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FUNCTIONAL-DIFFERENTIAL EQUATIONS OF MIXED TYPE, VIA WEAKLY PICARD OPERATORS

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Abstract. In this paper we apply the weakly Picard operators technique to study the following second order functional-differential equations of mixed type

$$-x''(t) = f(t, x(t), x(g(t)), x(h(t))), \quad t \in [a, b],$$

where $g([a, b] \cap (-\infty, a) \neq \emptyset$ and $h([a, b]) \cap (b, +\infty) \neq \emptyset$.
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1. Introduction

The purpose of this paper is to study the following boundary value problem (see [1], [3], [5], [7], [10], [12], [19]-[21])

$$-x''(t) = f(t, x(t), x(g(t)), x(h(t))), \quad t \in [a, b]; \tag{1.1}$$

$$\begin{cases} x(t) = \varphi(t), & t \in [a_1, a], \\ x(t) = \psi(t), & t \in [b, b_1], \end{cases} \tag{1.2}$$

where
(H_1) $a_1 \leq a < b \leq b_1$;
(H_2) $g, h \in C([a, b], [a_1, b_1])$;
(H_3) $f \in C([a, b] \times R^3)$;
(H_4) there exists $L_f > 0$ such that:

$$|f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3)| \leq L_f \left(\sum_{i=1}^{3} |u_i - v_i| \right),$$

for all $t \in [a, b]$, $u_i, v_i \in R$, $i = 1, 2, 3$;
(H_5) $\varphi \in [a_1, a]$ and $\psi \in C[b, b_1]$.

Some problems concerning equation (1.1) was study in the following particular cases (see [4], [14], [2], [5], [6], [17], [24], [25], [26]...)

$$g(t) = t - h, \quad h(t) = t + h, \quad h > 0,$$

and ([16])

$$g(t) = \lambda t, \quad h(t) = \frac{1}{\lambda} t, \quad 0 < \lambda < 1.$$

For other considerations on the functional-differential equations we mention: [1], [5], [6], [8], [9], [11], [13], [14], [18], [23], [27].

Let G be the Green function of the following problem

$$-x'' = \chi, \quad x(a) = 0, \quad x(b) = 0.$$

From the definition of the Green function we have that, the problem

$$(1.1) + (1.2), \quad x \in C[a_1, b_1] \cap C^2[a, b],$$

is equivalent with the fixed point equation

$$(1.3) \qquad x(t) = \begin{cases} \varphi(t), \ t \in [a_1, a], \\ w(\varphi, \psi)(t) + \displaystyle\int_a^b G(t,s) f(s, x(s), x(g(s)), x(h(s))) ds, \ t \in [a, b], \\ \psi(t), \ t \in [b, b_1], \end{cases}$$

$$x \in C[a_1, b_1],$$

where

$$w(\varphi, \psi)(t) := \frac{t-a}{b-a} \psi(b) + \frac{b-t}{b-a} \varphi(a).$$

The equation (1.1) is equivalent with

(1.4)

$$x(t) = \begin{cases} x(t), \ t \in [a_1, a], \\ w(x|_{[a_1,a]}, x|_{[b,b_1]})(t) + \displaystyle\int_a^b G(t,s) f(s, x(s), x(g(s)), x(h(s))) ds, \ t \in [a, b] \\ x(t), \ t \in [b, b_1]. \end{cases}$$

Consider the following operators

$$B_f, E_f : C[a_1, b_1] \to C[a_1, b_1],$$

where

$$B_f(x)(t) := \text{second part of } (1.3)$$

and

$$E_f(x)(t) := \text{second part of } (1.4).$$

Let $X := C[a_1, b_1]$ and $X_{\varphi,\psi} := \{x \in X |\ x|_{[a_1,a]} = \varphi, \ x|_{[b,b_1]} = \psi\}$. Then

$$X = \bigcup_{\substack{\varphi \in C[a_1, a] \\ \psi \in C[b, b_1]}} X_{\varphi,\psi}$$

is a partition of X.

We have

Lemma 1.1. *We suppose that the conditions* (H_1), (H_2), (H_3) *and* (H_5) *are satisfied. Then*

(a) $B_f(X) \subset X_{\varphi,\psi}$; $B_f(X_{\varphi,\psi}) \subset X_{\varphi,\psi}$;

(b) $B_f|_{X_{\varphi,\psi}} = E_f|_{X_{\varphi,\psi}}$.

In this paper we shall prove that, if L_f is small enough, then the operator E_f is weakly Picard operator and we study the equation (1.1) in the terms of this operator.

2. Weakly Picard operators

Let (X, d) be a metric space and $A : X \to X$ an operator. We shall use the following notations:

$F_A := \{x \in X | \ A(x) = x\}$ - the fixed point set of A;

$I(A) := \{Y \subset X | \ A(Y) \subset Y, \ Y \neq \emptyset\}$ - the family of the nonempty invariant subsets of A;

$$A^{n+1} := A \circ A^n, \quad A^0 = 1_X, \quad A^1 = A, \quad n \in N.$$

Definition 2.1. ([22], [23]) An operator A is weakly Picard operator (WPO) if the sequence

$$(A^n(x))_{n \in N}$$

converges, for all $x \in X$, and the limit (which may depend on x) is a fixed point of A.

Definition 2.2. ([22], [23]) If the operator A is WPO and $F_A = \{x^*\}$, then by definition, the operator A is Picard operator (PO).

Definition 2.3. ([22], [23]) If A is WPO, then we consider the operator A^∞ defined by

$$A^\infty : X \to X, \quad A^\infty(x) := \lim_{n \to \infty} A^n(x).$$

It is clear that

$$A^\infty(X) = F_A \text{ and } \omega_A(x) = \{A^\infty(x)\},$$

where $\omega_A(x)$ is the ω-limit point set of A.

For some examples of WPOs see [22] and [23].

3. Boundary value problem

Consider the problem (1.1)+(1.2). We have

Theorem 3.1. ([7], [19]) *We suppose that*

(a) the conditions $(H_1) - (H_5)$ *are satisfied,*

(b) $\frac{3}{8} L_f (b-a)^2 < 1$.

Then the problem (1.1)+(1.2) has a unique solution which is the uniform limit of the successive approximations.

Proof. Consider the Banach space $C[a_1, b_1]$ with Chebyshev norm. The problem (1.1)+(1.2) is equivalent with the fixed point equation

$$B_f(x) = x, \quad x \in C[a_1, b_1].$$

From the condition (H_4), the operator B_f is an α-contraction, with

$$\alpha = \frac{3}{8} L_f (b-a)^2.$$

The proof follows from the contraction principle.

Remark 3.1. From the Theorem 3.1 we have the operator B_f is PO. But

$$B_f|_{X_{\varphi,\psi}} = E_f|_{X_{\varphi,\psi}},$$

and

$$X := C[a_1, b_1] = \bigcup_{\varphi,\psi} X_{\varphi,\psi}, \quad X_{\varphi,\psi} \in I(E_f).$$

So, the operator E_f is WPO and

$$F_{E_f} \cap X_{\varphi,\psi} = \{x^*_{\varphi,\psi}\},\ \forall\ \varphi \in C[a_1, a],\ \psi \in C[b, b_1],$$

where $x^*_{\varphi,\psi}$ is the unique solution of the problem (1.1)+(1.2).

4. Inequalities of Čaplygin type

We have

Theorem 4.1. *We suppose that*

(a) the conditions $(H_1) - (H_5)$ *are satisfied;*

(b) $\frac{3}{8} L_f (b-a)^2 < 1$*;*

(c) $u_i, v_i \in R,\ u_i \leq v_i,\ i = 1, 2, 3,$ *imply that*

$$f(t, u_1, u_2, u_3) \leq f(t, v_1, v_2, v_3),$$

for all $t \in [a, b]$.

Let x *be a solution of the equation (1.1) and* y *a solution of the inequality*

$$-y''(t) \leq f(t, y(t), y(g(t)), y(h(t))), \quad t \in [a, b].$$

Then

$$y(t) \leq x(t),\ \forall\ t \in [a_1, a] \cup [b, b_1] \Rightarrow y \leq x.$$

Proof. In the terms of the operator E_f, we have

$$x = E_f(x) \text{ and } y \leq E_f(y)$$

and

$$w(y|_{[a_1,a]}, y|_{[b,b_1]}) \leq w(x|_{[a,a_1]}, x|_{[b,b_1]}).$$

On the other hand, from the condition (c), we have that the operator E_f^∞ is monotone increasing, and we have (see [22])

$$y \leq E_f^\infty(y) = E_f^\infty(\widetilde{w}(y)) \leq E_f^\infty(\widetilde{w}(x)) = x,$$

where, for $z \in X$,

$$\widetilde{w}(z)(t) := \begin{cases} z(t),\ t \in [a_1, a] \\ w(z|_{[a_1,a]}, z|_{[b,b_1]})(t),\ t \in [a, b], \\ z(t),\ t \in [b, b_1]. \end{cases}$$

So, $y \leq x$.

Remark 4.1. Let Y be an ordered Banach space. We consider the problem (1.1)+(1.2), where

(H_1') $a_1 \leq a < b \leq b_1$;

(H_2') $g, h \in C([a, b], [a_1, b_1])$;

(H_3') $f \in C([a, b] \times Y \times Y \times Y, Y)$;

(H_4') there exists $L_f > 0$, such that

$$f\|(t,u_1,u_2,u_3) - f(t,v_1,v_2,v_3)\| \le L_f \sum_{i=1}^{3} \|u_i - v_i\|,$$

for all $t \in [a,b]$, $u_i, v_i \in Y$, $i = 1,2,3$;

(H_5') $\varphi \in C([a_1,a],Y)$, $\psi \in C([b,b_1],Y)$.

As in the case $Y = R$, we consider the operators

$$B_f, E_f : C([a_1,b_1],Y) \to C([a_1,b_1],Y).$$

By a similar way we have

Theorem 4.2. *We suppose that*

(a) the condition $(H_1') - (H_5')$ *are satisfied;*

(b) $\frac{3}{8}L_f(b-a)^2 < 1$.

Then the corresponding problem, (1.1)+(1.2), has in $C([a_1,b_1],Y) \cap C^2([a,b],Y)$ *a unique solution* x_f^**, and* $F_{B_f} = \{x_f^*\}$.

Theorem 4.3. *We suppose that*

(i) f, g *and* h *are as in the Theorem 4.2,*

(ii) the operator $f(t,\cdot,\cdot,\cdot) : Y^3 \to Y^3$ *is monotone increasing.*

Let x *be a solution of the corresponding equation (1.1) and* y *a solution of the inequality*

$$-y'' \le f(t,y(t),y(g(t)),y(h(t))), \quad t \in [a,b].$$

Then

$$y(t) \le x(t), \ \forall\ t \in [a_1,a] \cup [b,b_1] \Rightarrow y \le x.$$

Remark 4.2. In the case $Y = R^n$, the corresponding equation, (1.3), is the following system of functional-integral equations ($f = (f_1,\dots,f_n), \varphi = (\varphi_1,\dots,\varphi_n), \psi = (\psi_1,\dots,\psi_n), x = (x_1,\dots,x_n)$)

$$x_i(t) = \begin{cases} \varphi_i(t), \ t \in [a_1,a], \\ w(\varphi_i,\psi_i)(t) + \displaystyle\int_a^b G(t,s) f_i(s,x(s),x(g(s)),x(h(s)))ds, \ t \in [a,b], \ i = \overline{1,n} \\ \psi_i(t), \ t \in [b,b_1]. \end{cases}$$

Remark 4.3. In the problem (1.1)+(1.3), instead of, $-x''$, we can put

$$-(p(t)x')' + q(t)x$$

if $p > 0$ and $q \ge 0$.

In this case, instead of the condition

$$\frac{3}{8}L_f(b-a)^2 < 1,$$

we must put

$$3L_f \int_a^b G(t,s)ds \le \alpha < 1$$

where G is the Green function of the problem

$$-(p(t),x')' + q(t)x = \chi, \quad x(a) = 0, \quad x(b) = 0.$$

5. Data dependence: monotony

Now we shall study the monotony of the solution of the problem (1.1)+(1.2), with respect to φ, ψ and f. For this study we need the following abstract result ([22]).

Abstract comparison lemma. *Let $(X, d, \leq)$ be an ordered metric space and $A, B, C : X \to X$ be such that:*

(i) $A \leq B \leq C$;

(ii) the operators A, B, C are WPOs;

(iii) the operator B is monotone increasing.

Then

$$x \leq y \leq z \Rightarrow A^\infty(x) \leq B^\infty(y) \leq C^\infty(z).$$

We have

Theorem 5.1. *Let $f_i \in C([a,b] \times R^3)$, $i = 1,2,3$, g and h be as in the Theorem 3.1. We suppose that*

(a) $f_2(t,\cdot,\cdot,\cdot) : R^3 \to R^3$ is monotone increasing;

(b) $f_1 \leq f_2 \leq f_3$.

Let x_i be a solution of the equation

$$-x'' = f_i(t, x(t), x(g(t)), x(h(t))), \quad t \in [a,b].$$

If

$$x_1(t) \leq x_2(t) \leq x_3(t), \ \forall\ t \in [a_1, a] \cup [b, b_1],$$

then

$$x_1 \leq x_2 \leq x_3.$$

Proof. The operators E_{f_i}, $i = 1,2,3$, are WPOs. From the condition (a) the operator E_{f_2} is monotone increasing. From (b) it follows that

$$E_{f_1} \leq E_{f_2} \leq E_{f_3}.$$

We remark that

$$x_i = E_{f_i}^\infty(\widetilde{w}(x_i)), \quad i = 1,2,3.$$

Now the proof follows from the Abstract comparison lemma.

6. Data dependence: continuity

Consider the boundary value problem (1.1)+(1.2) in the conditions of the Theorem 3.1. Denote by

$$x(\cdot; \varphi, \psi, f)$$

the solution of this problem. We have

Theorem 6.1. *Let φ_i, ψ_i, f_i, $i = 1,2$, be as in the Theorem 3.1. We suppose that*

(i) there exists $\eta_1 > 0$, such that

$$|\varphi_1(t) - \varphi_2(t)| \leq \eta_1, \ \forall\ t \in [a_1, a],$$

and

$$\|\psi_1(t) - \varphi_2(t) \leq \eta_2,\ \forall\ t \in [b, b_1];$$

(ii) there exists $\eta_2 > 0$ *such that*

$$|f_1(t, u_1, u_2, u_3) - f_2(t, u_1, u_2, u_3)| \leq \eta_2,\ \forall\ t \in [a, b],\ \forall\ u_i \in R.$$

Then

$$|x(t; \varphi_1, \psi_1, f_1) - x(t; \varphi_2, \psi_2, f_2)| \leq \frac{8\eta_1 + \eta_2(b-a)^2}{8 - 3L_f(b-a)^2}$$

where $L_f = \max(L_{f_1}, L_{f_2})$.

Proof. Consider the operators $B_{\varphi_i,\psi_i}, f_i,\ i = 1, 2$. These operators are contractions. Moreover

$$\|B_{\varphi_1,\psi_1,f_1}(x) - B_{\varphi_2,\psi_2,f_2}(x)\|_C \leq \eta_1 + \eta_2\frac{(b-a)^2}{\varphi},\ \forall\ x \in C[a_1, b_1].$$

Now, the proof follows from the following well known result (see [23]).

Theorem 6.2. *Let* (X, d) *be a complete metric space and* $A, B : X \to X$ *two operators. We suppose that*

(i) the operator A *is an a-contraction;*

(ii) $F_B \neq \emptyset$*;*

(iii) there exists $\eta > 0$ *such that*

$$d(A(x), B(x)) \leq \eta,\ \forall\ x \in X.$$

Then if $F_A = \{x_A^*\}$ *and* $x_B^* \in F_B$*, we have*

$$d(x_A^*, x_B^*) \leq \frac{\eta}{1-a}.$$

From the Theorem 6.1 we have

Theorem 6.3. *Let* $\varphi_i, \psi_i, f_i,\ i \in N$ *and* φ, ψ, f *be as in the Theorem 3.1. We suppose that*

$$\varphi_i \overset{unif.}{\longrightarrow} \varphi \text{ as } i \to \infty,$$
$$\psi_i \overset{unif.}{\longrightarrow} \psi \text{ as } i \to \infty,$$
$$f_i \overset{unif.}{\longrightarrow} f \text{ as } i \to \infty.$$

Then

$$x(\cdot, \varphi_i, \psi_i, f_i) \overset{unif.}{\longrightarrow} x(\cdot, \varphi, \psi, f),\ \text{ as } i \to \infty.$$

In what follow we shall use the c-WPOs technique to give some data dependence results.

Definition 6.1. Let A be an WPO and $c > 0$. The operator A is c-WPO if

$$d(x, A^\infty(x)) \leq cd(x, A(x)),\ \forall\ x \in X.$$

Example 6.1. Let (X, d) be a complete metric space and $A : X \to X$ an operator. We suppose that there exists $a \in [0, 1[$ such that

$$d(A^2(x), A(x)) \leq ad(x, A(x)),\ \forall\ x \in X.$$

Then A is c-WPO with $c = (1-a)^{-1}$.

We have (see [22])

Theorem 6.4. *Let* (X, d) *be a metric space and* $A_i : X \to X$, $i = 1, 2$. *We suppose that*

(i) the operator A_i *is* $c_i - WPO$, $i = 1, 2$;

(ii) there exists $\eta > 0$ *such that*

$$d(A_1(x), A_2(x)) \leq \eta, \ \forall \ x \in X.$$

Then

$$H(F_{A_1}, F_{A_2}) \leq \eta \max(c_1, c_2).$$

Here H *stands for Pompeiu-Hausdorff functional.*

From the Remark 3.1 and the Theorem 6.4, we have

Theorem 6.5. *Let* f_1 *and* f_2 *be as in the Theorem 3.1. Let* S_i *be the solution set of equation (1.1) corresponding to* f_i, $i = 1, 2$. *If* $\eta > 0$ *is such that*

$$|f_1(t, u_1, u_2, u_3) - f(t, u_1, u_2, u_3)| \leq \eta,$$

for all $t \in [a, b], \ u_i \in R, \ i = 1, 2$, *then*

$$H(S_1, S_2) \leq \frac{\eta(b-a)^2}{8 - 3L(b-a)^2}$$

where $L := \max(L_{f_1}, L_{f_2})$.

Proof. In the condition of the Theorem 3.1 the operators E_{f_i}, $i = 1, 2$, are $c_i - WPOs$ with

$$c_i = (1 - \alpha_i)^{-1}$$

where $\alpha_i = \dfrac{3}{8} L_{f_i}(b-a)^2$.

Now, we are in the conditions of the Theorem 6.4.

7. Smooth dependence on parameters

Consider the following boundary value problem with parameter

$$-x''(t) = f(t, x(t), x(g(t)), x(h(t))); \lambda), \quad t \in [a, b], \tag{7.1}$$

$$\begin{cases} x(t) = \varphi(t), & t \in [a_1, a], \\ x(t) = \psi(t), & t \in [b, b_1]. \end{cases} \tag{7.2}$$

We suppose that

(C_1) $a_1 \leq a < b \leq b_1$; $J \subset R$, a compact interval;

(C_2) $g, h \in C([a, b], [a_1, b_1])$;

(C_3) $f \in C^1([a, b] \times R^3 \times J)$;

(C_4) there exists $L_f > 0$, such that

$$\left| \frac{\partial f(t, u_1, u_2, u_3; \lambda)}{\partial u_i} \right| \leq L_f,$$

for all $t \in [a, b], \ u_i \in R, \ i = 1, 2, 3, \ \lambda \in J$;

(C_5) $\varphi \in C[a_1, a], \ \psi \in C[b, b_1]$;

(C_6) $\dfrac{3}{8} L_f (b-a)^2 < 1$.

In the above conditions, from the Theorem 3.1, we have that the problem (7.1)+(7.2) has a unique solution, $x^*(\cdot;\lambda)$.

Now we prove that

$$x^*(t;\cdot) \in C^1(J), \text{ for all } t \in [a_1, b_1].$$

For this, we consider the equation

$$-x''(t;\lambda) = f(t, x(t;\lambda), x(g(t);\lambda), x(h(t);\lambda);\lambda), \ t \in [a,b], \ \lambda \in J, \tag{7.3}$$
$$x \in C([a_1, b_1] \times J).$$

The problem, (7.3)+(7.2) is equivalent with the following functional-integral equation

$$x(t;\lambda) = \begin{cases} \varphi(t), \ t \in [a_1, a], \ \lambda \in J, \\ w(\varphi,\psi)(t) + \displaystyle\int_a^b G(t,s) f(s, x(s;\lambda), x(g(s);\lambda), x(h(s);\lambda);\lambda) ds, \\ \qquad t \in [a,b], \ \lambda \in J \\ \psi(t), \ t \in [b, b_1], \ \lambda \in J. \end{cases} \tag{7.4}$$

We consider the operator

$$B : C([a_1, b_1] \times J) \to C([a_1, b_1] \times J),$$

where $B(x)(t;\lambda) :=$ second part of (7.4).

Let $X := C([a_1, b_1] \times J)$ and let, $\|\cdot\|$, be the Chebyshev norm on X. It is clear that, in the conditions $(C_1) - (C_6)$, the operator B is Picard operator. Let x^* be the unique fixed point of B.

We suppose that there exists $\dfrac{\partial x^*}{\partial \lambda}$. Then from (7.4) we have that

$$\frac{\partial x^*(t;\lambda)}{\partial \lambda} = \int_a^b G(t,s) \frac{\partial f(s, x^*(s;\lambda), x^*(g(s);\lambda), x^*(h(s);\lambda);\lambda)}{\partial u_1} \cdot \frac{\partial x^*(s;\lambda)}{\partial \lambda} ds +$$
$$+ \int_a^b G(t,s) \frac{\partial f(s, x^*(s;\lambda), x^*(g(s);\lambda), x^*(h(s);\lambda);\lambda)}{\partial u_2} \cdot \frac{\partial x^*(g(s);\lambda)}{\partial \lambda} ds +$$
$$+ \int_a^b G(t,s) \frac{\partial f(s, x^*(s;\lambda), x^*(g(s);\lambda), x^*(h(s);\lambda);\lambda)}{\partial u_3} \cdot \frac{\partial x^*(h(s);\lambda)}{\partial \lambda} ds +$$
$$+ \int_a^b G(t,s) \frac{\partial f(s, x^*(s;\lambda), x^*(g(s);\lambda), x^*(h(s);\lambda);\lambda)}{\partial \lambda} ds, \quad t \in [a,b], \ \in J.$$

This relation suggest us to consider the following operator

$$C : X \times X \to X$$
$$(x, y) \mapsto C(x, y)$$

where

$$C(x,y)(t;\lambda) := \int_a^b G(t,s) \frac{\partial f(s, x(s;\lambda), x(g(s);\lambda), x(h(s);\lambda);\lambda)}{\partial u_1} y(s;\lambda) ds +$$
$$+ \int_a^b G(t,s) \frac{\partial f(s, x(s;\lambda), x(g(s);\lambda), x(h(s);\lambda);\lambda)}{\partial u_2} y(g(s);\lambda) ds +$$

$$+\int_a^b G(t,s)\frac{\partial f(s,x(s;\lambda),x(g(s);\lambda),x(h(s);\lambda);\lambda)}{\partial u_3}y(h(s);\lambda)ds+$$

$$+\int_a^b G(t,s)\frac{\partial f(s,x(s;\lambda),x(g(s);\lambda),x(h(s);\lambda);\lambda)}{\partial \lambda}ds,$$

for $t\in[a,b],\ \lambda\in J$ and

$$C(x,y)(t,\lambda):=0,\ \text{for } t\in[a_1,a]\cup[b,b_1],\ \lambda\in J.$$

In this way we have the triangular operator

$$A: X\times X\to X\times X$$

$$(x,y)\mapsto (B(x),C(x,y)),$$

where B is a Picard operator and $C(x,\cdot):X\to X$ is an α-contraction, with $\alpha=\frac{3}{8}L_f(b-a)^2$.

From the theorem of fibre contraction (see [22], [23]) we have that the operator A is Picard operator. So, the sequences

$$\begin{array}{l} x_{n+1}:=B(x_n),\\ y_{n+1}:=C(x_n,y_n), \end{array}\qquad n\in N$$

converges uniformly (with respect to $t\in[a_1,b_1],\ \lambda\in J$) to $(x^*,y^*)\in F_A$, for all $x_0,y_0\in C([a_1,b_1]\times J)$.

If we take, $x_0=0$, $y_0=\frac{\partial x_0}{\partial\lambda}=0$, then

$$y_1=\frac{\partial x_1}{\partial\lambda}.$$

By induction we prove that

$$y_n=\frac{\partial x_n}{\partial\lambda},\ \forall\ n\in N.$$

Thus

$$x_n\overset{unif.}{\longrightarrow}x^*\text{ as } n\to\infty,$$

$$\frac{\partial x_n}{\partial\lambda}\to y^*\text{ as } n\to\infty.$$

These imply that there exists $\frac{\partial x^*}{\partial\lambda}$ and $\frac{\partial x^*}{\partial\lambda}=y^*$.

From the above considerations, we have that

Theorem 7.1. *Consider the problem (7.3)+(7.2), in the conditions* $(C_1)-(C_6)$*. Then*

(i) The problem, (7.3)+(7.2), has in $C([a_1,b_1]\times J)$ *a unique solution,* x^**.*

(ii) $x^*(t,\cdot)\in C^1(J),\ \forall\ t\in[a_1,b_1]$.

Remark 7.1. By the same arguments we have that, if $f(t,\cdot,\cdot,\cdot)\in C^k$, then $x^*(t,\cdot)\in C^k(J),\ \forall\ t\in[a_1,b_1]$.

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