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EXISTENCE RESULTS FOR NEUTRAL FUNCTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. In this paper we prove an existence theorem for a Carathéodory neutral functional differential inclusion under the mixed Lipschitz and Carathéodory conditions. The existence of extremal solutions is also proved under certain monotonic conditions.

Key Words and Phrases: Functional differential inclusions, fixed point theorems, existence theorem, extremal solutions.

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1. Statement of problem

Let $\mathbb{R}$ denote the real line and let $\mathbb{R}^n$ be an n-dimensional Euclidean space. We define a norm $|\cdot|$ in $\mathbb{R}^n$ by

$$|x| = |x_1| + \cdots + |x_n|$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Let $I_0 = [-r, 0]$ and $I = [0, a]$ be two closed and bounded intervals in $\mathbb{R}$. Let $C = C(I_0, \mathbb{R}^n)$ denote the Banach space of all continuous $\mathbb{R}^n$-valued functions on I_0 with the usual supremum norm $\|\cdot\|_C$ given by

$$\|\phi\|_C = \sup\{|\phi(\theta)| : -r \le \theta \le 0\}.$$

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For any continuous function x defined on the interval $J = [-r, a] = I_0 \cup I$ and any $t \in I$ we denote by x_t the element of C defined by

$$x_t(\theta) = x(t+\theta), \quad -r \le \theta \le 0, \;\; 0 \le t \le a.$$

Given a function $\phi \in C$, consider the neutral functional differential inclusion (in short FDI)

$$\left.\begin{aligned} \frac{d}{dt}[x(t) - f(t, x_t)] \in G(t, x_t) \;\; \text{a.e.} \;\; t \in I, \\ x_0 = \phi, \end{aligned}\right\} \tag{1}$$

where $f : I \times C \to \mathbb{R}^n$ and $G : I \times C \to \mathcal{P}_f(\mathbb{R}^n)$ and $\mathcal{P}_f(\mathbb{R}^n)$ denotes the class of all nonempty subsets of $\mathbb{R}^n$.

Definition 1.1. *A function $x \in C(J, \mathbb{R}^n)$ is said to be a solution of the neutral FDI (1) if*

(i) $x(t) = \phi(t)$ *if* $t \in I_0$,
(ii) $x_t \in C$ *for* $t \in I$, *and*
(iii) *the difference* $x(t) - f(t, x_t)$ *is absolutely continuous and satisfies (1) on* J,

where $C(J, \mathbb{R}^n)$ *is the space of all continuous* $\mathbb{R}^n$*-valued functions on* J.

In the special case of FDE (1), when $G(t, x_t) = \{g(t, x_t)\}$, we obtain a neutral functional differential equation (FDE) of first order, viz.,

$$\left.\begin{aligned} \frac{d}{dt}[x(t) - f(t, x_t)] = g(t, x_t) \;\; \text{a.e.} \;\; t \in I, \\ x_0 = \phi. \end{aligned}\right\} \tag{2}$$

where $f, g : I \times C \to \mathbb{R}^n$.

The neutral FDE (2) has been studied in Ntouyas et. al. [5] for the existence theorems under some compactness conditions on both of the functions f and g. Again when $f \equiv 0$ on $I \times C$, the neutral FDI (1) reduces to

$$\left.\begin{aligned} x'(t) \in G(t, x_t) \;\; \text{a.e.} \;\; t \in I, \\ x_0 = \phi, \end{aligned}\right\} \tag{3}$$

where $G : I \times C \to P_f(\mathbb{R}^n)$.

The FDI (3) has already been discussed in the literature via different methods. In this article we shall prove the existence results for Carathéodory neutral FDI (1). The main tools used in the study are the fixed point theorems of

Dhage [1, 2]. As the special cases to our main results, we obtain the existence results for neutral FDE (2), and the neutral FDI (3). In the following section we give some auxiliary results needed in the subsequent part of the paper.

2. Auxiliary results

Throughout this paper X will be a Banach space and let $\mathcal{P}(X)$ denote the class of all subsets of X. Let $\mathcal{P}_f(X)$, $\mathcal{P}_{bd,cl}(X)$ and $\mathcal{P}_{cp,cv}(X)$ denote respectively the classes of all nonempty, bounded-closed and compact-convex subsets of X. For $x \in X$ and $Y, Z \in \mathcal{P}_{bd,cl}(X)$ we denote by $D(x, Y) = \inf\{\|x - y\| \mid y \in Y\}$, and $\rho(Y, Z) = \sup_{a \in Y} D(a, Z)$.

Define a function $H : \mathcal{P}_{bd,cl}(X) \times \mathcal{P}_{bd,cl}(X) \to \mathbb{R}^+$ by

$$H(A, B) = \max\{\rho(A, B), \rho(B, A)\}.$$

The function H is called a Hausdorff metric on X. Note that $\|Y\| = H(Y, \{0\})$.

A correspondence $T : X \to \mathcal{P}_f(X)$ is called a multi-valued mapping on X. A point $x_0 \in X$ is called a *fixed point of the multi-valued operator* $T : X \to \mathcal{P}_f(X)$ *if* $x_0 \in T(x_0)$. The fixed points set of T will be denoted by $\mathrm{Fix}(T)$.

Definition 2.1. *Let $T : X \to \mathcal{P}_{bd,cl}(X)$ be a multi-valued operator. Then T is called a multi-valued contraction if there exists a constant $k \in (0, 1)$ such that for each $x, y \in X$ we have*

$$H(T(x), T(y)) \le k\|x - y\|.$$

The constant k is called a contraction constant of T.

A multi-valued mapping $T : X \to \mathcal{P}_f(X)$ is called *lower semi-continuous* (shortly l.s.c.) (resp. *upper semi-continuous* (shortly u.s.c.)) if B is any open subset of X then $\{x \in X \mid Gx \cap B \neq \emptyset\}$(resp.$\{x \in X \mid Gx \subset B\}$) is an open subset of X. The multi-valued operator T is called *compact* if $\overline{T(X)}$ is a compact subset of X. Again T is called *totally bounded* if for any bounded subset S of X, $T(S)$ is a totally bounded subset of X. A multi-valued operator $T : X \to \mathcal{P}_f(X)$ is called *completely continuous* if it is upper semi-continuous and totally bounded on X, for each bounded $A \in \mathcal{P}_f(X)$. Every compact multi-valued operator is totally bounded but the converse may not be true. However the two notions are equivalent on a bounded subset of X.

We apply the following form of the fixed point theorem of Dhage [1] in the sequel.

Theorem 2.2. *Let X be a Banach space, $A : X \to \mathcal{P}_{cl,cv,bd}(X)$ and $B : X \to \mathcal{P}_{cp,cv}(X)$ two multi-valued operators satisfying*

(a) *A is contraction with a contraction constant k, and*
(b) *B is completely continuous.*

Then either

(i) *the operator inclusion $\lambda x \in Ax + Bx$ has a solution for $\lambda = 1$, or*
(ii) *the set $\mathcal{E} = \{u \in X \mid \lambda u \in Au + Bu,\ \lambda > 1\}$ is unbounded.*

In the following section we give our main results of this paper.

3. Existence theory

Let $C(J, \mathbb{R}^n), AC(J, \mathbb{R}^n), BM(J, \mathbb{R}^n), M(J, \mathbb{R}^n)$ and $B(J, \mathbb{R}^n)$ denote respectively the spaces of all continuous, absolutely continuous, bounded and measurable, measurable and bounded $\mathbb{R}^n$-valued functions on J. Then we have

$$C(J, \mathbb{R}^n) \subset AC(J, \mathbb{R}^n) \subset BM(J, \mathbb{R}^n) \subset B(J, \mathbb{R}^n).$$

Define a norm $\|\cdot\|$ in $C(J, \mathbb{R}^n)$ by

$$\|x\| = \max_{t \in J} |x(t)|. \tag{4}$$

Clearly $C(J, \mathbb{R}^n)$ is a Banach space with respect to this maximum norm.

Now the neutral FDI (1) is equivalent to the integral inclusion

$$\left. \begin{aligned} x(t) &\in [\phi(0) - f(0, \phi)] + f(t, x_t) + \int_0^t G(s, x_s)\, ds, \text{ if } t \in I, \\ x(t) &= \phi(t), \text{ if } t \in I_0. \end{aligned} \right\} \tag{5}$$

Define two operators $A : C(J, \mathbb{R}^n) \to C(J, \mathbb{R}^n)$ by

$$Ax(t) = \begin{cases} \{-f(0, \phi) + f(t, x_t)\}, \text{ if } t \in I, \\ 0, \text{ if } t \in I_0 \end{cases} \tag{6}$$

and the multi-valued operator $B : C(J, \mathbb{R}^n) \to \mathcal{P}_f(C(J, \mathbb{R}^n))$ by

$$Bx = \begin{cases} \left\{ u \in C(I, \mathbb{R}^n) : u(t) = \phi(0) + \displaystyle\int_0^t v(s)\, ds, \;\; v \in S_G^1(x) \right\}, \\ \hfill \text{if } \; t \in I, \\ \phi(t) \quad \text{if } \; t \in I_0 \end{cases} \tag{7}$$

where

$$S_G^1(x) = \{v \in L^1(I, \mathbb{R}^n) : v(t) \in G(t, x_t) \;\; a.e. \;\; t \in I\}.$$

Then the neutral FDI (1) is equivalent to the operator inclusion

$$x(t) \in Ax(t) + Bx(t), \;\; t \in J. \tag{8}$$

We shall discuss the operator inclusion (8) for the existence theorems under some suitable conditions on the function and the multi-functions involved in FDI (1).

We prove the existence theorem for the FDI (1) under the Carathéodory condition on the multi-function G in it. We need the following definitions in the sequel.

Definition 3.1. *A multi-valued map map $G : J \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ is said to be measurable if for every $y \in \mathbb{R}^n$, the function $t \to d(y, G(t)) = \inf\{\|y - x\| : x \in G(t)\}$ is measurable.*

Definition 3.2. *A multi-valued map $G : I \times C \to \mathcal{P}_{cl}(\mathbb{R}^n)$ is said to be L^1-Carathéodory if*

(i) *$t \mapsto G(t, x)$ is measurable for each $x \in C$,*
(ii) *$x \mapsto G(t, x)$ is upper semi-continuous for almost all $t \in I$, and*
(iii) *for each real number $\rho > 0$, there exists a function $h_\rho \in L^1(I, \mathbb{R}^+)$ such that*

$$\|G(t, u)\| = \sup\{|v| : v \in G(t, u)\} \le h_\rho(t), \quad a.e. \;\; t \in I$$

for all $u \in C$ with $\|u\|_C \le \rho$.

Then we have the following lemmas due to Lasota and Opial [4].

Lemma 3.3. *If $\dim(X) < \infty$ and $F : J \times X \to \mathcal{P}_f(X)$ is L^1-Carathéodory, then $S_G^1(x) \neq \emptyset$ for each $x \in X$.*

Lemma 3.4. *Let X be a Banach space, G an L^1-Carathéodory multi-valued map with $S^1_G \neq \emptyset$ and $\mathcal{K} : L^1(J,X) \to C(J,X)$ be a linear continuous mapping. Then the operator*

$$\mathcal{K} \circ S^1_G : C(J,X) \longrightarrow \mathcal{P}_{cp,cv}(C(J,X))$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

We consider the following set of assumptions in the sequel.

(H_1) There exists a function $k \in B(I, \mathbb{R}^+)$ such that

$$|f(t,x) - f(t,y)| \le k(t)\|x-y\|_C \quad \text{a.e.} \quad t \in I,$$

for all $x, y \in C$ and $\|k\| < 1$.

(H_2) The multi $G(t,x)$ has compact and convex values for each $(t,x) \in I \times C$.

(H_3) G is L^1-Carathéodory.

(H_4) There exists a function $q \in L^1(I, \mathbb{R})$ with $q(t) > 0$ for a.e. $t \in I$ and a nondecreasing function $\psi : \mathbb{R}^+ \to (0, \infty)$ such that

$$\|G(t,x)\| \le q(t)\psi(\|x\|_C) \ \text{a.e.} \ t \in I,$$

for all $x \in C$.

Theorem 3.5. *Assume that (H_1)-(H_4) hold. Suppose that*

$$\int_{c_1}^{\infty} \frac{ds}{\psi(s)} > c_2 \|\gamma\|_{L^1} \tag{9}$$

where $c_1 = \frac{F}{1-\|k\|}$, $c_2 = \frac{1}{1-\|k\|}$ and $F = \|\phi\|_C + |\phi(0) - f(0,\phi)| + \sup_{t \in I} |f(t,0)|$. Then the FDI (1) has a solution on J.

Proof : Let $X = C(J, \mathbb{R}^n)$ and we study the operator inclusion (8) in the space X of all continuous $\mathbb{R}^n$-valued functions on J with a supremum norm $\|\cdot\|$. We shall show that the operators A and B satisfy all the conditions of Theorem 2.2 on J.

Step I. Since Ax is singleton for each $x \in X$, A has closed, convex values on X. Also A has bounded values for bounded sets in X. To show this, let S be a bounded subset of X. Then, for any $x \in S$ one has

$$\begin{aligned} \|Ax\| &\le \|Ax - A0\| + \|A0\| \\ &\le \|k\|\|x\| + \|A0\| \\ &\le \|k\|\rho + \|A0\|. \end{aligned}$$

Hence A is bounded on bounded subsets of X.

Step II. Next we prove that Bx is a convex subset of X for each $x \in X$. Let $u_1, u_2 \in Bx$. Then there exists v_1 and v_2 in $S^1_G(x)$ such that

$$u_j(t) = \phi(0) + \int_0^t v_j(s)\,ds, \quad j = 1, 2.$$

Since $G(t, x)$ has convex values, one has for $0 \leq \mu \leq 1$,

$$[\mu v_1 + (1 - \mu)v_2](t) \in S^1_G(x)(t), \;\; \forall t \in J.$$

As a result we have

$$[\mu u_1 + (1 - \mu)u_2](t) = \phi(0) + \int_0^t [\mu v_1(s) + (1 - \mu)v_2(s)]\,ds.$$

Therefore $[\mu u_1 + (1 - \mu)u_2] \in Bx$ and consequently Bx has convex values in X. Thus we have $B : X \to P_{cv}(X)$.

Step III. We show that A is a contraction on X. Let $x, y \in X$. By (H_1),

$$\begin{aligned} |Ax(t) - Ay(t)| &\leq |f(t, x_t) - f(t, y_t)| \\ &\leq k(t)\|x_t - y_t\|_C \\ &\leq \|k\|\|x - y\|. \end{aligned}$$

Taking supremum over t,

$$\|Ax - Ay\| \leq \|k\|\|x - y\|.$$

This shows that A is a multi-valued contraction, since $\|k\| < 1$.

Step IV. Now we show that the multi-valued operator B is completely continuous on X. First we show that B maps bounded sets into bounded sets in X. To see this, let S be a bounded set in X. Then there exists a real number $\rho > 0$ such that $\|x\| \leq \rho, \forall x \in S$.

Now for each $u \in Bx$, there exists a $v \in S^1_G(x)$ such that

$$u(t) = \phi(0) + \int_0^t v(s)\,ds.$$

Then for each $t \in I$,

$$\begin{aligned} |u(t)| &\le |\phi(0)| + \int_0^t |v(s)|\,ds \\ &\le \|\phi\|_C + \int_0^t h_\rho(s)\,ds \\ &\le \|\phi\|_C + \|h_\rho\|_{L^1}. \end{aligned}$$

This further implies that

$$\|u\| \le \|\phi\|_C + \|h_\rho\|_{L^1}$$

for all $u \in Bx \subset \bigcup B(S)$. Hence $\bigcup B(S)$ is bounded.

Next we show that B maps bounded sets into equi-continuous sets. Let S be, as above, a bounded set and $u \in Bx$ for some $x \in S$. Then there exists $v \in S_G^1(x)$ such that

$$u(t) = \phi(0) + \int_0^t v(s)\,ds.$$

Then for any $t_1, t_2 \in I$ with $t_1 \le t_2$ we have

$$\begin{aligned} |u(t_1) - u(t_2)| &\le \left| \int_0^{t_1} v(s)\,ds - \int_0^{t_2} v(s)\,ds \right| \\ &\le \int_{t_1}^{t_2} |v(s)|\,ds \\ &\le \int_{t_1}^{t_2} h_\rho(s) ds \\ &\le |p(t_1) - p(t_2)| \end{aligned}$$

where $p(t) = \displaystyle\int_0^t h_\rho(s) ds$.

If $t_1, t_2 \in I_0$ then $|u(t_1) - u(t_2)| = |\phi(t_1) - \phi(t_2)|$. For the case where $t_1 \le 0 \le t_2$ we have that

$$\begin{aligned} |u(t_1) - u(t_2)| &\le \left| \phi(t_1) - \phi(0) - \int_0^{t_2} v(s)\,ds \right| \\ &\le |\phi(t_1) - \phi(0)| + \int_0^{t_2} |v(s)|\,ds \\ &\le |\phi(t_1) - \phi(0)| + \int_0^{t_2} h_\rho(s) ds \\ &\le |\phi(t_1) - \phi(0)| + |p(t_2) - p(0)|. \end{aligned}$$

Hence, in all cases, we have

$$|u(t_1) - u(t_2)| \to 0 \ \text{ as } \ t_1 \to t_2.$$

As a result $\bigcup B(S)$ is an equicontinuous set in X. Now an application of Arzelá-Ascoli theorem yields that the multi B is totally bounded on X.

Step V. Next we prove that B *has a closed graph.* Let $\{x_n\} \subset X$ be a sequence such that $x_n \to x_*$ and let $\{y_n\}$ be a sequence defined by $y_n \in Bx_n$ for each $n \in \mathbb{N}$ such that $y_n \to y_*$. We will show that $y_* \in Bx_*$. Since $y_n \in Bx_n$, there exists a $v_n \in S^1_G(x_n)$ such that

$$y_n(t) = \begin{cases} \phi(0) + \int_0^t v_n(s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases}$$

Consider the linear and continuous operator $\mathcal{K} : L^1(X) \to C(X)$ defined by

$$\mathcal{K}v(t) = \int_0^t v_n(s)\,ds.$$

Now

$$\begin{aligned} |y_n(t) - \phi(0) - (y_*(t) - \phi(0))| &\le |y_n(t) - y_*(t)| \\ &\le \|y_n - y_*\| \to 0 \ \text{ as } \ n \to \infty. \end{aligned}$$

From Lemma 3.4 it follows that $(\mathcal{K} \circ S^1_G)$ is a closed graph operator and from the definition of $\mathcal{K}$ one has

$$y_n(t) - \phi(0) \in (\mathcal{K} \circ S^1_F(x_n)).$$

As $x_n \to x_*$ and $y_n \to y_*$, there is a $v \in S^1_G(x_*)$ such that

$$y_*(t) = \begin{cases} \phi(0) + \int_0^t v_*(s)\,ds, & t \in I, \\ \phi(t), & t \in I_0. \end{cases}$$

Hence the multi B is an upper semi-continuous operator on X.

Step VI. Finally we show that the set

$$\mathcal{E} = \{u \in X : \lambda u \in Au + Bu \ \text{ for some } \ \lambda > 1\}$$

is bounded.

Let $u \in \mathcal{E}$ be any element. Then there exists $v \in S^1_G(u)$ such that

$$u(t) = \lambda^{-1}[\phi(0) - f(0,\phi)] + \lambda^{-1} f(t, u_t) + \lambda^{-1} \int_0^t v(s)\,ds.$$

Then

$$\begin{aligned}
|u(t)| &\le \|\phi\|_C + |\phi(0) - f(0,\phi)| + |f(t,u_t)| + \int_0^t |v(s)|\,ds \\
&\le \|\phi\|_C + |\phi(0) - f(0,\phi)| + |f(t,u_t) - f(t,0)| + |f(t,0)| \\
&\qquad + \int_0^t q(s)\psi(\|u_s\|_C)\,ds \\
&\le \|\phi\|_C + |\phi(0) - f(0,\phi)| + |f(t,0)| + k(t)\|u_t\|_C + \int_0^t q(s)\psi(\|u_s\|_C)\,ds \\
&\le \|\phi\|_C + |\phi(0) - f(0,\phi)| + \sup_{t\in I}|f(t,0)| + \|k\|\|u_t\|_C + \int_0^t q(s)\psi(\|u_s\|_C)\,ds \\
&\le F + \|k\|\|u_t\|_C + \int_0^t q(s)\psi(\|u_s\|_C)\,ds.
\end{aligned}$$

Put $w(t) = \max\{|u(s)| : -r \le s \le t\},\ t \in I$. Then $\|u_t\|_C \le w(t)$ for all $t \in I$ and there is a point $t^* \in [-r,t]$ such that $w(t) = u(t^*)$. Hence we have

$$\begin{aligned}
w(t) &= |u(t^*)| \\
&\le F + \|k\|\|u_t\|_C + \int_0^t q(s)\psi(\|u_s\|_C)\,ds \\
&\le F + \|k\|w(t) + \int_0^t q(s)\psi(w(s))\,ds,
\end{aligned}$$

or

$$(1 - \|k\|)w(t) \le F + \int_0^t q(s)\psi(w(s))\,ds$$

and

$$w(t) \le c_1 + c_2\int_0^t q(s)\psi(w(s))\,ds, \quad t \in I.$$

Let

$$m(t) = c_1 + c_2\int_0^t q(s)\psi(w(s))\,ds, \quad t \in I.$$

Then we have $w(t) \le m(t)$ for all $t \in I$. Differentiating w.r.t. to t, we obtain

$$m'(t) = c_2 q(t)\psi(w(t)),\ \text{a.e. } t \in I,\ m(0) = c_1.$$

This further implies that

$$m'(t) \le c_2 q(t)\psi(m(t)),\ \text{a.e. } t \in I,\ m(0) = c_1,$$

that is,

$$\frac{m'(t)}{\psi(m(t))} \leq c_2 q(t) \text{ a.e. } t \in J,\ m(0) = c_1.$$

Integrating from 0 to t we get

$$\int_0^t \frac{m'(s)}{\psi(m(t))}\, ds \leq c_2 \int_0^t q(s) ds.$$

By the change of variable,

$$\int_{c_1}^{m(t)} \frac{ds}{\psi(s)} \leq c_2 \|q\|_{L^1} < \int_{c_1}^{\infty} \frac{ds}{\psi(s)}.$$

Hence there exists a constant M such that

$$w(t) \leq m(t) \leq M \text{ for all } \ t \in I.$$

Now from the definition of w it follows that

$$\|u\| = \sup_{t \in [-r,a]} |u(t)| = w(a) \leq m(a) \leq M,$$

for all $u \in \mathcal{E}$. This shows that the set $\mathcal{E}$ is bounded in X. As a result the conclusion (ii) of Theorem 2.2 does not hold. Hence the conclusion (i) holds and consequently (5) or equivalently FDI (1) has a solution x on J. This completes the proof. $\square$

4. Existence of extremal solutions

In this section we shall prove the existence of maximal and minimal solutions of the FDI (1) under suitable monotonicity conditions on the multi-functions involved in it. We define the usual co-ordinate-wise order relation " $\leq$ " in $\mathbb{R}^n$ as follows. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ be any two elements. Then by " $x \leq y$ " we mean $x_i \leq y_i$ for all $\forall i$, $i = 1, \cdots, n$. We equip the space $C(J, \mathbb{R}^n)$ with the order relation $\leq$ defined by the cone K in $C(J, \mathbb{R}^n)$, that is,

$$K = \big\{x \in C(J, \mathbb{R}^n) \mid x(t) \geq 0,\ \ \forall\ t \in J\big\}. \tag{10}$$

It is known that the cone K is normal in $C(J, \mathbb{R}^n)$. The details of cones and their properties may be found in Heikkila and Lakshmikantham [3]. Let $a, b \in C(J, \mathbb{R}^n)$ be such that $a \leq b$. Then by an order interval $[a, b]$ we mean a set of points in $C(J, \mathbb{R}^n)$ given by

$$[a, b] = \{x \in C(J, \mathbb{R}^n) \mid a \leq x \leq b\}. \tag{11}$$

Let $D, Q \in \mathcal{P}_{cl}(C(J, \mathbb{R}^n))$. Then by $D \leq Q$ we mean $a \leq b$ for all $a \in D$ and $b \in Q$. Thus $a \leq D$ implies that $a \leq b$ for all $b \in Q$ in particular, if $D \leq D$, then it follows that D is a singleton set.

Definition 4.1. *Let X be an ordered Banach space. A mapping $T : X \to \mathcal{P}_{cl}(X)$ is called isotone increasing if $x, y \in X$ with $x < y$, then we have that $Tx \leq Ty$.*

We use the following fixed point theorem in the proof of main existence result of this section.

Theorem 4.2. (Dhage [2]). *Let $[a, b]$ be an order interval in a Banach space and let $A, B : [a, b] \to \mathcal{P}_{cl}(X)$ be two multi-valued operators satisfying*

(a) *A is multi-valued contraction,*
(b) *B is completely continuous,*
(c) *A and B are isotone increasing, and*
(d) *$Ax + Bx \subset [a, b], \ \forall\, x \in [a, b]$.*

Further if the cone K in X is normal, then the operator inclusion $x \in Ax + Bx$ has a least fixed point x_* and a greatest fixed point x^* in $[a, b]$. Moreover $x_* = \lim_n x_n$ and $x^* = \lim_n y_n$, where $\{x_n\}$ and $\{y_n\}$ are the sequences in $[a, b]$ defined by

$$x_{n+1} \in Ax_n + Bx_n, \ x_0 = a \quad \text{and} \quad y_{n+1} \in Ay_n + By_n, \ y_0 = b.$$

We need the following definitions in the sequel.

Definition 4.3. *A function $a \in C(J, \mathbb{R}^n)$ is called a lower solution of the FDI (1) if $\frac{d}{dt}[a(t) - f(t, a_t)] \leq v(t)$ a.e. $t \in I$, $\quad a_0 \leq \phi$, for all $v \in L^1(I, \mathbb{R}^n)$ such that $v(t) \in G(t, a_t)$ almost everywhere $t \in I$. Similarly an upper solution b of the FDI (1) is defined.*

Definition 4.4. *A solution x_M of the FDI (1) is said to be maximal if x is any other solution of FDI (1) on J, then we have $x(t) \leq x_M(t)$ for all $t \in J$. Similarly a minimal solution of the FDI (1) is defined.*

We consider the following assumptions in the sequel.

(H_5) The function $f(t, x)$ and the multi-function $G(t, x)$ are nondecreasing in x almost everywhere for $t \in I$.

(H_6) The FDI (1) has a lower solution a and an upper solution b with $a \leq b$.

Theorem 4.5. *Assume that the hypotheses* (H_1)-(H_6) *hold. Then the FDI (1) has minimal and maximal solutions on* J.

Proof : Let $X = C(J, \mathrm{I\!R}^n)$ and consider the order interval $[a,b]$ in X which is well defined in view of hypothesis (H_7). Define two operators A, B by (6) and (7) respectively. It can be shown, as in the proof of Theorem 3.5, that A and B define the operators $A : [a,b] \to X$ and $B : [a,b] \to \mathcal{P}_{cp,cv}(X)$. It is also similarly shown that A and B are respectively contraction and completely continuous on $[a,b]$. We shall show that A and B are isotone increasing on $[a,b]$. Let $x \in [a,b]$ be such that $x \leq y, x \neq y$. Then by (H_5), we have

$$\begin{aligned} Ax(t) &= -f(0,\phi) + f(t,x_t) \\ &\leq -f(0,\phi)] + f(t,y_t) \\ &= Ay(t), \end{aligned}$$

for all $t \in I$ and $Ax(t) = 0 = Ay(t)$ for all $t \in I_0$. Hence $Ax \leq Ay$. Similarly by (H_5), we have

$$\begin{aligned} Bx(t) &= \left\{ u(t) : u(t) = \phi(0) + \int_0^t v(s)\, ds,\ v \in S_G^1(x) \right\} \\ &\leq \left\{ u(t) : u(t) = \phi(0) + \int_0^t v(s)\, ds,\ v \in S_G^1(y) \right\} \\ &= By(t), \end{aligned}$$

for all $t \in I$ and $Bx(t) = \phi(t) = By(t)$ for all $t \in I_0$. Hence $Bx \leq By$. Thus A and B are monotone increasing on $[a,b]$. Finally let $x \in [a,b]$ be any element. Then by (H_6),

$$a \leq Aa + Ba \leq Ax + Bx \leq Ab + Bb \leq b,$$

which shows that $Ax + Bx \in [a,b]$ for all $x \in [a,b]$. Thus the multi-valued operator A and B satisfy all the conditions of Theorem 4.2 to yield that the operator inclusion and consequently the FDI (1) has maximal and minimal solutions on J. This completes the proof. $\square$

References

[1] B. C. Dhage, Multi-valued mappings and fixed points I, *Nonlinear Funct. Anal. Appl.* (to appear).

[2] B. C. Dhage, A fixed point theorem for multi-valued mappings with applications, (Preprint).

[3] S. Heikkila, V. Lakshmikantham, *Monotone Iterative Technique for Nonlinear Discontinuous Differential Equations*, Marcel Dekker Inc., New York, 1994.

[4] A. Lasota, Z. Opial, An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations, *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys.*, **13**(1965), 781-786.

[5] S. K. Ntouyas, Y. Sficas, P. Tsamatos, Existence results for initial value problems for neutral functional differential equations, *J. Differential Equations*, **114**(1994), 527-537.