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ON NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. In this paper, we prove a theorem on local existence and uniqueness of integral solutions to a class of partial neutral functional differential equations with infinite delay. Our method of proof is based on the integrated semigroup theory and the well known Banach fixed point theorem.

Key Words and Phrases: Contraction, integral solution, integrated semigroup, neutral type, infinite delay, phase space, local existence and uniqueness.

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1. Introduction

In this article, we consider the following class of nonlinear partial neutral functional differential equations with infinite delay

$$\begin{cases} \dfrac{\partial}{\partial t}\mathcal{D}u_t = A\mathcal{D}u_t + F(t, u_t), \ t \geq 0, \\ x_0 = \phi \in \mathcal{B}, \end{cases} \tag{1}$$

where $A : D(A) \subseteq E \to E$ is a linear operator on a Banach space $(E, |.|)$, $\mathcal{B}$ is the phase space of functions mapping $(-\infty, 0]$ into E, which will be specified later, $\mathcal{D}$ is a bounded linear operator from $\mathcal{B}$ into E defined by

$$\mathcal{D}\varphi = \varphi(0) - \mathcal{D}_0\varphi \text{ for any } \varphi \in \mathcal{B},$$

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$\mathcal{D}_0$ is a bounded linear operator from $\mathcal{B}$ into E and for each $u : (-\infty, b] \to E$, $b > 0$, and $t \in [0, b]$, u_t represents, as usual, the mapping defined from $(-\infty, 0]$ into E by

$$u_t(\theta) = u(t + \theta) \text{ for } \theta \in (-\infty, 0].$$

F is an E-valued nonlinear continuous mapping on $I\!\!R_+ \times \mathcal{B}$.

Throughout this paper, we suppose that $(\mathcal{B}, \|.\|_{\mathcal{B}})$ is a (semi)normed abstract linear space of functions mapping $(-\infty, 0]$ into E, and satisfies the following fundamental axioms which have been first introduced in [23] and widely discussed in [29].

(**A**) There exist a positive constant H and functions $K(.)$, $M(.) : I\!\!R^+ \to I\!\!R^+$, with K continuous and M locally bounded, such that for any $\sigma \in I\!\!R$ and $a > 0$, if $x : (-\infty, \sigma + a] \to E$, $x_\sigma \in \mathcal{B}$ and $x(.)$ is continuous on $[\sigma, \sigma + a]$, then for every t in $[\sigma, \sigma + a]$ the following conditions hold:
(**i**) $x_t \in \mathcal{B}$,
(**ii**) $|x(t)| \leq H \, \|x_t\|_{\mathcal{B}}$, which is equivalent to
(**ii**)′ $|\varphi(0)| \leq H \, ||\varphi||_{\mathcal{B}}$, for every $\varphi \in \mathcal{B}$,
(**iii**) $\|x_t\|_{\mathcal{B}} \leq K(t - \sigma) \sup\limits_{\sigma \leq s \leq t} |x(s)| + M(t - \sigma) \|x_\sigma\|_{\mathcal{B}}$.
(**A**1) For the function $x(.)$ in (**A**), $t \mapsto x_t$ is a $\mathcal{B}$-valued continuous function for t in $[\sigma, \sigma + a]$.
(**B**) The space $\mathcal{B}$ is complete.

Example 1. *Define for a constant γ the following standard space*

$$C_\gamma := \left\{ \phi : (-\infty, 0] \to E \textit{ continuous such that} \lim_{\theta \to -\infty} e^{\gamma\theta} \phi(\theta) \textit{ exists in } E \right\}.$$

It's known from [29] *that C_γ with the norm $\|\phi\|_\gamma = \sup\limits_{\theta \leq 0} e^{\gamma\theta} |\phi(\theta)|$, $\phi \in C_\gamma$, satisfies the axioms* (**A**)*,* (**A**1) *and* (**B**) *with $H = 1$, $K(t) = \max(1, e^{-\gamma t})$ and $M(t) = e^{-\gamma t}$ for all $t \geq 0$.*

Throughout, we also assume that the operator A satisfies the Hille-Yosida condition:

(**H1**) there exist $\bar{M} \geq 0$ and $\varpi \in I\!\!N$ such that $]\varpi, +\infty[\subset \rho(A)$ and

$$\sup \left\{ (\lambda - \varpi)^n \left\| (\lambda I - A)^{-n} \right\| : \ n \in I\!\!N, \ \lambda > \varpi \right\} \leq \bar{M}. \tag{2}$$

Let A_0 be the part of the operator A in $\overline{D(A)}$, which is defined by

$$\begin{cases} D(A_0) = \left\{x \in D(A) : Ax \in \overline{D(A)}\right\}, \\ A_0 x = Ax, \text{ for } x \in D(A_0). \end{cases}$$

It is well known that $\overline{D(A_0)} = \overline{D(A)}$ and the operator A_0 generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$.

Recall that ([34]) for all $x \in \overline{D(A)}$ and $t \geq 0$, one has $\int_0^t T_0(s)x \in D(A_0)$ and

$$\left(A \int_0^t T_0(s)xds\right) + x = T_0(t)x. \tag{3}$$

We also recall that $(T_0(t))_{t\geq 0}$ coincides on $\overline{D(A_0)}$ with the derivative of the locally Lipschitz integrated semigroup $(S(t))_{t\geq 0}$ generated by A on E. Which is, according to [10] and [31], a family of bounded linear operators on E, that satisfies

(i) $S(0) = 0$,
(ii) for any $y \in E$, $t \to S(t)y$ is strongly continuous with values in E,
(iii) $S(s)S(t) = \displaystyle\int_0^s (S(t+r) - S(r))dr$ for all $t, s \geq 0$,
and for any $\tau > 0$ there exists a constant $l(\tau) > 0$ such that

$$\|S(t) - S(s)\| \leq l(\tau)\,|t-s| \qquad \text{for all } t, s \in [0, \tau].$$

This integrated semigroup is exponentially bounded, that is, there exist two constants $\bar{M}$ and $\overline{\omega}$ such that $\|S(t)\| \leq \bar{M}e^{\overline{\omega}t}$ for all $t \geq 0$.

As known in Hale [19], Hale and Lunel [24] and the references therein, very much attention has been given to differential difference equations of neutral type. The reason was applications on lossless transmission lines. The development has concerned the general theory of partial neutral functional differential equations. The study of the special form (1) has been initiated in the case of finite delay by Hale in [21] and [22]. The motivation was a model for a continuous circular array of resistively coupled transmission lines with mixed initial boundary conditions introduced by Wu and Xia ([40], [41]). Based mainly on the last cited four articles and a detailed discussion in the book by Wu [39], Adimy and Ezzinbi have published a list of interesting papers ([6], [7], [8] and [9]) about Eq. (1) but with finite delay. This work (such as [2] and [3]) contributes to the construction of a complete theory about the infinite delay case. It can be seen as an extension to the case of neutral type of some of our earlier results about functional differential equations with infinite delay in [4] and [12]. Precisely here we address local existence and uniqueness of integral solutions in the case where F is Lipschitz continuous with respect to φ on the balls of $\mathcal{B}$. In general, the development may also be seen as a reproduction to the case of infinite delay of some equivalent results in the above papers by Adimy and Ezzinbi. The main theorem is proved in section 3.

2. Preliminaries

All the results of this section are proved in [5] in the case of infinite delay and [9] in the case of finite delay.

Consider the following system

$$\begin{cases} \dfrac{\partial}{\partial t}\mathcal{D}u_t = A\mathcal{D}u_t \text{ if } t \geq 0, \\ u(\theta) = \varphi(\theta) \text{ if } \theta \in (-\infty, 0] \text{ with } \varphi \in \mathcal{B}. \end{cases} \tag{4}$$

Using equality (3), we can see that a necessary condition for $u : (-\infty, b) \to E$, $b > 0$, to be a solution of Eq. (4) is that it verifies the following integrated one on $(-\infty, b)$

$$\begin{cases} \mathcal{D}u_t = T_0(t)\mathcal{D}\varphi, \ \ t \geq 0, \\ u_0 = \varphi, \end{cases} \tag{5}$$

where

$$\varphi \in \mathcal{Y} := \left\{ \varphi \in \mathcal{B} : \mathcal{D}\varphi \in \overline{D(A)} \right\}.$$

The following result is only the combination of Lemma 3 in [3] and Proposition 11 in [2] which are proved in a general framework. Precisely, here it suffices to take $h(t) := T_0(t)\mathcal{D}\varphi$.

Proposition 1. *Assume that Condition* $(\mathbf{H}1)$ *is satisfied and* $\|\mathcal{D}_0\| K(0) < 1$. *Then, for given* $\varphi \in \mathcal{Y}$ *there exists a unique function* u *which is continuous on* $[0, T)$ *and solves Eq. (5) on* $(-\infty, T)$. *Moreover, the family of operators* $(\mathcal{T}(t))_{t\geq 0}$ *defined on* $\mathcal{Y}$ *by* $\mathcal{T}(t)\varphi = u_t(., \varphi)$ *is a* C_0*-semigroup on* $\mathcal{Y}$.

We now define a fundamental integral solution $Z(t)$ associated to Eq. (1). Consider for given $c \in E$ the following equation

$$\begin{cases} \mathcal{D}z_t = S(t)c \ \text{ if } t \geq 0, \\ z(t) = 0 \ \text{ if } t \in (-\infty, 0]. \end{cases} \tag{6}$$

To our purpose, we make the following condition

(**H2**) There exists a continuous nondecreasing function $\delta : [0,+\infty) \to [0,+\infty[$, $\delta(0) = 0$ and a family of continuous linear operators $W_\varepsilon : \mathcal{B} \to E$, $\varepsilon \in [0,+\infty)$, such that

$$|\mathcal{D}_0\varphi - \mathcal{D}_\varepsilon\varphi| \leq \delta(\varepsilon) \|\varphi\|_{\mathcal{B}}, \text{ for } \varepsilon \in [0,+\infty) \text{ and } \varphi \in \mathcal{B},$$

where the linear operator $\mathcal{D}_\varepsilon : \mathcal{B} \to E$ is defined, for $\varepsilon \in [0,+\infty)$, by

$$\begin{cases} \mathcal{D}_\varepsilon = W_\varepsilon \circ \tau_\varepsilon, \\ \tau_\varepsilon(\varphi)(\theta) = \varphi(\theta - \varepsilon), \quad \text{for } \varphi \in \mathcal{B} \text{ and } \theta \in (-\infty, 0]. \end{cases}$$

Notice that Assumption (**H**2) means that the operator $\mathcal{D}_0$ does not depend very strongly upon $\varphi(0)$. It is the infinite delay version of the one introduced in [8] and [9].

Proposition 2. [5] *Assume that Conditions* (**H**1) *and* (**H**2) *are satisfied such that* $K(0)\|\mathcal{D}_0\| < 1$. *Then, for given* $c \in E$, *Eq. (6) has a unique integral solution* $z := z(.)c : (-\infty,+\infty) \to E$. *Moreover, the operator* $Z(t) : E \to \mathcal{B}$ *defined by*

$$Z(t)c = z_t(.)c$$

satisfies, for any continuous function $f : [0,+\infty) \to E$, *the following properties*

(i) *For each* $T > 0$, *there exists a function* $\alpha(.) \in L^\infty([0,T], I\!R^+)$ *and* $\beta \in I\!R$ *such that* $\|Z(t)\| \leq \alpha(t)e^{t\beta}$ *for all* $t \in [0,T]$;

(ii) $Z(t)(E) \subseteq \mathcal{Y}$, *for all* $t \geq 0$;

(iii) *For all* $\tau > 0$ *there exists a constant* $k(\tau) > 0$ *such that*

$$\|Z(t)c - Z(s)c\|_{\mathcal{B}} \leq k(\tau)\,|t-s|\,|c| \quad \text{for all } t,s \in [0,\tau] \text{ and } c \in E.$$

(iv) *For any continuous function* $f : [0,+\infty) \to E$, *the functions*

$$t \mapsto \int_0^t Z(t-s)f(s)\,ds \text{ and } t \mapsto \int_0^t S(t-s)f(s)\,ds$$

are continuously differentiable for all $t \geq 0$ *and satisfy*

$$\frac{d}{dt}\left(\int_0^t Z(t-s)f(s)\,ds\right) = \lim_{h\to 0^+} \frac{1}{h}\int_0^t \mathcal{T}(t-s)Z(h)f(s)ds \quad \text{for all } t \geq 0.$$

$$\begin{aligned}\mathcal{D}\left(\frac{d}{dt}\int_0^t Z(t-s)f(s)\,ds\right) &= \lim_{h\to0^+}\frac{1}{h}\int_0^t S'(t-s)S(h)f(s)ds\\ &= \frac{d}{dt}\int_0^t S(t-s)f(s)ds.\end{aligned}$$

For convenience of the reader about the main equation (1), we recall the following definition.

Definition 1. *Let $T>0$ and $\varphi\in\mathcal{B}$. We consider the following definitions. We say that a function $u:=u(.,\varphi):(-\infty,T)\to E,\ 0<T\le+\infty$, is an integral solution of Eq. (1) if*
(**i**) *u is continuous on $[0,T)$,*
(**ii**) $\int_0^t \mathcal{D}u_s\,ds\in D(A)$ *for $t\in[0,T)$,*
(**iii**) $\mathcal{D}u_t=\mathcal{D}\varphi+A\int_0^t\mathcal{D}u_s\,ds+\int_0^t F(s,u_s)\,ds$ *for $t\in[0,T)$,*
(**iv**) *$u(t)=\varphi(t)$, for all $t\in(-\infty,0]$.*

We deduce from [2] and [37] that integral solutions of Eq. (1) are given for $\varphi\in\mathcal{B}$ such that $\mathcal{D}\varphi\in\overline{D(A)}$ by the following system

$$\begin{cases}\mathcal{D}u_t=S'(t)\mathcal{D}\varphi+\dfrac{d}{dt}\displaystyle\int_0^t S(t-s)F(s,u_s)ds, & t\in[0,T)\,,\\ u(t)=\varphi(t), \quad t\in(-\infty,0]\,,\end{cases}\tag{7}$$

We also suppose that $\mathcal{B}$ is normed and satisfies one of the following two extra axioms.
(**C1**) If $(\phi_n)_{n\ge0}$ is a Cauchy sequence in $\mathcal{B}$ and if $(\phi_n)_{n\ge0}$ converges compactly to ϕ on $(-\infty,0]$, then ϕ is in $\mathcal{B}$ and $\|\phi_n-\phi\|_{\mathcal{B}}\to0$, as $n\to\infty$.
(**D1**) For a sequence $(\varphi_n)_{n\ge0}$ in $\mathcal{B}$, if $\|\varphi_n\|_{\mathcal{B}}\to0$, as $n\to\infty$, then $|\varphi_n(\theta)|\to0$, as $n\to\infty$, for each $\theta\in(-\infty,0]$.

Remark that Axiom (**D1**) implies that the space $\mathcal{B}$ is normed.

Proposition 3. [5] *Let $\mathcal{B}$ be a normed space which satisfies Axiom* (**C**1) *or Axiom* (**D1**). *If there exists an integral solution $u:=u(.,\varphi):(-\infty,T)\to E,\ 0<T\le+\infty$, of Eq. (1), then the function $[0,T)\ni t\mapsto u_t\in\mathcal{B}$ satisfies*

$$\begin{aligned}u_t &= \mathcal{T}(t)\varphi+\frac{d}{dt}\int_0^t Z(t-s)F(s,u_s)ds \qquad (8)\\ &= \mathcal{T}(t)\varphi+\lim_{h\to0^+}\frac{1}{h}\int_0^t\mathcal{T}(t-s)Z(h)F(s,u_s)ds.\end{aligned}$$

Conversely, if there exists a function $v \in \mathcal{C}([0,T),\mathcal{B})$ *such that*

$$v(t) = \mathcal{T}(t)\varphi + \frac{d}{dt}\int_0^t Z(t-s)F(s,v(s))ds, \ t \in [0,T) \tag{9}$$

then $v(t) = u_t$ *for all* $t \in [0,T)$, *where*

$$u(t) = \begin{cases} v(t)(0), & t \in [0,T) \\ \varphi(t) & t \in (-\infty,0] \end{cases}$$

and $u(.)$ *is an integral solution of Eq.(1).*

Remark 1. *The prove of the above proposition needs computing integrals in* $\mathcal{B}$ *from integrals in* E*. That's why we suppose that* $\mathcal{B}$ *is normed and satisfies one of the two extra axioms* (**C**1) *or* (**D1**)*. See* [32], [2] *or* [5] *for more details.*

3. Local existence and uniqueness of solutions

To obtain our main result on local existence and uniqueness of integral solutions to Eq. (1), we add an extra condition

(**H3**) $F : [0,+\infty[\times \mathcal{B}$ is Lipschitz continuous with respect to φ on the balls of $\mathcal{B}$, i.e., for each $r > 0$ there exists a constant $c_0(r) > 0$ such that if $t \geq 0$, $\varphi_1, \varphi_2 \in \mathcal{B}$ and $\|\varphi_1\|_{\mathcal{B}}, \|\varphi_2\|_{\mathcal{B}} \leq r$ then

$$|F(t,\varphi_1) - F(t,\varphi_2)| \leq c_0(r)\|\varphi_1 - \varphi_2\|_{\mathcal{B}}.$$

Theorem 4. *Let* $\mathcal{B}$ *be a normed space which satisfies Axiom* (**C**1) *or Axiom* (**D1**)*. Assume that* (**H**1), (**H**2) *and* (**H**3) *hold. Let* $\varphi \in \mathcal{B}$ *such that* $\mathcal{D}\varphi \in \overline{D(A)}$*. Then, there exists a maximal interval of existence* $(-\infty, b_\varphi)$, $b_\varphi > 0$*, and a unique mild solution* $u(.,\varphi)$ *of Eq. (1), defined on* $(-\infty, b_\varphi)$ *and either* $b_\varphi = +\infty$ *or*

$$\limsup_{t \to b_\varphi^-} |u(t,\varphi)| = +\infty.$$

Proof. Note that (**H**3) implies that for each $r > 0$ there exists $c_0(r) > 0$ such that

$$|F(t,\varphi)| \leq rc_0(r) + |F(t,0)| \quad \text{for } t \geq 0, \ \varphi \in \mathcal{B} \text{ and } \|\varphi\|_{\mathcal{B}} \leq r.$$

Let $b > 0$. Suppose that $\varphi \in \mathcal{B}, \ \mathcal{D}\varphi \in \overline{D(A)}, \ r = \|\varphi\|_{\mathcal{B}} + 1$ and

$$c_1 = rc_0(r) + \sup_{t \in [0,b]} |F(t,0)|.$$

Consider the following set

$$\Omega_{\varphi} := \left\{ v \in \mathcal{C}\left([0,b],\mathcal{B}\right) : \sup_{0\leq s\leq b} \|v(s)-\varphi\|_{\mathcal{B}} \leq 1 \right\},$$

where $\mathcal{C}\left([0,b],\mathcal{B}\right)$ is endowed with the uniform convergence topology. We can easily see that Ω_{φ} is a closed set of $\mathcal{C}\left([0,b],\mathcal{B}\right)$. Consider the mapping

$$H : \Omega_{\varphi} \to \mathcal{C}\left([0,b],\mathcal{B}\right),$$

defined for $v \in \Omega_{\varphi}$ and $t \in [0,b]$ by

$$\begin{aligned} H\left(v\right)\left(t\right) &:= \mathcal{T}(t)\varphi + \frac{d}{dt}\int_0^t Z(t-s)F(s,v(s))ds, \\ &= \mathcal{T}(t)\varphi + \lim_{h\to 0^+}\frac{1}{h}\int_0^t \mathcal{T}(t-s)Z(h)F(s,v(s))ds. \end{aligned}$$

Next we show that

$$H\left(\Omega_{\varphi}\right) \subseteq \Omega_{\varphi}.$$

One can remark, as in the proof of Proposition 2.2 of [31], that

$$\lim\sup_{h\to 0^+}\frac{1}{h}\|Z(h)\| < +\infty.$$

Then we can set

$$k := \lim\sup_{h\to 0^+}\frac{1}{h}\|Z(h)\|. \tag{10}$$

We have for suitable constants $\overline{M}$ and ω

$$\|H\left(v\right)\left(t\right)-\varphi\|_{\mathcal{B}} \leq \|\mathcal{T}(t)\varphi-\varphi\|_{\mathcal{B}} + \overline{M}e^{\omega t}\int_0^t e^{-\omega s}\frac{1}{h}\|Z(h)\|\left|F(s,v(s))\right|ds.$$

We can assume here without loss of generality that $\omega > 0$. Thus we obtain the estimate

$$\|H\left(v\right)\left(t\right)-\varphi\|_{\mathcal{B}} \leq \|\mathcal{T}(t)\varphi-\varphi\|_{\mathcal{B}} + \overline{M}ke^{\omega t}\int_0^t \left|F(s,v(s))\right|ds.$$

Since $\|v(s)-\varphi\|_{\mathcal{B}} \leq 1$, for $s \in [0,b]$ and $r = \|\varphi\|_{\mathcal{B}}+1$, we have that $\|v(s)\|_{\mathcal{B}} \leq r$, for $s \in [0,b]$. Then

$$\left|F(s,v(s))\right| \leq c_0(r)\|v(s)\|_{\mathcal{B}} + \left|F(s,0)\right| \leq c_1.$$

Consider $b > 0$ sufficiently small such that

$$\sup_{0 \le s \le b} \left\{ \|\mathcal{T}(s)\varphi - \varphi\|_{\mathcal{B}} + \overline{M} k e^{\omega s} c_1 s \right\} < 1.$$

Then, we deduce that for $t \in [0, b]$

$$\|H(v)(t) - \varphi\|_{\mathcal{B}} \le \|\mathcal{T}(t)\varphi - \varphi\|_{\mathcal{B}} + \overline{M} k e^{\omega t} c_1 t < 1.$$

Hence

$$H(\Omega_\varphi) \subseteq \Omega_\varphi.$$

On the other hand, let $u, v \in \Omega_\varphi$ and $t \in [0, b]$. We can easily see that

$$\|H(u)(t) - H(v)(t)\|_{\mathcal{B}} \le \overline{M} k e^{\omega b} c_0(r)\, b\, \|u - v\|_{\mathcal{C}([0,b],\mathcal{B})}.$$

Note that $r \ge 1$. Then, by definition of c_1, $c_0(r) \le c_1$. Consequently

$$\overline{M} k e^{\omega b} c_0(r) b \le \sup_{0 \le s \le b} \left\{ \|\mathcal{T}(s)\varphi - \varphi\|_{\mathcal{B}} + \overline{M} k e^{\omega s} c_1 s \right\} < 1.$$

We conclude that there exists a unique function $v \in \Omega_\varphi$ such that $H(v) = v$. Then, Eq. (1) has one and only one integral solution $u : (-\infty, b) \to E$ defined by

$$u(t) = \begin{cases} v(t)(0), & t \in [0, b) \\ \varphi(t), & t \in (-\infty, 0]. \end{cases}$$

Let $(-\infty, b_\varphi)$ be the maximal interval of existence of u. Assume that $b_\varphi < +\infty$ and

$$\limsup_{t \to b_\varphi} \|u(t, \varphi)\|_{\mathcal{B}} < +\infty.$$

Then, from Axiom $(\mathbf{A - iii})$, we can see that there exists a constant $r > 0$ such that $\|u_t(., \varphi)\|_{\mathcal{B}} \le r$ for all $t \in [0, b_\varphi)$. Consider $t, t + h \in [0, b_\varphi)$ and $h > 0$,

$$\begin{aligned} u_{t+h} &= \mathcal{T}(t+h)\varphi + \lim_{d \to 0^+} \frac{1}{d} \int_0^{t+h} \mathcal{T}(t+h-s) Z(d) F(s, u_s) ds \\ &= \mathcal{T}(t+h)\varphi + \lim_{d \to 0^+} \frac{1}{d} \left(\int_0^h \mathcal{T}(t)\mathcal{T}(h-s) Z(d) F(s, u_s) ds \right. \\ &\quad \left. + \int_h^{t+h} \mathcal{T}(t+h-s) Z(d) F(s, u_s) ds \right) \end{aligned}$$

and

$$u_t = \mathcal{T}(t)\varphi + \lim_{d \to 0^+} \frac{1}{d} \int_0^t \mathcal{T}(t-s) Z(d) F(s, u_s) ds.$$

Since

$$\int_h^{t+h} \mathcal{T}(t+h-s)Z(d)F(s,u_s)ds = \int_0^t \mathcal{T}(t-s)Z(d)F(s+h,u_{s+h})ds,$$

we have

$$\begin{aligned} u_{t+h} - u_t = \quad & \mathcal{T}(t+h)\varphi - \mathcal{T}(t)\varphi \\ & + \lim_{d\to 0^+} \frac{1}{d}\left(\mathcal{T}(t)\int_0^h \mathcal{T}(h-s)Z(d)F(s,u_s)ds \right. \\ & \left. + \int_0^t \mathcal{T}(t-s)Z(d)\left(F(s+h,u_{s+h}) - F(s,u_s)\right)ds\right). \end{aligned} \tag{11}$$

From **(iv)** in Proposition 2, we know that

$$\lim_{d\to 0^+} \frac{1}{d}\int_0^h \mathcal{T}(h-s)Z(d)F(s,u_s)ds = \frac{d}{dh}\int_0^h Z(h-s)F(s,u_s)ds.$$

Then (10) becomes

$$\begin{aligned} u_{t+h} - u_t = \quad & \mathcal{T}(t+h)\varphi - \mathcal{T}(t)\varphi \\ & + \lim_{d\to 0^+} \frac{1}{d}\mathcal{T}(t)\int_0^h \mathcal{T}(h-s)Z(d)F(s,u_s)ds \\ & + \lim_{d\to 0^+} \frac{1}{d}\int_0^t \mathcal{T}(t-s)Z(d)\left(F(s+h,u_{s+h}) - F(s,u_s)\right)ds. \end{aligned} \tag{12}$$

To estimate the above two limits

$$\begin{aligned} \left\|\mathcal{T}(t)\int_0^h \mathcal{T}(h-s)Z(d)F(s,u_s)ds\right\|_{\mathcal{B}} & \leq \overline{M}e^{\omega(t+h)}\int_0^h \|Z(d)\|\,|F(s,u_s)|\,ds, \\ & \leq \overline{M}e^{\omega(t+h)}\|Z(d)\|\,h\left(rc_0(r) + \sup_{s\in[0,b_\varphi)}|F(s,0)|\right). \end{aligned}$$

Notice that $\sup_{s\in[0,b_\varphi)}|F(s,0)| < \infty$ since $b_\varphi < \infty$. Hence

$$\left\|\lim_{d\to 0^+}\frac{1}{d}\mathcal{T}(t)\int_0^h \mathcal{T}(h-s)Z(d)F(s,u_s)ds\right\|_{\mathcal{B}} \leq \overline{M}e^{\omega b_\varphi}khc_2,$$

where

$$c_2 := rc_0(r) + \sup_{s\in[0,b_\varphi)}|F(s,0)|.$$

Let us make the following decomposition

$$\begin{aligned}
&\int_0^t \mathcal{T}(t-s)Z(d)\left(F(s+h,u_{s+h})-F(s,u_s)\right)ds\\
&\qquad\qquad = \int_0^t \mathcal{T}(t-s)Z(d)\left(F(s+h,u_{s+h})-F(s+h,u_s)\right)ds \qquad (13)\\
&\qquad\qquad + \int_0^t \mathcal{T}(t-s)Z(d)\left(F(s+h,u_s)-F(s,u_s)\right)ds.
\end{aligned}$$

Then, the first integral is estimated as

$$\begin{aligned}
&\left\|\int_0^t \mathcal{T}(t-s)Z(d)\left(F(s+h,u_{s+h})-F(s+h,u_s)\right)ds\right\|_{\mathcal{B}}\\
&\qquad \leq \int_0^t \overline{M}e^{\omega(t-s)}\left\|Z(d)\right\|c_0(r)\left\|u_{s+h}-u_s\right\|_{\mathcal{B}}ds \qquad (14)\\
&\qquad \leq \overline{M}e^{\omega b_\varphi}\left\|Z(d)\right\|c_0(r)\int_0^t\left\|u_{s+h}-u_s\right\|_{\mathcal{B}}ds,
\end{aligned}$$

and the second integral is estimated as

$$\begin{aligned}
&\left\|\int_0^t \mathcal{T}(t-s)Z(d)\left(F(s+h,u_s)-F(s,u_s)\right)ds\right\|\\
&\qquad \leq \int_0^t \overline{M}e^{\omega(t-s)}\left\|Z(d)\right\|\left|F(s+h,u_s)-F(s,u_s)\right|ds\\
&\qquad \leq \overline{M}e^{\omega b_\varphi}\left\|Z(d)\right\|\int_0^t\left|F(s+h,u_s)-F(s,u_s)\right|ds.
\end{aligned}$$

Set

$$f(t,h) := \int_0^t \left|F(s+h,u_s)-F(s,u_s)\right|ds.$$

Then

$$\begin{aligned}
&\left\|\lim_{d\to 0^+}\frac{1}{d}\int_0^t \mathcal{T}(t-s)Z(d)\left(F(s+h,u_{s+h})-F(s,u_s)\right)ds\right\|_{\mathcal{B}}\\
&\qquad \leq \overline{M}e^{\omega b_\varphi}k\left(c_0(r)\int_0^t\left\|u_{s+h}-u_s\right\|ds+f(t,h)\right).
\end{aligned}$$

Thus we infer

$$\begin{aligned}
\left\|u_{t+h}-u_t\right\|_{\mathcal{B}} \leq\ & \overline{M}e^{\omega b_\varphi}\left\|\mathcal{T}(h)\varphi-\varphi\right\|_{\mathcal{B}}+\overline{M}e^{\omega b_\varphi}khc_2\\
&+\overline{M}e^{\omega b_\varphi}kc_0(r)\int_0^t\left\|u_{s+h}-u_s\right\|_{\mathcal{B}}ds+\overline{M}e^{\omega b_\varphi}kf(t,h).
\end{aligned}$$

By Gronwall's lemma, it follows that

$$\begin{aligned}
&\|u_{t+h} - u_t\|_{\mathcal{B}} \\
&\qquad \leq \overline{M} e^{\omega b_\varphi} \left(\|\mathcal{T}(h)\varphi - \varphi\|_{\mathcal{B}} + khc_2 + kf(t,h) \right) \exp\left(\overline{M} e^{\omega b_\varphi} kc_0(r) t \right) \\
&\qquad \leq \overline{M} e^{\omega b_\varphi} \left(\|\mathcal{T}(h)\varphi - \varphi\|_{\mathcal{B}} + khc_2 + kf(b_\varphi,h) \right) \exp\left(\overline{M} e^{\omega b_\varphi} kc_0(r) b_\varphi \right).
\end{aligned}$$

The bounded convergence theorem by Lebesgue implies that

$$\lim_{h \to 0} f(b_\varphi, h) = 0.$$

Therefore,

$$\lim_{h \to 0} \|u_{t+h}(.,\varphi) - u_t(.,\varphi)\|_{\mathcal{B}} = 0$$

uniformly for $t \in [0, b_\varphi)$. Consequently, Axiom $(\mathbf{A} - \mathbf{ii})$ implies that

$$\lim_{h \to 0} |u(t+h,\varphi) - u(t,\varphi)| = 0.$$

Using the same reasoning, one can show a similar result for $h < 0$. We deduce that u is uniformly continuous on $[0, b_\varphi)$, and $\lim\limits_{t \to b_\varphi} |u(t,\varphi)|$ exists. This implies that the solution $u(.,\varphi)$ can be extended to the right in b_φ, which contradicts the maximality of $(-\infty, b_\varphi)$. This completes the proof of Theorem 4.

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