

FIXED POINTS, COINCIDENCE POINTS AND GENERALIZED G -KKM MAPPINGS

MIRCEA BALAJ, DANIEL ERZSE

Department of Mathematics, University of Oradea

3700, Oradea, Romania

E-mail: mbalaj@uoradea.ro, derzse@uoradea.ro

Abstract. In this paper we extend to G -convex spaces a characterization theorem of generalized KKM mappings obtained by Pathak and Khan. As applications we establish coincidence, fixed point and matching theorems.

Key Words and Phrases: Generalized convex space, generalized G -KKM mapping, coincidence point, fixed point.

2000 Mathematics Subject Classification: 54H25, 49J35.

1. INTRODUCTION

Motivated by the well-known works of Horvath [1,2], there have appeared many generalizations of the concept of convex subset of a topological vector space. The most general one seems to be that of generalized convex space or G -convex space introduced by Park and Kim [3] which extends many of topological spaces having generalized convexity structures. In [3] and in other earlier papers [4,5,6,7] a *generalized convex space* or a *G -convex space* $(X, D; \Gamma)$ consists of a topological space X , a nonempty subset D of X and a map $\Gamma : \langle D \rangle \rightarrow X$ with nonempty values such that:

- (1) for each $A, B \in \langle D \rangle$, $A \subset B$ implies $\Gamma(A) \subset \Gamma(B)$; and
- (2) for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\Phi_A : \Delta_n \rightarrow \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\Phi_A(\Delta_J) \subset \Gamma(J)$.

This paper was presented at International Conference on Nonlinear Operators, Differential Equations and Applications held in Cluj-Napoca (Romania) from August 24 to August 27, 2004.

Here $\langle D \rangle$ denotes the set of all nonempty finite subsets of D , Δ_n any n -dimensional simplex with vertices $\{e_j\}_{j=0}^n$ and Δ_J the face of Δ_n corresponding to $J \in \langle D \rangle$; that is if $A = \{z_0, z_1, \dots, z_n\}$ and $J = \{z_{i_0}, \dots, z_{i_k}\} \subset A$ then $\Delta_J = \text{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\}$.

Recently, [8,9,10] in the definition of generalized convex space (G -convex space) the set D is not necessarily subset of X and the isotonic condition (1) is dropped. More precisely, a generalized convex space is a triplet $(X, D; \Gamma)$, where X is a topological space, D is a nonempty set and $\Gamma : \langle D \rangle \rightarrow X$ is a set-valued mapping satisfying condition (2), above. This will be also the meaning of generalized convex space in our paper.

When $D \subset X$ the G -convex space will be denoted by $(X \supset D; \Gamma)$. In this case a set $K \subset Y$ is called G -convex if for any $A \in \langle K \cap D \rangle$, $\Gamma(A) \subset K$. When $D = X$, the G -convex space will be denoted by $(X; \Gamma)$.

The main example of G -convex space corresponds to the case when $X = D$ is a convex subset of a topological vector space and for each $A \in \langle X \rangle$, $\Gamma(A)$ is the convex hull of A . For other major examples of G -convex see [11,12].

Let $(X, D; \Gamma)$ be a G -convex space, Y a nonempty set and $T : X \rightarrow Y$, $S : D \rightarrow Y$ two mappings. We say that S is a *generalized G -KKM mapping* w.r.t. T if for each $A \in \langle D \rangle$, $T(\Gamma(A)) \subset S(A)$. The previous concept has been introduced by the first author in [13]. In the classical (convex) case it reduces to that of generalized KKM mapping (w.r.t. T) introduced by Park [14] and systematically studied by Chang and Yen [15], and when $D = X$ it coincides with the corresponding notion in [16, Definition 1].

For a mapping $T : X \rightarrow Y$ (X, Y arbitrary sets) we define $T^-, T^* : Y \rightarrow X$ by

$$T^-(y) = \{x \in X : y \in Tx\},$$

$$T^*(y) = X \setminus T^-(y) = \{x \in X : y \notin T(x)\}.$$

If $(X, D; \Gamma)$ is a G -convex space, Y a nonempty set and $R : Y \rightarrow D$ is a mapping, then the mapping $\Gamma \circ R : Y \rightarrow X$ is defined by

$$(\Gamma \circ R)(y) = \cup\{\Gamma(A) : A \in \langle R(y) \rangle\}.$$

2. MAIN RESULTS

The following theorem extends to G -convex spaces Proposition 3.1 in [17].

Theorem 1. *Let $(X, D; \Gamma)$ be a G -convex space, Y be a nonempty set and $T : X \multimap Y$, $S : D \multimap Y$ be two mappings. Then the following conditions are equivalent:*

- (i) S is a generalized G -KKM mapping w.r.t. T .
- (ii) $(\Gamma-S^*)(y) \subset T^*(y)$ for each $y \in Y$.
- (iii) $\Gamma-(S^* \circ T)$ has no fixed point.

Proof. (i) \Leftrightarrow (ii). Suppose that S is a generalized G -KKM mapping w.r.t. T . We now suppose that there exists $y \in Y$ such that $(\Gamma-S^*)(y) \not\subset T^*(y)$. Then there exists $A \in \langle S^*(y) \rangle$ and $x \in \Gamma(A)$ such that $x \notin T^*(y)$. This implies that $y \notin S(A)$ and $y \in T(x)$. Hence $T(\Gamma(A)) \not\subset S(A)$, which contradicts the assumption that S is a generalized G -KKM mapping w.r.t. T .

Conversely, suppose that S is not a generalized G -KKM mapping w.r.t. T . In doing so, we adopt the contrapositive approach. So, let there be a set $A \in \langle D \rangle$ such that $T(\Gamma(A)) \not\subset S(A)$. Then there exists a $x \in \Gamma(A)$ and $y \in T(x)$ such that $y \notin S(A)$, that is $A \subset S^*(y)$; hence $x \in (\Gamma-S^*)(y)$. From $y \in T(x)$ it follows that $x \notin T^*(y)$, which implies $(\Gamma-S^*)(y) \not\subset T^*(y)$. The obtained contradiction completes the proof.

(iii) \Leftrightarrow (i). Suppose that $x \in (\Gamma-(S^* \circ T))(x)$. Then there exists $y \in T(x)$ and $A \in \langle S^*(y) \rangle$ such that $x \in \Gamma(A)$. Since $A \subset S^*(y)$, $y \notin S(A)$. On the other hand we have $y \in T(x) \subset T(\Gamma(A))$. Thus

$$y \in T(\Gamma(A)) \setminus S(A)$$

hence S is not a generalized G -KKM mapping w.r.t. T .

Conversely, suppose that S is not a generalized G -KKM mapping w.r.t. T . Then there exist $A \in \langle D \rangle$, $x \in \Gamma(A)$ and $y \in T(x) \setminus S(A)$. Since $y \notin S(A)$, we have $A \subset S^*(y)$ and consequently

$$x \in \Gamma(A) \subset (\Gamma-S^*)(y) \subset (\Gamma-S^*)(T(x)),$$

hence x is a fixed pint for $\Gamma-(S^* \circ T)$. □

Let $(X, D; \Gamma)$ be a G -convex space and Y be a topological space. a mapping $T : X \multimap Y$ is said to have the G -KKM property [15] if for any closed-valued mapping $S : D \multimap Y$ generalized G -KKM w.r.t. T , the family $\{S(x)\}_{x \in D}$ has the finite intersection property.

From Theorem 1 we obtain a new proof for the following coincidence theorem due to Park [18]:

Theorem 2. *Let $(X, D; \Gamma)$ be a G -convex space, Y a topological space, $G : D \multimap Y$, $F : X \multimap Y$ and $T : X \multimap Y$. Suppose that*

- (i) T has the KKM property;
- (ii) G has open values;
- (iii) for each $y \in T(X)$, $(\Gamma - G^-)(y) \subset F^-(y)$;
- (iv) $\overline{T(X)} \subset G(B)$ for some $B \in \langle D \rangle$.

Then F and T have a coincidence point $x_0 \in X$, that is $F(x_0) \cap T(x_0) \neq \emptyset$.

Proof. Define a mapping $S : D \multimap Y$ by

$$S(z) = \overline{T(X)} \setminus G(z) \text{ for each } z \in D.$$

Then S has a closed values and

$$\bigcap_{z \in B} S(z) = \bigcap_{z \in B} (\overline{T(X)} \setminus G(z)) = \overline{T(X)} \setminus \bigcup_{z \in B} G(z) \subset \overline{T(X)} \setminus \overline{T(X)} = \emptyset.$$

Therefore $\{S(z)\}_{z \in D}$ does not have the finite intersection property. Since T has the KKM property, it follows that S is not a generalized G -KKM mapping w.r.t. T . By equivalence (i) \Leftrightarrow (iii) in Theorem 1, there is a $x_0 \in X$ such that $x_0 \in (\Gamma - (S^* \circ T))(x_0)$. Consequently there exist $y_0 \in T(x_0)$ and $A \in \langle S^*(y_0) \rangle$ such that $x_0 \in \Gamma(A)$. Since $A \subset S^*(y_0)$ we infer successively

$$y_0 \notin S(A) \Rightarrow y_0 \in \bigcap_{z \in A} G(z) \Rightarrow A \subset G^-(y_0).$$

By (iii) we get $x_0 \in \Gamma(A) \subset F^-(y_0)$, whence $y_0 \in F(x_0)$. Thus we have $y_0 \in F(x_0) \cap T(x_0)$, which completes the proof. \square

Remarks.1. Note that condition (iv) in Theorem 2 is satisfied if the following holds:

- (iv') T is compact and $\overline{T(X)} \subset G(D)$.
2. Instead of condition (iii) in Theorem 2, the following suffices:
 - (ii') $\overline{T(X)} \cap G(z)$ is open in $\overline{T(X)}$ for each $z \in D$.
 3. Theorem 2 generalizes Theorem 4 in [15].

Corollary 3. *Let $(X, D; \Gamma)$ be a compact G -convex space and $G : D \multimap X$, $F : X \multimap X$ be two mappings such that:*

- (i) G has open values;

(ii) for each $y \in X$, $G^-(y)$ is nonempty and $(\Gamma-G^-)(y) \subset F^-(y)$.

Then F has a fixed point.

Proof. For each $x \in X$ let $T(x) = \{x\}$. According to Corollary in [5], T has the KKM property. Since for each $y \in X$ G^- is nonempty, we have $X = \cup_{z \in D} G(z)$, hence $\{G(z)\}_{z \in D}$ is an open cover of the compact X . Consequently there exists a set $B \in \langle D \rangle$ such that

$$\overline{T(X)} = X = \cup_{z \in B} G(z) = G(B)$$

hence condition (iv) in Theorem 2 is fulfilled. By Theorem 2, there exists $x_0 \in X$ such that $F(x_0) \cap T(x_0) \neq \emptyset$, that is, $x_0 \in F(x_0)$. \square

From Theorem 1 we also immediately obtain the following matching theorem:

Theorem 4. *Let $(X, D; \Gamma)$ be a G -convex space, Y be a nonempty set and $T : X \multimap Y$, $R : D \multimap Y$ be two mappings. If $\Gamma - (R^{-1} \circ T)$ has a fixed point then there exists $A \in \langle D \rangle$ such that $T(\Gamma(A)) \cap \cap_{z \in A} R(z) \neq \emptyset$.*

Proof. Let $S = R^c$ i.e., $S : D \multimap Y$ is the mapping defined by

$$S(x) = Y \setminus R(x)$$

Then $S^*(y) = R^-(y)$. By hypothesis $\Gamma - (S^* \circ T)$ has a fixed point, hence by equivalence (i) \Leftrightarrow (iii) in Theorem 1 there exists $A \in \langle D \rangle$ such that $T(\Gamma(A)) \not\subset S(A)$, that is, $T(\Gamma(A)) \cap \cap_{z \in A} R(z) \neq \emptyset$. \square

A G -convex space (Y, Γ) is called an *LG-space* (or a locally G -convex space) (see [10]) if (Y, \mathcal{U}) is a uniform space and if there exists a basis $\{V_i\}_{i \in I}$ for the uniformity \mathcal{U} such that for each $i \in I$, $\{y \in Y : K \cap V_i[y] \neq \emptyset\}$ is G -convex whenever $K \subset Y$ is G -convex, where $V_i(y) = \{y' \in Y : (y, y') \in V_i\}$.

In proving our last result we need two lemmas. The first one is a particular case of assertion (i) in Theorem 1 in [19].

Lemma 5. *Let $(X, D; \Gamma)$ be a compact G -convex space, X be a topological space and $S : Y \multimap X$ be a mapping such that:*

- (i) for each $y \in Y$, $S(y)$ is G -convex;
- (ii) $Y = \cup_{z \in D} \text{Int} S^-(z)$.

Then S has a continuous selection, i.e., there is a continuous function $p : Y \multimap X$ such that $p(y) \in S(y)$ for each $y \in Y$.

The second lemma is Theorem 2 in [10].

Lemma 6. *Let $(Y; \Gamma)$ be a compact LG-space and $T : Y \multimap Y$ be a upper semicontinuous mapping with closed G -convex values. Then T has a fixed point.*

Theorem 7. *Let $(X \supset D; \Gamma_1)$ be a G -convex space, $(Y; \Gamma_2)$ be a compact LG-space and $T : X \multimap Y$, $R : D \multimap Y$ be two mappings such that:*

- (i) T is upper semicontinuous with closed G -convex values;
- (ii) for each $y \in Y$, $G^{-1}(y)$ is G -convex;
- (iii) $Y = \cup_{z \in D} \text{Int}R(z)$.

Then there exists $A \in \langle D \rangle$ such that $T(\Gamma_1(A)) \cap \cap_{z \in D} R(z) \neq \emptyset$.

Proof. By (ii) and (iii), using Lemmas 5, R^- has a continuous selection p . Since $T \circ p : Y \multimap Y$ is upper semicontinuous mapping with closed G -convex values, by Lemma 6, there exists $y_0 \in Y$ such that $y_0 \in (T \circ p)(y_0)$. Clearly y_0 is a fixed point for $T \circ R^-$ and by Theorem 4 we obtain the desired conclusion. \square

REFERENCES

- [1] C. D. Horvath, *Some results on multivalued mappings and inequalities without convexity*, in "Nonlinear and Convex Analysis - Proc. in Honor of Ky Fan" (B.-L. Lin and S. Simons, eds.), Marcel Dekker, New York, 1987, 99-106.
- [2] C. D. Horvath, *Contractibility and generalized convexity*, J. Math. Anal. Appl., **156** (1991), 341-357.
- [3] S. Park and H. Kim, *Admissible classes of multifunctions generalized convex spaces*, Proc. Coll. Natur. Sci. SNU, **18** (1993), 1-21.
- [4] S. Park and H. Kim, *Coincidence theorems for admissible multifunctions on generalized convex spaces*, J. Math. Anal. Appl., **197** (1996), 173-187.
- [5] S. Park, *Foundations of the KKM theory on generalized convex spaces*, J. Math. Anal. Appl., **209** (1997), 551-571.
- [6] S. Park and H. Kim, *Generalizations of the KKM type theorems on generalized convex spaces*, Indian J. Pure Appl. Math., **29** (1998), 121-132.
- [7] E. Kalmoun and H. Ricchi, *Topological KKM theorems and generalized vector equilibria on G -convex spaces with applications*, Proc. Amer. Math. Soc., **129** (2001), 1335-1348.
- [8] S. Park, *Elements of the KKM theory for generalized convex spaces*, Korean J. Comput. and Appl. Math., **7** (2000), 1-28.

- [9] S. Park and H. Kim, *generalized KKM maps on generalized convex spaces*, *Nonlinear Analysis Forum*, **5** (2000), 15-34.
- [10] S. Park, *Fixed point theorems in locally G -convex spaces*, *Nonlinear Anal.*, **48** (2002), 869-879.
- [11] S. Park, *New subclasses of generalized convex spaces*, in: Y. J. Cho (Ed.), *Fixed point Theory and Applications*, Nova Sci. Publ., New York, 2000, 91-98.
- [12] S. Park, *Remarks on fixed point theorem for generalized convex spaces*, in: Y. J. Cho (Ed.), *Fixed Point Theory and Applications*, Nova Sci. Publ., New York, 2000, pp. 135-144.
- [13] M. Balaj, *Weakly G -KKM mapping, G -KKM property and minimax inequalities*, *J. Math. Anal. Appl.*, **294** (2004), 237-245.
- [14] S. Park, *Generalizations of Ky Fan's matching theorem and their applications*, *J. Math. Anal. Appl.*, **141** (1989), 164-176.
- [15] T. H. Cheng and C. L. Yen, *KKM property and fixed point theorems*, *J. Math. Anal. Appl.*, **203** (1996), 224-235.
- [16] L. J. Lin, C. J. Ko and S. Park, *Coincidence theorems for set-valued maps with G -KKM property and fixed point theorems*, *Discussiones Math. - Diff. Inclusions*, **18** (1998), 69-85.
- [17] H. K. Pathak and M. S. Khan, *On D -KKM theorem and its applications*, *Bull. Austral Math. Soc.*, **67** (2003), 67-77.
- [18] S. Park, *New topological versions of the Fan-Browder fixed point theorem*, *Nonlinear Anal.*, **47** (2001), 595-606.
- [19] S. Park, *Continuous selection theorems in generalized convex spaces*, *Numer. Funct. Anal. and Optimiz.*, **20** (1999), 567-583.

Received October 2, 2004; Revised December 13, 2004.