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EXISTENCE RESULTS FOR SEMILINEAR FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH INFINITE DELAY

E. P. GATSORI[1], L. GÓRNIEWICZ[2], S. K. NTOUYAS[1] AND G. Y. SFICAS[1]

[1]Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
e-mail: egatsori@yahoo.gr, sntouyas@cc.uoi.gr

[2] Faculty of Mathematic and Informatic Science, Nicholas Copernicus University, Chopina 12/18, 87-100 Torun, Poland
e-mail:gorn@mat.uni.torun.pl

Abstract. In this paper we shall establish sufficient conditions for the existence of integral solutions for some nondensely defined first order impulsive semilinear functional differential inclusions with infinite dalay in Banach spaces.

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1. Introduction

In this paper we shall be concerned with the existence of solutions for first order impulsive semilinear functional differential inclusions with infinite delay in a Banach space of the form:

$$y'(t) \in Ay(t) + F(t, y_t), \quad \text{a.e. } t \in J := [0, b], \quad t \neq t_k, \ k = 1, \ldots, m \tag{1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \ k = 1, \ldots, m \tag{2}$$

$$y(t) = \phi(t), \quad t \leq 0, \tag{3}$$

where $A : D(A) \subset X \to X$ is nondensely defined closed linear operator, $F : J \times \mathcal{B} \to \mathcal{P}(X)$ is a multivalued map, ($\mathcal{P}(X)$ is the family of all nonempty subsets of X), $0 \leq t_1 < \ldots < t_p \leq b$, $p \in \mathbb{N}$, $\phi \in \mathcal{B}$, $I_k : X \to X, \ k = 1, \ldots, m$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$,

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$y(t_k^+) = \lim\limits_{h\to 0^+} y(t_k+h)$ and $y(t_k^-) = \lim\limits_{h\to 0^+} y(t_k-h)$ represent the right and left limits of $y(t)$ at $t = t_k$ and X a real separable Banach space with norm $|\cdot|$.

We assume that the histories $y_t : (-\infty, 0] \to X$, $y_t(\theta) = y(t+\theta)$, belong to some abstract phase space $\mathcal{B}$, that is a phase space defined axiomatically. Thus $\mathcal{B}$ is a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$.

Consider the following space

$$\begin{aligned}\mathcal{B}_b \quad &= \Big\{ y : (-\infty, b] \to X, \quad y_k \in C(J_k, X) \text{ and there exist } y(t_k^-), y(t_k^+) \\ &\qquad \text{with } \ y(t_k) = y(t_k^-), \ y(t) = \phi(t), \ t \le 0 \Big\}\end{aligned}$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}], \ k = 0, \ldots, m$. Let $\|\cdot\|_b$ be the seminorm in $\mathcal{B}_b$ defined by

$$\|y\|_b = \|y_0\|_{\mathcal{B}} + \sup\{|y(s)| : 0 \le s \le b\}, \ y \in \mathcal{B}_b.$$

We will assume that $\mathcal{B}$ satisfies the following axioms suggested by Hale and Kato [15]:

(A) If $y : (-\infty, b] \to X, b > 0$ is such that $y|_{[0,b]} \in \mathcal{B}_b$ and $y_0 \in \mathcal{B}$, then for every t in $[0, b)$ the following conditions hold :
  (i) y_t is in $\mathcal{B}$;
  (ii) $\|y_t\|_{\mathcal{B}} \le K(t) \sup\{|y(s)| : 0 \le s \le t\} + M(t)\|y_0\|_{\mathcal{B}}$,
  where $K : [0, \infty) \to [0, \infty)$ is continuous, $M : [0, \infty) \to [0, \infty)$ is locally bounded and K, M are independent of $y(\cdot)$.

(A-1) For the function $y(\cdot)$ in (A), y_t is a $\mathcal{B}$-valued continuous function on $[0, b)$.

(B) The space $\mathcal{B}$ is complete.

The theory of impulsive differential equations has been emerging as an important area of investigation in recent years, because all the structure of its emergence has deep physical background and realistic mathematical model. The theory of impulsive differential equations appears as a natural description of several real processes subject to certain perturbations whose duration is negligible in comparison with the duration of the process. It has seen considerable development in the last decade; see the monographs of Bainov and Simeonov [5], Lakshmikantham, *et al.* [19], and Samoilenko and Perestyuk [22] where numerous properties of their solutions are studied, and detailed bibliographies are given. The case of impulsive inclusions was considered by Ahmed [1]. The case of infinite delay was described by Gori *et al.* in [13].

The differential inclusion (1), when A generates a C_0 semigroup of bounded linear operators, or equivalently,

(α) $\overline{D(A)} = E$, (D means domain),

(β) the Hille-Yosida condition that is, there exists $M \geq 0$ and $\tau \in \mathbb{R}$ such that $(\tau, \infty) \subset \rho(A)$, $\sup\{(\lambda I - \tau)^n |(\lambda I - A)^{-n}| : \lambda > \tau,\ n \in \mathbb{N}\} \leq M$,

where $\rho(A)$ is the resolvent operator set of A and I is the identity operator, has been studied extensively. See for instance [7], [8] and the recent monograph [17] where the general theory is expounded.

In this work we investigate the case when the operator A satisfies only the assumption (β), that is, *when A is nondensely defined*. Related studies and examples concerning nondensely defined operators can be found in references such as [6], [9], [11], [12].

This paper will be organized as follows. In Section 2 we state some facts from multivalued analysis, integrated semigroups and integral solutions that will be used later. In Section 3 we establish an existence theorem for the problem (1)–(3) using Ma's fixed point theorem [21].

2. Preliminaries

We will briefly recall some basic definitions and facts from multivalued analysis and integrated semigroups that are used throughout this paper.

We denote by $\mathcal{P}(X)$ the set of all subsets of X normed by $\|\cdot\|_{\mathcal{P}}$. $C(J, X)$ is the Banach space of continuous functions from J into X normed by

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

$B(X)$ denotes the Banach space of bounded linear operators from X into X with norm

$$\|N\|_{B(X)} := \sup\{|N(y)| : |y| = 1\}.$$

A measurable function $y : J \longrightarrow X$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of Bochner integral see Yosida [23]).

$L^1(J, X)$ denotes the linear space of equivalence classes of all measurable functions $y : J \to X$ which are normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|dt.$$

A multivalued map $G : X \longrightarrow \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for any bounded set B of X, that is, $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$.

G is called upper semicontinuous (u.s.c.) on X if for each $y_1 \in X$ the set $G(y_1)$ is a nonempty, closed subset of X, and if for each open set B of X containing $G(y_1)$, there exists an open neighborhood $\mathcal{A}$ of y_1 such that $G(\mathcal{A}) \subseteq B$.

G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*,\ y_n \longrightarrow y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

Let $P(X) = \{Y \in \mathcal{P}(X) : Y \neq \emptyset\}$, $P_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in P(X) : Y \text{ bounded}\}$, $P_c(X) = \{Y \in P(X) : Y \text{ convex}\}$, and $P_{cp}(X) = \{Y \in P(X) : Y \text{ compact}\}$.

A multivalued map $G : J \longrightarrow P_{cl}(X)$ is said to be measurable if for each $x \in X$ the distance between x and $G(t)$ is a measurable function on J.

For more details on multivalued maps we refer to the books of Deimling [10], Gorniewicz [14], Hu and Papageorgiou [16] and Kamenski *et al.* [17].

Definition 2.1. *([2]). Let X be a Banach space. An integrated semigroup is a family of operators $(S(t))_{t\geq 0}$ of bounded linear operators $S(t)$ on X with the following properties:*

- (i) $S(0) = 0$;
- (ii) $t \to S(t)$ *is strongly continuous;*
- (iii) $S(s)S(t) = \int_0^s (S(t+r) - S(r))dr$, *for all* $t, s \geq 0$.

Definition 2.2. *([18]). An operator A is called a generator of an integrated semigroup if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ ($\rho(A)$ is the resolvent set of A) and there exists a strongly continuous exponentially bounded family $(S(t))_{t\geq 0}$ of bounded operators such that $S(0) = 0$ and $R(\lambda, A) := (\lambda I - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} S(t)dt$ exists for all λ with $\lambda > \omega$.*

Proposition 2.3. *([2]). Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then for all $x \in X$ and $t \geq 0$,*

$$\int_0^t S(s)xds \in D(A) \quad \textit{and} \quad S(t)x = A\int_0^t S(s)xds + tx.$$

Theorem 2.4. *([18]). The following assertions are equivalent:*

- (i) *A is the generator of a non degenerate, locally Lipschitz continuous integrated semigroup;*
- (ii) *A satisfies the Hille-Yosida condition.*

If A is the generator of an integrated semigroup $(S(t))_{t\geq 0}$ which is locally Lipschitz, then from [2], $S(\cdot)x$ is continuously differentiable if and only if $x \in \overline{D(A)}$ and $(S'(t))_{t\geq 0}$ is a C_0 semigroup on $\overline{D(A)}$.

Definition 2.5. *We say that $y : (-\infty, b] \to X$ is an integral solution of (1)-(3) if*

(i) $y \in \mathcal{B}_b$,
(ii) $\int_0^t y(s)ds \in D(A)$ *for* $t \in J$,
(iii) $\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1, \dots, m$,
(iv) *there exists a function $v \in L^1(J, X)$ such that $v(t) \in F(t, y_t)$ a.e. on $t \in J$ and*

$$y(t) = \begin{cases} \phi(t), & t \leq 0 \\ \phi(0) + A\displaystyle\int_0^t y(s)ds + \int_0^t v(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)), & t \in J. \end{cases}$$

From (ii) it follows that $y(t) \in \overline{D(A)},\ \forall\, t \geq 0$ and from (iv) that $\phi(0) \in \overline{D(A)}$.

Here and hereafter we assume that

(H1) A satisfies the Hille-Yosida condition.

Let $B_\lambda = \lambda R(\lambda, A)$, then for all $x \in \overline{D(A)},\ B_\lambda x \to x$ as $\lambda \to \infty$.

Remark 2.6. *If y is an integral solution of (1)-(3), then it is given by*

$$y(t) = \begin{cases} \phi(t), & t \leq 0, \\ S'(t)\phi(0) + \dfrac{d}{dt}\displaystyle\int_0^t S'(t-s)v(s)ds & \\ + \displaystyle\sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)), & t \in J. \end{cases} \tag{4}$$

For the multifunction F and for each $y \in C(J, X)$ we define $S_{F,y}$ by

$$S_{F,y} := \Big\{g \in L^1(J, X) :\ g(t) \in F(t, y_t) \text{ for a.e. } t \in J\Big\}.$$

This set is not empty if $F(t, y)$ is measurable and satisfies (H5) below (see [10] and [17]).

For the proof of our main theorem we will need the following:

Lemma 2.7. *(*[20]*). Let X be a Banach space. Let $F : J \times \mathcal{B} \longrightarrow P_{b,cl,c}(X)$; $(t, y) \longmapsto F(t, y)$ be measurable with respect to t, for each $y \in X$, u.s.c. with respect to y, for each $t \in J$, and let Γ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$. Then the operator*

$$\Gamma \circ S_F : C(J, X) \longrightarrow P_{cp,c}(C(J, X)),\ y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

3. Existence result

In this section we are concerned with the existence of solutions for problem (1)–(3). Let $\mathcal{B}_{b_*}$ be the set of all functions that belong in $\mathcal{B}_b$ and have values in $\overline{D(A)}$.

Let us introduce the following hypotheses:

(H2) The operator $S'(t)$ is compact in $\overline{D(A)}$ whenever $t > 0$.

(H3) $F : J \times \mathcal{B} \longrightarrow P_{cp,c}(X)$, $(t, v) \longmapsto F(t, v)$ is measurable with respect to t for each $y \in \mathcal{B}$.

(H4) $I_k : X \to \overline{D(A)}$ and there exist constants $d_k,\ k = 1, \ldots, p$ such that

$$|I_k(x)| \leq d_k, \qquad x \in X.$$

(H5) There exists a continuous nondecreasing function $\psi : [0, \infty) \longrightarrow (0, \infty)$ and $p \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, u)\| \leq p(t)\psi(\|u\|_{\mathcal{B}}) \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B},$$

with

$$\int_0^b m(s)ds < \int_c^\infty \frac{d\tau}{\psi(\tau)},$$

where

$$c := M_0\|\phi\|_{\mathcal{B}} + K_0M^*\|\phi\|_{\mathcal{B}} + K_0M^*\sum_{k=1}^{p} e^{-\omega t_k}d_k,\ m(t) := K_0M^*e^{-\omega t}p(t)$$

and

$$K_0 := \max_{t\in J} K(t), \quad M_0 := \max_{t\in J} M(t), \quad M^* := M\max\{e^{\omega b}, 1\}.$$

Theorem 3.1. *Assume that hypotheses (H1)-(H5) hold and that* $\phi(0) \in \overline{D(A)}$. *Then the IVP (1)–(3) has at least one solution.*

Proof. Transform the problem (1)-(3) into a fixed point problem. Consider the multivalued operator $N : \mathcal{B}_{b_*} \longrightarrow \mathcal{P}(\mathcal{B}_{b_*})$ defined by:

$$N(y) = \left\{ h \in \mathcal{B}_{b_*} : h(t) = \left\{ \begin{array}{ll} \phi(t), & t \leq 0, \\ S'(t)\phi(0) + \lim\limits_{\lambda\to\infty} \displaystyle\int_0^t S'(t-s)B_\lambda v(s)ds & \\ \quad + \displaystyle\sum_{0<t_k<t} S'(t-t_k)I_k(y(t_k^-)), & t \in J, \end{array} \right. \right\}$$

where $v \in S_{F,y}$.

We will show that $N(\cdot)$ has a fixed point. For $\phi \in \mathcal{B}$ we define

$$\tilde{\phi}(t) = \begin{cases} \phi(t), & -\infty < t \le 0, \\ S'(t)\phi(0) & 0 < t \le b, \end{cases}$$

Then $\tilde{\phi} \in \mathcal{B}_b$. Set

$$y(t) = x(t) + \tilde{\phi}(t).$$

It is clear to see that y satisfies (4) if and only if x satisfies $x_0 = 0$ and

$$x(t) = \lim_{\lambda\to\infty} \int_0^t S'(t-s)B_\lambda v(s)ds + \sum_{0<t_k<t} S'(t-t_k)I_k(x(t_k^-) + \tilde{\phi}(t_k^-)).$$

Let $\mathcal{B}_b^0 = \{x \in \mathcal{B}_{b_*} :\ x_0 = 0 \in \mathcal{B}\}$. For any $x \in \mathcal{B}_b^0$ we have

$$\|x\|_b = \|x_0\|_{\mathcal{B}} + \sup\{|x(s)| : 0 \le s \le b\} = \sup\{|x(s)| : 0 \le s \le b\}.$$

Thus $(\mathcal{B}_b^0, \|\cdot\|_b)$ is a Banach space.

Define the multivalued operator $\bar{N} : \mathcal{B}_b^0 \longrightarrow \mathcal{P}(\mathcal{B}_b^0)$ by:

$$\bar{N}(x) = \left\{ \bar{h} \in \mathcal{B}_b^0 : \bar{h}(t) = \begin{cases} 0, & t \le 0, \\ \lim\limits_{\lambda\to\infty} \displaystyle\int_0^t S'(t-s)B_\lambda v(s)ds & \\ + \displaystyle\sum_{0<t_k<t} S'(t-t_k)I_k((x(t_k^-) + \tilde{\phi}(t_k^-)), & t \in J, \end{cases} \right\}$$

where $v \in S_{F,y}$.

Step 1: *$\bar{N}(x)$ is convex for every $x \in \mathcal{B}_b^0$.*

The proof of this step is obvious therefore we omit the details.

Step 2: *$\bar{N}$ sends bounded sets into bounded sets.*

We will show that for any $r > 0$ there exists a positive constant ℓ such that for each $h \in \bar{N}(x), x \in B_r = \{x \in \mathcal{B}_b^0 : \|x\|_b \le r\}$ one has $\|\bar{N}(x)\| := \sup\{\|h\|_b : h \in \bar{N}(x)\} \le \ell$. We remark that by (H1) $\|S'(t)\|_{B(X)} \le Me^{\omega t}$, for every $t \in J$ (see [18]). For every $x \in B_r$, we have

$$\begin{aligned} \|x_t + \tilde{\phi}_t\|_{\mathcal{B}} &\le \|x_t\|_{\mathcal{B}} + \|\tilde{\phi}_t\|_{\mathcal{B}} \\ &\le K(t)\sup\{|x(s)| : 0 \le s \le t\} + M(t)\|x_0\|_{\mathcal{B}} \\ &\quad + K(t)\sup\{|\tilde{\phi}(s)| : 0 \le s \le t\} + M(t)\|\tilde{\phi}_0\|_{\mathcal{B}} \\ &\le K_0 r + K_0 M|\phi(0)| + M_0\|\phi\|_{\mathcal{B}} := r^*. \end{aligned}$$

By (H1), (H2), (H4) and (H5), for each $t \in J$, we have that

$$\begin{aligned} |h(t)| &\leq M^* \int_0^t e^{-\omega s} p(s)\psi(\|x_s + \tilde{\phi}_s\|_{\mathcal{B}})ds + M\sum_{k=1}^{p} d_k e^{-\omega t_k} \\ &\leq M^* \int_0^b e^{-\omega s} p(s)\psi(r^*)ds + M\sum_{k=1}^{p} d_k e^{-\omega t_k}. \end{aligned}$$

Then for each $h \in \bar{N}(\mathcal{B}_q)$ we have

$$\|\bar{N}(y)\| \leq M^* \int_0^b e^{-\omega s} p(s)\psi(r^*)ds + M^* \sum_{k=1}^{p} d_k e^{-\omega t_k} := \ell.$$

Step 3: *$\bar{N}$ sends bounded sets into equicontinuous sets*

Let $\tau_1, \tau_2 \in J,\ 0 < \tau_1 < \tau_2$. Then we have

$$\begin{aligned} |h(\tau_2) - h(\tau_1)| &\leq \left| [S'(\tau_2 - \tau_1) - I] \lim_{\lambda\to\infty} \int_0^{\tau_1} S'(\tau_1 - s)B_\lambda v(s)ds \right| \\ &+ \sum_{0<t_k<\tau_1} |[S'(\tau_1 - s) - S'(\tau_2 - s)](I_k(x(t_k^-) + \tilde{\phi}(t_k^-)))| \\ &+ M^* \int_{\tau_1}^{\tau_2} e^{-\omega s} p(s)\psi(r^*)ds + M^* \sum_{\tau_1 \leq t_k < \tau_2} d_k e^{-\omega t_k}. \end{aligned}$$

The right-hand side tends to zero as $\tau_2 \to \tau_1$ and ϵ sufficiently small, since $S'(t)$ is a strongly continuous operator and the compactness of $S'(t)$ for $t > 0$ implies the continuity in the uniform operator topology. The equicontinuity for the cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 \leq 0 \leq \tau_2$ is obvious.

This proves the equicontinuity for the case where $t \neq t_i \quad i = 1, \ldots, m$. It remains to examine the equicontinuity at $t = t_i$.

First we prove equicontinuity at $t = t_i^+$. Fix $\delta_1 > 0$ such that $\{t_k : k \neq i\} \cap [t_i - \delta_1, t_i + \delta_1] = \emptyset$.

For $0 < h < \delta_1$ we have that

$$\begin{aligned} |h(t_i + h) - h(t_i)| &\leq \left| [S'(h) - I] \lim_{\lambda\to\infty} \int_0^{t_i} S'(t_i - s)B_\lambda v(s)ds \right| \\ &+ \sum_{0<t_k<t_i} |[S'(t_i - t_k) - S'(t_i + h - t_k)](I_k(x(t_k^-) + \ tilde\phi(t_k^-)))| \\ &+ M^* \int_{t_i}^{t_i+h} e^{-\omega s} p(s)\psi(r^*)ds + M^* \sum_{t_i \leq t_k < t_i + h} d_k e^{-\omega t_k}. \end{aligned}$$

The right-hand side tends to zero as $h \to 0$.

Next we prove equicontinuity at $t = t_i^-$. Fix $\delta_1 > 0$ such that $\{t_k : k \neq i\} \cap [t_i - \delta_1, t_i + \delta_1] = \emptyset$.

For $0 < h < \delta_1$ we have that

$$\begin{aligned} |h(t_i) - h(t_i - h)| &\leq \left| \lim_{\lambda\to\infty} \int_0^{t_i - h} [S'(t_i - s) - S'(t_i - h - s)] B_\lambda v(s) ds \right| \\ &+ \sum_{k=1}^{i-1} \left| [S'(t_i - t_k) - S'(t_i - h - t_k)] (I_k(x(t_k^-) + \tilde{\phi}(t_k^-))) \right| \\ &+ M^* \int_{t_i - h}^{t_i} e^{-\omega s} p(s) \psi(r^*) ds. \end{aligned}$$

The right-hand side tends to zero as $h \to 0$.

As a consequence of Steps 1-3 together with the Arzelá-Ascoli theorem, it suffices to show that $\bar{N}$ maps B_r into precompact sets.

Let $0 < t \leq b$ be fixed and let ε be a real number satisfying $0 < \varepsilon < t$. Then for each $x \in B_r$ we have that

$$h_\varepsilon(t) = \lim_{\lambda\to\infty} \int_0^{t-\varepsilon} S'(t-s) B_\lambda v_n(s) ds + \sum_{0<t_k<t} S'(t - t_k) I_k(x_n(t_k^-) + \tilde{\phi}(t_k^-)).$$

Since $S'(t)$ is a compact operator the set $H_\varepsilon(t) = \{h_\varepsilon(t) : h_\varepsilon \in \bar{N}(x)\}$ is precompact in $\overline{D(A)}$ for every ε, $0 < \varepsilon < t$. Moreover, for every $h \in \bar{N}(x)$ we have that

$$|h(t) - h_\varepsilon(t)| \leq M^* \int_{t-\varepsilon}^{t} e^{-\omega s} p(s) \psi(r^*) ds.$$

Therefore, there are precompact sets arbitrarily close to the set $H(t) = \{h(t) : h \in \bar{N}(x)\}$. Hence $H(t)$ is precompact. Hence the operator is completely continuous.

Step 4: *$\bar{N}$ has closed graph.*

Let $x_n \longrightarrow x_*$, $h_n \in \bar{N}(x_n)$, $x_n \in \mathcal{B}_b^0$ and $h_n \longrightarrow h_*$. We shall prove that $h_* \in \bar{N}(x_*)$.

$h_n \in N(x_n)$ means that there exists $v_n \in S_{F,x_n}$ such that for each $t \in J$

$$h_n(t) = \lim_{\lambda\to\infty} \int_0^{t} S'(t-s) B_\lambda v_n(s) ds + \sum_{0<t_k<t} S'(t - t_k) I_k(x_n(t_k^-) + \tilde{\phi}(t_k^-)).$$

We must prove that there exists $v_* \in S_{F,y_*}$ such that for each $t \in J$

$$h_*(t) = \lim_{\lambda\to\infty} \int_0^{t} S'(t-s) B_\lambda v_*(s) ds + \sum_{0<t_k<t} S'(t - t_k) I_k(x_*(t_k^-) + \tilde{\phi}(t_k^-)).$$

Clearly since $I_k,\ k=1,\dots,m$ are continuous we have that

$$\Big\|\Big(h_n - \sum_{0<t_k<t} S'(t-t_k)I_k(x_n(t_k^-)+\tilde{\phi}(t_k^-)\Big)$$

$$-\Big(h_* - \sum_{0<t_k<t} S'(t-t_k)I_k(x_*(t_k^-)+\tilde{\phi}(t_k^-))\Big)\Big\|_b \longrightarrow 0,$$

as $n\to\infty$.

Consider the linear continuous operator

$$\Gamma : L^1(J,X) \longrightarrow C(J,X),$$

$$v \longmapsto \Gamma(v)(t) = \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda v(s)ds.$$

From Lemma 2.7 and Theorem 5.1.2 of [17], it follows that $\Gamma\circ S_F$ is a closed graph operator. Moreover, we have that

$$h_n(t) - \sum_{0<t_k<t} S'(t-t_k)I_k(x_n(t_k^-)+\tilde{\phi}(t_k^-)) \in \Gamma(S_{F,x_n}).$$

Since $y_n \longrightarrow y_*$, it follows from Lemma 2.7 and Theorem 5.1.2 of [17] that

$$h_*(t) - \sum_{0<t_k<t} S'(t-t_k)I_k(x_*(t_k^-)+\tilde{\phi}(t_k^-)) = \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda v_*(s)ds$$

for some $v_*\in S_{F,x_*}$.

Step 5: *The set*

$$\mathcal{M} := \{x\in\mathcal{B}_b^0 : \lambda x\in\bar{N}(x), \quad \textit{for some} \quad \lambda>1\}$$

is bounded.

Let $x\in\mathcal{M}$. Then $\lambda x\in N(x)$ for some $\lambda>1$. Thus for each $t\in J$

$$x(t) = \lambda^{-1}\lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda w(s)ds + \lambda^{-1}\sum_{0<t_k<t} S'(t-t_k)I_k(x(t_k^-)+\tilde{\phi}(t_k^-)).$$

By (H1), (H4) and (H5) we have that

$$|x(t)| \le M^*\int_0^t e^{-\omega s}p(s)\psi(\|x_s+\tilde{\phi}_s\|_{\mathcal{B}})ds + M^*\sum_{0<t_k<t} e^{-\omega t_k}d_k. \tag{5}$$

But

$$\begin{aligned}\|x_t+\tilde{\phi}_t\|_{\mathcal{B}} &\leq \|x_t\|_{\mathcal{B}}+\|\tilde{\phi}_t\|_{\mathcal{B}}\\ &\leq K(t)\sup\{|x(s)|:0\leq s\leq t\}+M(t)\|x_0\|_{\mathcal{B}}\\ &\quad +K(t)\sup\{|\tilde{\phi}(s)|:0\leq s\leq t\}+M(t)\|\tilde{\phi}_0\|_{\mathcal{B}}\\ &\leq K_0\sup\{|x(s)|:0\leq s\leq t\}+M^*K_0|\phi(0)|+M_0\|\phi\|_{\mathcal{B}}.\end{aligned}$$

If we set $w(t)$ the right hand side of the above inequality we have that

$$\|x_t+\tilde{\phi}_t\|_{\mathcal{B}}\leq w(t)$$

and therefore (5) becomes

$$|x(t)|\leq M^*\int_0^t e^{-\omega s}p(s)\psi(w(s))ds+M^*\sum_{0<t_k<t}e^{-\omega t_k}d_k. \tag{6}$$

Using (6) in the definition of w, we have that

$$w(t)\leq K_0\left[M^*\int_0^t e^{-\omega s}p(s)\psi(w(s))ds+M^*\sum_{0<t_k<t}e^{-\omega t_k}d_k\right]+K_0M^*|\phi(0)|+M_0\|\phi\|_{\mathcal{B}}.$$

Denoting by $\beta(t)$ the right-hand side of the above inequality we have

$$w(t)\leq\beta(t),\ \ t\in J,$$
$$\beta(0)=K_0M^*\sum_{0<t_k<t}e^{-\omega t_k}d_k+K_0M^*|\phi(0)|+M_0\|\phi\|_{\mathcal{B}}$$

and

$$\beta'(t)=K_0M^*e^{-\omega t}p(t)\psi(w(t)),\ \ t\in J.$$

By using the increasing character of ψ we get

$$\begin{aligned}\beta'(t)\ &\leq K_0M^*e^{-\omega t}p(t)\psi(\beta(t))\\ &\leq m(t)\psi(\beta(t)),\qquad t\in J.\end{aligned}$$

Then for each $t\in J$ we have

$$\int_{\beta(0)}^{\beta(t)}\frac{du}{\psi(u)}\leq\int_0^b m(s)ds<\int_{\beta(0)}^{\infty}\frac{du}{\psi(u)}.$$

This inequality implies that there exists a constant K such that $\beta(t)\leq K,\ t\in J$, and hence $\|x_t+\tilde{\phi}_t\|_{\mathcal{B}}\leq w(t)\leq K,\ t\in J$, where K depends on ϕ and on the functions p and ψ. From equation (5) we have that

$$|x(t)|\leq M^*\int_0^b e^{-\omega s}p(s)\psi(K)ds+M^*\sum_{0<t_k<t}e^{-\omega t_k}d_k:=L. \tag{7}$$

Thus $\|x\|_b \leq L$.

As a consequence of Theorem 16.1 of Ma [21] we deduce that $\bar{N}$ has a fixed point, $x^* \in \mathcal{B}_b^0$. Let $y(t) = x^*(t) + \tilde{\phi}(t),\ t \in (-\infty, b]$. Then, $y(t)$ is a fixed point of the operator N which gives rise to an integral solution of the problem (1)-(3).

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