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ON A THEOREM OF ROLEWICZ TYPE FOR LINEAR SKEW-PRODUCT SEMIFLOWS

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Abstract. Theorems of characterization for uniform exponential stability of linear skew-product semiflows on locally compact spaces, in terms of Banach function spaces, are given. Some theorems due to Neerven and Rolewicz are generalized for the case of linear skew-product semiflows.
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1. Introduction

One of the most notable results in the theory of stability of evolution equations has been proved by Neerven in [12]. It connects the uniform exponential stability of a C_0 - semigroup with the ownership of its orbits to a certain Banach function space. This theorem can be considered as a reformulation of a well-known stability theorem due to Datko [6], which says that an evolution operator $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ is uniformly exponentially stable if and only if for every $x \in X$ and $s \geq 0$ the mapping $t \mapsto U(t+s,s)x$ belongs to $L^p(\mathbf{R}_+, X)$ and these orbits are uniformly bounded in $L^p(\mathbf{R}_+, X)$.

An another important step in the theory of evolution operators has been made by Rolewicz [14], which expressed the uniform exponential stability as follows:

Theorem 1.1. *(Rolewicz) Let $\varphi : \mathbf{R}_+^* \times \mathbf{R}_+ \to \mathbf{R}$ be a function with the following properties:*

(i) for every $t > 0, s \to \varphi(t,s)$ is a continuous, non-decreasing function with $\varphi(t,0) = 0$ and $\varphi(t,s) > 0$, for all $s > 0$;

(ii) for every $s \geq 0, t \to \varphi(t,s)$ is non-decreasing.

Let X be a Banach space and let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be an evolution family on X. If for every $x \in X$, there is $\alpha(x) > 0$ such that

$$\sup_s \int_s^\infty \varphi(\alpha(x), ||U(t,s)x||)\, dt < \infty$$

then $\mathcal{U}$ is uniformly exponentialy stable.

It is easy to see that if α is constant, then the property of uniform exponential stability from above proceed from the fact that all the orbits of the evolution operator of the type $U(\cdot+s,s)x$ lie in a certain Orlicz space and they are uniformly bounded.

In this paper we shall extend the ideeas presented above to a more general situation described by linear skew-product semiflows on locally compact spaces.

The central purpose is to obtain a variant of Rolewicz's theorem for linear skew-product semiflows. This is done by employing a Banach function space technique. It is important to mention that the methods used in the proofs are completely different from those used by Neerven and Rolewicz.

2. Preliminary results

2.1. Banach function spaces

In this section we recall some facts about Banach function spaces over $\mathbf{R}_+$. For the proofs we refer to [11].

Let $(\mathbf{R}_+,\mathcal{L},m)$, where $\mathcal{L}$ is the σ-algebra of all Lebesgue measurable sets $A\subset\mathbf{R}_+$ and m the Lebesgue measure. We shall denote by $\mathcal{M}$ the linear space of all m-measurable functions $f:\mathbf{R}_+\to\mathbf{C}$, identifying functions which are equal a.e.

A *Banach function norm* is a function $N:\mathcal{M}\to\bar{\mathbf{R}}_+=[0,\infty]$ with the following properties:

$n_1)$ $N(f)=0$ if and only if $f=0$ a.e.;

$n_2)$ if $|f|\le|g|$ a.e. then $N(f)\le N(g)$;

$n_3)$ $N(\alpha f)=|\alpha|N(f)$, for all scalars $\alpha\in\mathbf{C}$ and all f with $N(f)<\infty$;

$n_4)$ $N(f+g)\le N(f)+N(g)$, for all $f,g\in\mathcal{M}$.

Let $B=B_N$ be the set defined by

$$B:=\{f\in\mathcal{M}:|f|_B:=N(f)<\infty\}.$$

It is easy to see that $(B,|\cdot|_B)$ is a normed linear space. If B is complete, then B is called *Banach function space over* $\mathbf{R}_+$.

Remark 2.1. B is an ideal in $\mathcal{M}$, i.e. if $|f|\le|g|$ a.e. with $g\in B$, then also $f\in B$ and $|f|_B\le|g|_B$.

Remark 2.2. If $f_n\to f$ in B, then there is a subsequence (f_{k_n}) converging to f pointwise a.e. (see [11]).

For a Banach function space B over $\mathbf{R}_+$ we define

$$F_B:\mathbf{R}_+\to\bar{\mathbf{R}}_+,\quad F_B(t):=\begin{cases}|\chi_{[0,t)}|_B & , \quad \text{if } \chi_{[0,t)}\in B\\ \infty & , \quad \text{if } \chi_{[0,t)}\notin B\end{cases}$$

where $\chi_{[0,t)}$ denotes the characteristic function of the interval $[0,t)$. The function F_B is called *the fundamental function* of the Banach function space B.

Remark 2.3. F_B is a non-decreasing function.

In what follows we denote by $\mathcal{B}(\mathbf{R}_+)$ the set of all Banach function spaces B with the property

$$\lim_{t\to\infty} F_B(t) = \infty$$

and with $\mathcal{E}(\mathbf{R}_+)$ the set of all Banach function spaces $B \in \mathcal{B}(\mathbf{R}_+)$ with the property that there exists a strictly increasing sequence $(t_n)_n \subset \mathbf{R}_+$ such that

$$t_n \to \infty, \ \sup_{n\in\mathbf{N}} (t_{n+1} - t_n) < \infty \quad \text{and} \quad \inf_{n\in\mathbf{N}} |\chi_{[t_n,t_{n+1})}|_B > 0.$$

Example 2.1. We consider the Banach function norm $N : \mathcal{M} \to \bar{\mathbf{R}}_+$ defined by

$$N(f) = \sum_{n=1}^{\infty} \frac{1}{n} \int_{n-1}^{n} |f(s)|\, ds.$$

We observe that if $B = B_N$ then

$$F_B(n) = \sum_{j=1}^{n} \frac{1}{j}, \qquad \text{for all } n \in \mathbf{N}^*,$$

so $B \in \mathcal{B}(\mathbf{R}_+)$, but $B \notin \mathcal{E}(\mathbf{R}_+)$.

Example 2.2. For every $p \in [1, \infty)$ the space $L^p(\mathbf{R}_+, \mathbf{C})$ with respect to the norm

$$|f|_p := \left(\int_0^{\infty} |f(t)|^p \, dt \right)^{\frac{1}{p}}$$

is a Banach function space. It is easy to see that $F_{L^p}(t) = t^{1/p}$, for all $t > 0$ and for $t_n = n, |\chi_{[n,n+1)}|_p = 1$, for all $n \in \mathbf{N}$. So we obtain that $L^p(\mathbf{R}_+, \mathbf{C})$ belongs to $\mathcal{E}(\mathbf{R}_+)$.

Example 2.3. (*Orlicz spaces*) Let $\varphi : \mathbf{R}_+ \to \bar{\mathbf{R}}_+$ be a non-decreasing and left-continuous function which is not identically 0 or ∞ on $(0, \infty)$. The *Young function* associated to φ is given by

$$Y_\varphi(t) := \int_0^t \varphi(s)\, ds.$$

Let $f : \mathbf{R}_+ \to \mathbf{C}$ be a measurable function. We define

$$M_\varphi(f) := \int_0^{\infty} Y_\varphi(|f(s)|)\, ds.$$

The set L_φ, of all functions f with the property that there exists a $k > 0$ such that $M_\varphi(kf) < \infty$, is easily checked to be a linear space. With respect to the norm

$$|f|_\varphi := \inf \{k > 0 : M_\varphi(\frac{1}{k} f) \leq 1\}$$

$(L_\varphi, |\cdot|_\varphi)$ is a Banach function space over $\mathbf{R}_+$ called the *Orlicz space* associated to φ.

Trivial examples of Orlicz spaces are $L^p(\mathbf{R}_+, \mathbf{C}), 1 \le p \le \infty$. They are obtained for

$$\varphi(t) = pt^{p-1}, \text{ for } 1 \le p < \infty \text{ and } \varphi(t) = \begin{cases} 0 & , \quad 0 \le t \le 1 \\ \infty & , \quad t > 1 \end{cases} \quad \text{for } p = \infty.$$

Proposition 2.1. *If $0 < \varphi(t) < \infty$ for all $t > 0$ then the Orlicz space L_φ has the following properties*

i) the Young function Y_φ is bijective;

ii) the fundamental function F_{L_φ} can be expressed in terms of the Y_φ^{-1} by

$$F_{L_\varphi}(t) = \frac{1}{Y_\varphi^{-1}(\frac{1}{t})}, \qquad \textit{for all } t > 0;$$

iii) $\lim\limits_{t\to\infty} F_{L_\varphi}(t) = \infty$ and hence $L_\varphi \in \mathcal{B}(\mathbf{R}_+)$;

iv) $L_\varphi \in \mathcal{E}(\mathbf{R}_+)$.

Proof. i) It is easy to see that Y_φ is strictly increasing, continuous with $Y_\varphi(0) = 0$ and $Y_\varphi(t) \ge (t-1)\varphi(1)$, for all $t > 1$, so $\lim\limits_{t\to\infty} Y_\varphi(t) = \infty$. Hence Y_φ is bijective.

ii) Let $t > 0$. Since

$$M_\varphi(\frac{1}{k}\chi_{[0,t)}) = tY_\varphi(\frac{1}{k}),$$

for all $k > 0$, it follows that $M_\varphi(\frac{1}{k}\chi_{[0,t)}) \le 1$ if and only if $1/Y_\varphi^{-1}(\frac{1}{t}) \le k$. So

$$F_{L_\varphi}(t) = \frac{1}{Y_\varphi^{-1}(\frac{1}{t})},$$

for all $t > 0$.

iii) Since $Y_\varphi^{-1}(0) = 0$, using (ii) it follows that $\lim\limits_{t\to\infty} F_{L_\varphi}(t) = \infty$.

iv) We observe that for every $n \in \mathbf{N}$

$$|\chi_{[n,n+1)}|_\varphi = \frac{1}{Y_\varphi^{-1}(1)}.$$

2.2. Linear skew-product semiflows

Let X be a fixed Banach space, let (Θ, d) be a locally compact metric space and let $\mathcal{E} = X \times \Theta$. We shall denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators from X into itself.

Definition 2.1. A mapping $\sigma : \Theta \times \mathbf{R}_+ \to \Theta$ is called a *semiflow* on Θ, if it has the following properties:

(i) $\sigma(\theta, 0) = \theta$, for all $\theta \in \Theta$;

(ii) $\sigma(\theta, s+t) = \sigma(\sigma(\theta, s), t)$, for all $(\theta, s, t) \in \Theta \times \mathbf{R}_+^2$;
(iii) σ is continuous.

Definition 2.2. A pair $\pi = (\Phi, \sigma)$ is called a *linear skew-product semiflow* on $\mathcal{E} = X \times \Theta$ if σ is a semiflow on Θ and $\Phi : \Theta \times \mathbf{R}_+ \to \mathcal{B}(X)$ satisfies the following conditions:

(i) $\Phi(\theta, 0) = I$, the identity operator on X, for all $\theta \in \Theta$;
(ii) $\Phi(\theta, t+s) = \Phi(\sigma(\theta, t), s)\Phi(\theta, t)$, for all $(\theta, t, s) \in \Theta \times \mathbf{R}_+^2$ (*the cocycle identity*);
(iii) $(\theta, t) \mapsto \Phi(\theta, t)x$ is continuous, for every $x \in X$;
(iv) there are $M \geq 1$ and $\omega > 0$ such that

$$||\Phi(\theta, t)|| \leq Me^{\omega t} \tag{2.1}$$

for all $(\theta, t) \in \Theta \times \mathbf{R}_+$.

The mapping Φ is called *the cocycle* associated to the linear skew-product semiflow $\pi = (\Phi, \sigma)$.

Remark 2.4. If $\pi = (\Phi, \sigma)$ is a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$ then for every $\beta \in \mathbf{R}$ the pair $\pi_\beta = (\Phi_\beta, \sigma)$, where $\Phi_\beta(\theta, t) = e^{-\beta t}\,\Phi(\theta, t)$ for all $(\theta, t) \in \Theta \times \mathbf{R}_+$, is also a linear skew-product semiflow on $\mathcal{E}$.

Example 2.4. Let Θ be a locally compact metric space, let σ be a semiflow on Θ and let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a C_0 - semigroup on X. Then the pair $\pi_T = (\Phi_T, \sigma)$, where

$$\Phi_T(\theta, t) = T(t)$$

for all $(\theta, t) \in \Theta \times \mathbf{R}_+$, is a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$, which is called *the linear skew-product semiflow generated by the C_0 - semigroup $\mathbf{T}$ and the semiflow σ.*

Example 2.5. Let $\Theta = \mathbf{R}_+$, $\sigma(\theta, t) = \theta + t$ and let $\mathcal{U} = \{U(t, s)\}_{t\geq s\geq 0}$ be an evolution family on the Banach space X. We define

$$\Phi(\theta, t) = U(t+\theta, \theta)$$

for all $(\theta, t) \in \mathbf{R}_+^2$. Then $\pi = (\Phi, \sigma)$ is a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$ called *the linear skew-product semiflow generated by the evolution family $\mathcal{U}$ and the semiflow σ.*

Example 2.6. Let X be a Banach space, let Θ be a compact metric space and let $\sigma : \Theta \times \mathbf{R}_+ \to \Theta$ be a semiflow on Θ. Let $A : \Theta \to \mathcal{B}(X)$ be a continuous mapping. If $\Phi(\theta, t)$ denotes the solution of the linear differential system

$$\dot{u}(t) = A(\sigma(\theta, t))\,u(t), \quad t \geq 0$$

then the pair $\pi = (\Phi, \sigma)$ is a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$. In fact, these equations arise from the linearization of nonlinear equations (see e.g. [15] and the references therein).

Example 2.7. On the Banach space X, we consider the nonautonomous differential equation

$$\dot{x}(t) = a(t)\, x(t), \quad t \geq 0$$

where $a : \mathbf{R}_+ \to \mathbf{R}_+$ is an uniformly continuous function such that there exists $\alpha := \lim\limits_{t\to\infty} a(t) < \infty$.

Let $C(\mathbf{R}_+, \mathbf{R})$ be the space of all continuous functions $f : \mathbf{R}_+ \to \mathbf{R}$. This space is metrizable with the metric

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x,y)}{1 + d_n(x,y)}\,,$$

where $d_n(x,y) = \sup\limits_{t\in[0,n]} |x(t) - y(t)|$.

If we denote by $a_s(t) = a(t+s)$ and by $\Theta =$ closure $\{a_s : s \in \mathbf{R}_+\}$ then

$$\sigma : \Theta \times \mathbf{R}_+ \to \Theta, \quad \sigma(\theta,t)(s) := \theta(t+s)$$

is a semiflow on Θ,

$$\Phi : \Theta \times \mathbf{R}_+ \to \mathcal{B}(X), \quad \Phi(\theta,t)x = exp\,(\int_0^t \theta(\tau)\, d\tau)\, x$$

is a cocycle and hence $\pi = (\Phi, \sigma)$ is a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$.

Definition 2.3. A linear skew-product semiflow $\pi = (\Phi, \sigma)$ on $\mathcal{E} = X \times \Theta$ is called *uniformly exponentially stable* if there are $N \geq 1$ and $\nu > 0$ such that

$$||\Phi(\theta,t)|| \leq Ne^{-\nu t}$$

for all $(\theta, t) \in \Theta \times \mathbf{R}_+$.

Example 2.8. Consider the linear skew-product semiflow $\pi_\beta = (\Phi_\beta, \sigma)$, where

$$\Phi_\beta(\theta,t) = e^{-\beta t}\, \Phi(\theta,t), \quad \beta \in \mathbf{R}_+$$

and $\pi = (\Phi, \sigma)$ is the linear skew-product semiflow given in Example 2.7. It is easy to see that π_β is uniformly exponentially stable if and only if $\beta > \alpha$.

A sufficient condition for uniform exponential stability of linear skew-product semiflows is given by:

Proposition 2.2. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$. If there are $t_0 > 0$ and $c \in (0,1)$ such that*

$$||\Phi(\theta, t_0)|| \leq c,$$

for all $\theta \in \Theta$, then $\pi = (\Phi, \sigma)$ is uniformly exponentially stable.

Proof. Let $M \geq 1$ and $\omega > 0$ given by (2.1) and let $\nu > 0$ such that $c = e^{-\nu t_0}$.

Let $\theta \in \Theta$. For all $t \in \mathbf{R}_+$ there are $n \in \mathbf{N}$ and $r \in [0, t_0)$ such that $t = nt_0 + r$. Then we obtain:

$$||\Phi(\theta,t)|| \leq ||\Phi(\sigma(\theta, nt_0), r)||\, ||\Phi(\theta, nt_0)|| \leq Me^{\omega t_0}\, e^{-n\nu t_0} \leq Ne^{-\nu t},$$

where $N = Me^{(\omega+\nu)t_0}$. So π is uniformly exponentially stable.

3. The main results

In this section we shall give necessary and sufficient conditions for uniform exponential stability of linear skew-product semiflows in terms of Banach function spaces.

Theorem 3.1. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$. Then π is uniformly exponentially stable if and only if there exists a Banach function space $B \in \mathcal{E}(\mathbf{R}_+)$ such that:*

(i) for every $x \in X$ and $\theta \in \Theta$ the function

$$f_{\theta,x} : \mathbf{R}_+ \to \mathbf{R}_+, \quad f_{\theta,x}(t) = ||\Phi(\theta, t)x||$$

belongs to B;

(ii) there exists a function $K : X \to (0, \infty)$ such that

$$|f_{\theta,x}|_B \leq K(x),$$

for all $(x, \theta) \in \mathcal{E}$.

Proof. Necessity. Let $N, \nu > 0$ such that

$$||\Phi(\theta, t)|| \leq N\, e^{-\nu t},$$

for all $(\theta, t) \in \Theta \times \mathbf{R}_+$ and $B = L^p(\mathbf{R}_+, \mathbf{C})$, where $p \in [1, \infty)$. Then for every $(x, \theta) \in \mathcal{E}$ we have that

$$|f_{\theta,x}|_p \leq \frac{N}{(\nu\, p)^{1/p}}\, ||x||.$$

Sufficiency. Since $B \in \mathcal{E}(\mathbf{R}_+)$ there exists a strictly increasing sequence $(t_n) \subset (0, \infty)$ with $t_n \to \infty$ and

$$\delta = \sup_n\,(t_{n+1} - t_n) < \infty \quad \text{ and } \quad c = \inf_n\, |\chi_{[t_n, t_{n+1})}|_B > 0. \tag{3.1}$$

Let $n \in \mathbf{N}$ and $\theta \in \Theta$. For every $t \in [t_n, t_{n+1})$ we have that:

$$||\Phi(\theta, t_{n+1})x|| \leq M\, e^{\omega\, \delta}\, ||\Phi(\theta, t)x||$$

where M and ω are given by the relation (2.1). It follows that:

$$\chi_{[t_n, t_{n+1})}(t)\, ||\Phi(\theta, t_{n+1})x|| \leq M\, e^{\omega\, \delta}\, ||\Phi(\theta, t)x||$$

for all $t \geq 0$. Using the relation (3.1) and the hypothesis we obtain that:

$$c\, ||\Phi(\theta, t_{n+1})x|| \leq M\, e^{\omega\, \delta}\, K(x)$$

for every $(x, \theta, n) \in \mathcal{E} \times \mathbf{N}$. By the uniform boundedness principle it results that there exists $L_1 > 0$ such that

$$||\Phi(\theta, t_{n+1})|| \leq L_1$$

for all $(\theta, n) \in \Theta \times \mathbf{N}$.

Let $\theta \in \Theta$ and $t \geq t_1$. Then there exists an unique $n \in \mathbf{N}^*$ such that $t_n \leq t < t_{n+1}$. Hence we deduce that:

$$||\Phi(\theta, t)|| \leq ||\Phi(\sigma(\theta, t_n), t - t_n)||\, ||\Phi(\theta, t_n)|| \leq M\, e^{\omega\, \delta}\, L_1.$$

Denoting by $L = max\{M\, e^{\omega\,\delta}\, L_1, M\, e^{\omega\, t_1}\}$ we obtain that

$$||\Phi(\theta, t)|| \leq L,$$

for all $t \geq 0$ and $\theta \in \Theta$

Let $x \in X$ and $n \in N^*$. For $t \in [0, n]$ we have:

$$||\Phi(\theta, n)x|| \leq L\, ||\Phi(\theta, t)x||$$

so

$$\chi_{[0,n]}(t)\, ||\Phi(\theta, n)x|| \leq L\, ||\Phi(\theta, t)x||$$

for all $t \geq 0$. It follows that

$$F_B(n)\, ||\Phi(\theta, n)x|| \leq L\, |f_{\theta,x}|_B \leq L\, K(x).$$

By the uniform boundedness principle we obtain that there exists $K > 0$ such that

$$F_B(n)\, ||\Phi(\theta, n)|| \leq K,$$

for all $\theta \in \Theta$ and $n \in \mathbf{N}^*$.

Since $B \in \mathcal{E}(\mathbf{R}_+)$ there exists $n_0 \in \mathbf{N}$ with $F_B(n_0) > 2K$. Then we deduce that

$$||\Phi(\theta, n_0)|| \leq \frac{1}{2},$$

for all $\theta \in \Theta$.

Using Proposition 2.2. it follows that π is uniformly exponentially stable.

Remark 3.1. It is easy too see that in the particular case when the linear skew-product semiflow is given by Example 2.4., the theorem from above generalizes the theorem of Neerven [12], because the condition (ii) becomes trivial.

A theorem of Rolewicz's type, for linear skew-product semiflows is given by

Theorem 3.2. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$. Then π is uniformly exponentially stable if and only if there exist a non-decreasing function $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ and $K > 0$ such that:*

(i) $\varphi(0) = 0$ and $\varphi(t) > 0$, for all $t > 0$;

(ii) for every $(x, \theta) \in \mathcal{E}$ with $||x|| \leq 1$

$$\int_0^\infty \varphi(||\Phi(\theta, t)x||)\, dt \leq K.$$

Proof. Necessity. Let $\varphi(t) = t$, for all $t \geq 0$. Let $N, \nu \in (0, \infty)$ such that

$$||\Phi(\theta, t)|| \leq N\, e^{-\nu t},$$

for all $t \geq 0$ and $\theta \in \Theta$. Then for every $(x, \theta) \in \mathcal{E}$ with $||x|| \leq 1$ we have that

$$\int_0^\infty ||\Phi(\theta, t)x||\, dt \leq \frac{N}{\nu}.$$

Sufficiency. Let M, ω given by relation (2.1), $t_0 > 0$ such that $K < t_0\, \varphi(1)$ and $\delta = 1/\, M\, e^{\omega\, t_0}$.

Let $(x,\theta)\in\mathcal{E}$ with $||x||\leq 1$ and $t\geq t_0$. We have that:

$$||\Phi(\theta,t)\delta\, x||\leq||\Phi(\theta,u)x||,$$

for all $u\in[t-t_0,t]$.

Since φ is non-decreasing using the relation from above it follows that

$$t_0\,\varphi(||\Phi(\theta,t)\delta\, x||)\leq\int_{t-t_0}^{t}\varphi(||\Phi(\theta,u)x||)\,du\leq K.$$

Taking in account the way that t_0 was chosen, from the last inequality we obtain

$$||\Phi(\theta,t)\delta\, x||\leq 1,$$

for all $t\geq t_0$ and $\theta\in\Theta$, so

$$||\Phi(\theta,t)||\leq\frac{1}{\delta},\tag{3.2}$$

for all $t\geq t_0$ and $\theta\in\Theta$. Denoting by $L=\frac{1}{\delta}+Me^{\omega t_0}$ and using the relation (3.2) it follows that:

$$||\Phi(\theta,t)||\leq L,$$

for all $t\geq 0$ and $\theta\in\Theta$.

Without lost of generality we can suppose that φ is left-continuous (if not we consider the function $\tilde{\varphi}(t)=\lim\limits_{s\nearrow t}\varphi(s)$, for $t>0$ and the proof is the same).

Let L_φ be the Orlicz space associated to φ. For every $(x,\theta)\in\mathcal{E}$ let

$$f_{\theta,x}:\mathbf{R}_+\to\mathbf{R}_+,\quad f_{\theta,x}(t)=||\Phi(\theta,t)x||.$$

If $x\in X\setminus\{0\}$ and $\tilde{x}=\frac{x}{(K+1)\,L\,||x||}$ we have:

$$Y_\varphi(f_{\theta,\tilde{x}}(t))=Y_\varphi(||\Phi(\theta,t)\tilde{x}||)\leq||\Phi(\theta,t)\tilde{x}||\,\varphi(||\Phi(\theta,t)\tilde{x}||)\leq$$

$$\leq\frac{1}{K+1}\,\varphi(||\Phi(\theta,t)\tilde{x}||).$$

It follows that

$$M_\varphi(f_{\theta,\tilde{x}})<1$$

so $f_{\theta,\tilde{x}}\in L_\varphi$ and $|f_{\theta,\tilde{x}}|_\varphi\leq 1$. Because L_φ is a linear space and $f_{\theta,\tilde{x}}=\frac{1}{(K+1)\,L\,||x||}\,f_{\theta,x}$, it results that $f_{\theta,x}\in L_\varphi$ and

$$|f_{\theta,x}|_\varphi\leq(K+1)\,L\,||x||.$$

By applying Proposition 2.2. and Theorem 3.1. we conclude that π is uniformly exponentially stable.

Remark 3.2. If in Theorem 3.2. $\varphi(t)=t^2$ and π is given by Example 2.5.,then we obtain the theorem of Datko ([6]).

Remark 3.3. Another proofs for the theorems from this section are presented in [10], where the results are obtained for discrete-time cases and the techniques involved are based on Banach sequence spaces.

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