(T, \mathbb{R}^{\mathbb{N}})$. $\square$

Let $u \in C^1(T, \mathbb{R}^{\mathbb{N}})$ be a normalized eigenfunction corresponding to $\lambda_1 > 0$ the first nonzero eigenfunction of $(-\triangle_p, W^{1,p}_{per}(T, \mathbb{R}^{\mathbb{N}}))$.

Proposition 2.2. $\|u(\cdot)\|^{p-2}u(\cdot) \in W^{1,p}_{per}(T, \mathbb{R}^{\mathbb{N}})$ and $\int_0^b \|u(t)\|^{p-2}u(t)dt = 0$.

Proof. Let $\xi = \|u\|_\infty$. Then for all $t \in T$ we have $\left\| \|u(t)\|^{p-2}u(t) \right\| \le \xi^{p-1}$, hence $\|u(\cdot)\|^{p-2}u(\cdot) \in L^p(T, \mathbb{R}^{\mathbb{N}})$. Also the function $t \to \|u(t)\|^{p-2}u(t)$ is differentiable a.e. on T and

$$\frac{d}{dt}\left(\|u(t)\|^{p-2}u(t)\right) = \left(\frac{d}{dt}\|u(t)\|^{p-2}\right)u(t) + \|u(t)\|^{p-2}u'(t) \text{ a.e. on } T.$$

Note that

$$\begin{aligned} &\left\| \|u(t)\|^{p-2}u'(t) \right\| \le \xi^{p-2}\|u'(t)\| \\ \Rightarrow &\|u(\cdot)\|^{p-2}u'(\cdot) \in L^p(T, \mathbb{R}^{\mathbb{N}}). \end{aligned} \tag{2.12}$$

Also from the chain rule and since $\{t \in T : u(t) = 0\}$ is Lebesgue-null, we have

$$\begin{aligned} \left(\frac{d}{dt}\|u(t)\|^{p-2}\right)u(t) &= (p-2)\|u(t)\|^{p-3}\frac{(u'(t), u(t))_{\mathbb{R}^{\mathbb{N}}}}{\|u_n(t)\|}u(t) \\ &= \frac{(p-2)}{2}\|u(t)\|^{p-4}\left(\frac{d}{dt}\|u(t)\|^2\right)u(t) \text{ a.e. on } T, \\ \Rightarrow \left\| \left(\frac{d}{dt}\|u(t)\|^{p-2}\right)u(t) \right\|^p &= \frac{(p-2)^p}{2^p}\|u(t)\|^{(p-4)p}\|u(t)\|^p\left|\frac{d}{dt}\|u(t)\|^2\right|^p \\ &\le \frac{(p-2)^p}{2^{p-1}}\xi^{(p-3)p}\left|\|u(t)\|\frac{d}{dt}\|u(t)\|\right|^p \\ &\le \frac{(p-2)^p}{2^{p-1}}\xi^{(p-2)p}\left|\frac{d}{dt}\|u(t)\|\right|^p \\ &\le \frac{(p-2)^p}{2^{p-1}}\xi^{(p-2)p}\|u'(t)\|^p \text{ a.e. on } T \\ \Rightarrow \left(\frac{d}{dt}\|u(\cdot)\|^{p-2}\right)u(\cdot) &\in L^p(T, \mathbb{R}^{\mathbb{N}}). \end{aligned} \tag{2.13}$$

From (2.12) and (2.13) it follows that $\frac{d}{dt}\left(\|u(\cdot)\|^{p-2}u(\cdot)\right) \in L^p(T, \mathbb{R}^{\mathbb{N}})$ and so we infer that $\|u(\cdot)\|^{p-2}u(\cdot) \in W^{1,p}_{per}(T, \mathbb{R}^{\mathbb{N}})$. Also directly from the eigenvalue equation we have $\int_0^b \|u(t)\|^{p-2}u(t)dt = 0$. ☐

Remark 2.3. According to this proposition $\|u(\cdot)\|^{p-2}u(\cdot) \in V$.

Next we set $h(t) = \|u(t)\|^{p-2}u(t)$ and we introduce the cylinder set

$$C_{\mathbb{R}} = \left\{ x \in W^{1,p}_{per}(T, \mathbb{R}^{\mathbb{N}}) : x(t) = c + \lambda\frac{h(t)}{\|h'\|_p},\ c \in \mathbb{R}^{\mathbb{N}},\ \|c\| \le \mathbb{R},\ 0 \le \lambda \le \mathbb{R} \right\}.$$

By the Poincare-Wirtinger inequality $\|h'\|_p$ is an equivalent norm on V.

Proposition 2.4. If hypotheses $H(j)$ hold and $\mathbb{R} > 0$ is large enough, then $\varphi|_{\partial C_{\mathbb{R}}} \leq 0$.

Proof. First we check the lower base (i.e. $\lambda = 0$) of the cylinder set. In this case if $x \in C_{\mathbb{R}}$, then $x = c \in \mathbb{R}^{\mathbb{N}}$ and we have

$$\varphi(c) = \int_0^b j(t,c)dt < 0 \text{ (see hypothesis } H(j)(v)).$$

For any $x \in C_{\mathbb{R}}$ we have

$$\begin{aligned} &\|x\|_p^p \leq b\|c\|_{\mathbb{R}^{\mathbb{N}}}^p + \lambda^p \frac{\|h\|_p^p}{\|h\|^p} \leq b\|c\|_{\mathbb{R}^{\mathbb{N}}}^p + \lambda^p \\ &\text{(since } \|h\|_p \leq \gamma\|h'\|_p \text{ for some } \gamma > 0) \end{aligned} \tag{2.14}$$

Because of (2.1) for $x \in C_{\mathbb{R}}$ we have

$$\int_0^b j(t,x(t))dt \leq \xi_5 - \xi_6\|x\|_p^\eta \text{ with } \xi_5, \xi_6 > 0.$$

Thus for $x \in C_{\mathbb{R}}$ we can write that

$$\begin{aligned} \varphi(x) &= \frac{1}{p}\|x'\|_p^p + \int_0^b j(t,x(t))dt \leq \frac{\lambda^p}{p} + \xi_5 - \xi_6\|x\|_p^\eta \\ &\leq \frac{\lambda^p}{p} + \xi_5 - \xi_6 \left(b\|c\|^p + \lambda^p\gamma^p\right)^{\frac{\eta}{p}}. \end{aligned}$$

Then for $x \in C_{\mathbb{R}}$ located on the lateral boundary $\|c\| = \mathbb{R}$ of the cylinder set and for $x \in C_{\mathbb{R}}$ on the upper base $\lambda = \mathbb{R}$ of the cylinder set we have

$$\varphi(x) \leq \frac{R^p}{p} - \xi_7\mathbb{R}^\eta + \xi_8 \text{ for some } \xi_7, \xi_8 > 0.$$

Because $\eta > p$, for $R > 0$ large we have

$$\varphi|_{\partial C_{\mathbb{R}}} \leq 0.$$

□

Next let $E = \left\{v \in V : \|v'\|_p = \frac{1}{b^{p-1}}\right\}$.

Proposition 2.5. If hypotheses $H(j)$ hold, then $0 \leq \inf\limits_E \varphi$.

Proof. From the Poincare-Wirtinger inequality we know that for all $v \in V$

$$\begin{aligned} &\|v\|_\infty^p \le b^{p-1}\|v'\|_p^p \\ \Rightarrow &\|v\|_\infty^p \le b^{p-1}\|v'\|_p^p = 1 \text{ for all } v \in E. \end{aligned}$$

Then by hypothesis $H(j)(v)$ we have that for all $v \in E$ and almost all $t \in T$

$$j(t, v(t)) \ge -\frac{1}{pb^p}.$$

Therefore for all $v \in E$ we have

$$\varphi(v) = \frac{1}{pb^{p-1}} - \frac{1}{pb^p} b = 0$$

□

Now we are ready for the existence theorem concerning problem (1.1).

Theorem 2.6. *If hypotheses $H(j)$ hold, then problem (1.1) has a nonconstant solution $x \in C^1(T, \mathbb{R}^{\mathbb{N}})$.*

Proof. By virtue of Theorem 1.1 there exists $x \in W^{1,p}_{per}(T, \mathbb{R}^{\mathbb{N}})$ such that

$$0 \in \partial\varphi(x) \text{ and } 0 \le \varphi(x).$$

Evidently $x \in W^{1,p}_{per}(T, \mathbb{R}^{\mathbb{N}})$ is nonconstant because for every $c \in \mathbb{R}^{\mathbb{N}}$, $\varphi(c) < 0$ (see hypothesis $H(j)(iv)$). We have

$$A(x) = u \text{ with } u \in L^q(T, \mathbb{R}^{\mathbb{N}}), \ \ u(t) \in \partial j(t, x(t)) \text{ a.e. on } T.$$

For every $\psi \in C_0^\infty((0,b), \mathbb{R}^{\mathbb{N}})$ we have

$$\int_0^b \|x'(t)\|^{p-2} \left(x'(t), \psi'(t)\right)_{\mathbb{R}^{\mathbb{N}}} dt = \int_0^b (u(t), \psi(t))_{\mathbb{R}^{\mathbb{N}}} dt.$$

Recalling that $\|x'(\cdot)\|^{p-2}x'(\cdot) \in W^{-1,q}(T, \mathbb{R}^{\mathbb{N}}) = W_0^{1,p}(T, \mathbb{R}^{\mathbb{N}})^*$ (see Adams [1], p.50) we have that

$$\langle \left(\|x'\|^{p-2}x'\right)', \psi\rangle_0 = \int_0^b (u(t), \psi(t))_{\mathbb{R}^{\mathbb{N}}} dt = \langle u, \psi\rangle_0,$$

where $\langle \cdot, \cdot \rangle_0$ denotes the duality brackets for the pair $(W_0^{1,p}(t, \mathbb{R}^{\mathbb{N}}), W^{-1,q}(T, \mathbb{R}^{\mathbb{N}}))$. Since $C_0^\infty(T, \mathbb{R}^{\mathbb{N}})$ is dense in $W^{1,p}_{per}(T, \mathbb{R}^{\mathbb{N}})$, it follows that

$$\left(\|x'(t)\|^{p-2}x'(t)\right)' = u(t) \in \partial j(t, x(t)) \text{ a.e. on } T. \tag{2.15}$$

Also for every $y \in W^{1,p}_{per}(T, \mathbb{R}^{\mathbb{N}})$, using Green's identity (integration by parts), we have

$$\begin{aligned}\langle A(x), y\rangle &= \big(\|x'(b)\|^{p-2}x'(b), y(b)\big)_{\mathbb{R}^{\mathbb{N}}} - \big(\|x'(0)\|^{p-2}x'(0), y(0)\big)_{\mathbb{R}^{\mathbb{N}}} \\ &\quad - \int_0^b \big((\|x'(t)\|^{p-2}x'(t))', y(t)\big)_{\mathbb{R}^{\mathbb{N}}}\, dt.\end{aligned}$$

Because $A(x) = u$ and using (2.15) we obtain

$$\big(\|x'(0)\|^{p-2}x'(0), y(0)\big)_{\mathbb{R}^{\mathbb{N}}} = \big(\|x'(b)\|^{p-2}x'(b), y(b)\big)_{\mathbb{R}^{\mathbb{N}}}$$

for all $y \in W^{1,p}_{per}(T, \mathbb{R}^{\mathbb{N}}) \ \Rightarrow x'(0) = x'(b)$.

Also remark that because $x \in W^{1,p}_{per}(T, \mathbb{R}^{\mathbb{N}})$, we have $x(0) = x(b)$. Finally since $\|x'\|^{p-2}x' \in W^{1,q}_{per}(T, \mathbb{R}^{\mathbb{N}})$, we have $\|x'(\cdot)\|^{p-2}x'(\cdot) \in C(T, \mathbb{R}^{\mathbb{N}})$. Because the map $y \to \|y\|^{p-2}y$ is a homeomorphism of $\mathbb{R}^{\mathbb{N}}$ it follows that $x' \in C_{per}(T, \mathbb{R}^{\mathbb{N}})$ and so $x \in C^1_{per}(T, \mathbb{R}^{\mathbb{N}})$ and it solves (1.1). $\square$

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