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# RESULTS ON QUASISTATIC ANTIPLANE CONTACT PROBLEMS WITH SLIP DEPENDENT FRICTION

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**Abstract.** In this paper we present same recent results on quasistatic antiplane contact problems, where general versions of Tresca's friction law are considered.
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## 1. Introduction

This paper is a survey on our recent results on quasistatic antiplane contact problems, where general versions of Tresca's friction law (see [1] for details) are considered. First, we recall in Section 2 an abstract result on evolution variational inequalities obtained in [4], then we apply it in Section 3 in the study of an elastic contact problem with slip dependent friction and provide a result obtained in [3]. Further, in Section 4 we slightly generalize a result obtained in [2] which expresses the convergence of the viscoelastic solution to the solution of the elastic problem studied in Section 3.

## 2. An abstract existence and uniqueness result in [4]

In this section we recall an existence and uniqueness result which was established in [4] in the study of the following evolution problem.

**Problem $P$.** *Find $u : [0,T] \to V$ such that*

$$a(u(t), v - \dot{u}(t)) + j(u(t), v) - j(u(t), \dot{u}(t)) \geq (f(t), v - \dot{u}(t))_V \quad \forall v \in V,\ a.e.\ t \in (0,T),$$
$$u(0) = u_0.$$

Here, $V$ denotes a real Hilbert space and suppose that

$(i_1)$ $a : V \times V \to I\!R$ is a bilinear, continuous, symmetric form for which there exists $m > 0$ such that $a(v, v) \geq m\|v\|_V^2,\ \forall v \in V$.

$(i_2)$ $j : V \times V \to I\!R$ is positively homogeneous and subadditive with respect to the second argument.

$(i_3)$ $f \in W^{1,\infty}(0,T;V)$, $u_0 \in V$, $a(u_0,v) + j(u_0,v) \geq (f(0),v)_V \quad \forall v \in V$.

Consider the properties below.

$(j_1)$ For every sequence $\{u_n\} \subset V$ with $\|u_n\|_V \to \infty$, every sequence $\{t_n\} \subset [0,1]$ and each $\overline{u} \in V$, one has

$$\liminf_{n\to\infty} \Big[\frac{1}{\|u_n\|_V^2} j_2'(t_n u_n, u_n - \overline{u}; -u_n)\Big] < m.$$

$(j_2)$ For every sequence $\{u_n\} \subset V$ with $\|u_n\|_V \to \infty$, every bounded sequence $\{\eta_n\} \subset V$ and each $\overline{u} \in V$ one has

$$\liminf_{n\to\infty} \Big[\frac{1}{\|u_n\|_V^2} j_2'(\eta_n, u_n - \overline{u}; -u_n)\Big] < m.$$

$(j_3)$ For all sequences $\{u_n\} \subset V$ and $\{\eta_n\} \subset V$ such that $u_n \rightharpoonup u \in V$, $\eta_n \rightharpoonup \eta \in V$ weakly in $V$ and for every $v \in V$, the inequality below holds

$$\limsup_{n\to\infty} \,[j(\eta_n, v) - j(\eta_n, u_n)] \leq j(\eta, v) - j(\eta, u).$$

$(j_4)$ There exists $c_0 \in (0,m)$ such that

$$j(u, v-u) - j(v, v-u) \leq c_0 \|u - v\|_V^2 \quad \forall u, v \in V.$$

$(j_5)$ There exist two functions $a_1 : V \to I\!R$ and $a_2 : V \to I\!R$ which map bounded sets in $V$ into bounded sets in $I\!R$ such that $a_1(0_V) < m - c_0$ and

$$|j(\eta, u)| \leq a_1(\eta)\|u\|_V^2 + a_2(\eta) \quad \forall \eta, u \in V.$$

$(j_6)$ For every sequence $\{\eta_n\} \subset V$ with $\eta_n \rightharpoonup \eta \in V$ weakly in $V$ and every bounded sequence $\{u_n\} \subset V$ one has $\lim\limits_{n\to\infty}[j(\eta_n, u_n) - j(\eta, u_n)] = 0$.

$(j_7)$ For every $s \in (0,T]$ and every functions $u, v \in W^{1,\infty}(0,T;V)$ with $u(0) = v(0)$, $u(s) \neq v(s)$, the inequality below holds

$$\int_0^s [j(u(t), \dot{v}(t)) - j(u(t), \dot{u}(t)) + j(v(t), \dot{u}(t)) - j(v(t), \dot{v}(t))]dt < \frac{m}{2}\|u(s) - v(s)\|_V^2.$$

$(j_8)$ There exists $\alpha \in (0, \frac{m}{2})$ such that for every $s \in (0,T]$ and every functions $u, v \in W^{1,\infty}(0,T;V)$ with $u(s) \neq v(s)$, the inequality below holds

$$\int_0^s [j(u(t), \dot{v}(t)) - j(u(t), \dot{u}(t)) + j(v(t), \dot{u}(t)) - j(v(t), \dot{v}(t))]dt < \alpha\|u(s) - v(s)\|_V^2.$$

In $(j_1)$-$(j_2)$, $j_2'$ denotes the directional derivative with respect to the second variable, i.e.

$$j_2'(\eta, u; v) = \lim_{\lambda\to 0^+} \frac{1}{\lambda}\Big[j(\eta, u + \lambda v) - j(\eta, u)\Big] \quad \forall \eta,\ u,\ v \in V,$$

which exists since $j(\eta, \cdot) : V \to I\!R$ is a convex functional for all $\eta \in V$.

In the study of Problem $P$ the following result was obtained.

**Theorem 1.** (D. Motreanu and M. Sofonea [4]) *Assume* $(i_1)$-$(i_3)$.

(*i*) *If* $(j_1)$-$(j_6)$ *hold then there exists at least a solution* $u \in W^{1,\infty}(0,T;V)$ *to Problem*

*P.*

*(ii) If* $(j_1)$-$(j_7)$ *hold then there exists a unique solution* $u \in W^{1,\infty}(0,T;V)$ *to Problem* $P$.

*(iii) Under the assumptions* $(j_1)$-$(j_6)$ *and* $(j_8)$ *there exists a unique solution* $u = u(f,u_0) \in W^{1,\infty}(0,T;V)$ *to Problem* $P$ *and the mapping* $(f,u_0) \longmapsto u$ *is Lipschitz continuous from* $W^{1,\infty}(0,T;V) \times V$ *to* $L^\infty(0,T;V)$.

The proof of Theorem 1 is based on a time discretization method. We resume here the main ingredients of the proof: first, Problem $P$ is replaced by a sequence of quasivariational inequalities which have a unique solution; then, the discrete solution is interpolated in time and, using compactness and lower semicontinuity arguments, the existence of a solution to Problem $P$ is derived; the uniqueness of the solution as well as its Lipschitz continuous dependence with respect to the data is proved by using Gronwall-type arguments.

## 3. Application of Theorem 1 to an antiplane problem

The rest of the paper deals with antiplane contact problems, specifically the contact between a cylinder and a rigid foundation. The cylinder is supposed to have the generators sufficiently long, parallel with the $x_3$-axis of a fixed Cartesian coordinate system $Ox_1x_2x_3$ in $\mathbb{R}^3$ with a regular, bounded cross-section $\Omega$ in the $x_1, x_2$-plane. The boundary $\Gamma$ of $\Omega$ is divided into three disjoint measurable parts $\Gamma_1, \Gamma_2, \Gamma_3$ with $|\Gamma_1| > 0$. The body is fixed on $\Gamma_1 \times (-\infty, +\infty)$. The contact between the cylinder and the foundation is frictional, bilateral on $\Gamma_3 \times (-\infty, +\infty)$.

Assume that in the time interval $[0,T]$ the cylinder is submitted to volume forces of density $\boldsymbol{f}_0 = (0,0,f_0) : \Omega \times (0,T) \to \mathbb{R}^3$ and surface tractions of density $\boldsymbol{f}_2 = (0,0,f_2) : \Gamma_2 \times (0,T) \to \mathbb{R}^3$. The forces give rise to a deformation of the cylinder whose displacement $\boldsymbol{u}$ is parallel to the generators, independent on the axial coordinate, i.e. $\boldsymbol{u} = (0,0,u)$, with $u : \Omega \times (0,T) \to \mathbb{R}$. Denote $\boldsymbol{\nu}$ the unit normal on $\Gamma \times (-\infty, +\infty)$. We have $\boldsymbol{\nu} = (\nu_1, \nu_2, 0)$, with $\nu_1, \nu_2 : \Gamma \to \mathbb{R}$. We use the notation $\partial_\nu u = (\partial u/\partial x_1)\nu_1 + (\partial u/\partial x_2)\nu_2$.

Suppose now the cylinder elastic, homogeneous, isotropic, then it follows the law $\boldsymbol{\sigma} = \lambda(\mathrm{tr}\, \boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u})$, where $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ is the infinitesimal strain tensor, that is $\varepsilon_{ij}(\boldsymbol{u}) = (1/2)(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$, $i,j = 1,2,3$, $\mathrm{tr}\, \boldsymbol{\varepsilon}(\boldsymbol{u}) = \varepsilon_{ii}(\boldsymbol{u})$, $\boldsymbol{I}$ is the unit tensor in $\mathbb{R}^3$, $\lambda > 0$ and $\mu > 0$ are the Lamé coefficients. The law permits to determine the stress field $\boldsymbol{\sigma}$ when the displacement $\boldsymbol{u}$ is known and to consider the following contact problem.

**Problem** $P_0$**.** *Find the displacement field* $u : \Omega \times [0, T] \to I\!R$ *such that*

$$
\begin{array}{l}
\mu\Delta u + f_0 = 0 \quad \text{on } \Omega \times (0,T),\\
u = 0 \quad \text{on } \Gamma_1 \times (0,T),\\
\mu\, \partial_\nu u = f_2 \quad \text{on } \Gamma_2 \times (0,T),\\
\partial_\nu u \le 0 \Rightarrow \left\{ \begin{array}{l} \mu\partial_\nu u \ge g_1(|u|)\\ \mu\partial_\nu u > g_1(|u|) \Rightarrow \dot{u} = 0\\ \mu\partial_\nu u = g_1(|u|) \Rightarrow \exists\, \beta > 0 \text{ a.e. on } \Gamma_3 \text{ such that } \mu\partial_\nu u = -\beta\dot{u} \end{array}\right.\\
\partial_\nu u \ge 0 \Rightarrow \left\{ \begin{array}{l} \mu\partial_\nu u \le g_2(|u|)\\ \mu\partial_\nu u < g_2(|u|) \Rightarrow \dot{u} = 0\\ \mu\partial_\nu u = g_2(|u|) \Rightarrow \exists\, \beta > 0 \text{ a.e. on } \Gamma_3 \text{ such that } \mu\partial_\nu u = -\beta\dot{u}\\ \hfill \text{on } \Gamma_3 \times (0,T), \end{array}\right.\\
u(0) = u_0 \quad \text{on } \Omega.
\end{array}
$$

Here $u_0$ is given and $g_1, g_2 : \Gamma_3 \times I\!R_+ \to I\!R$ are assumed to satisfy

$$
\left\{ \begin{array}{l} g_1(x,r) \le 0,\ g_2(x,r) \ge 0 \quad \text{a.e. } x \in \Gamma_3,\ \forall r \in I\!R_+,\\ g_i(\cdot, r) \text{ is Lebesgue measurable on } \Gamma_3 \ \ \forall r \in I\!R_+, \quad g_i(\cdot, 0) \in L^2(\Gamma_3),\\ |g_i(x,r_1) - g_i(x,r_2)| \le L_i|r_1 - r_2| \quad \text{a.e. } x \in \Gamma_3,\ \forall r_1, r_2 \in I\!R_+, \end{array}\right. \tag{1}
$$

for some positive constants $L_i$, where $i = 1, 2$.

Consider the Hilbert space

$$V = \{v \in H^1(\Omega) \,|\, v = 0 \ \text{ on } \ \Gamma_1\}, \quad (u, v)_V = \int_\Omega \nabla u \cdot \nabla v \, dx \quad \forall u, v \in V.$$

By Sobolev's trace theorem we find a constant $C_0 = C_0(\Omega, \Gamma_1, \Gamma_3) > 0$ such that

$$\|v\|_{L^2(\Gamma_3)} \le C_0\|v\|_V \quad \forall v \in V.$$

In view of (1), let the functional $j : V \times V \to I\!R$ defined by

$$j(u, v) = \int_{\Gamma_3} [g_2(|u|)v^- - g_1(|u|)v^+]\, da \quad \forall u, v \in V, \tag{2}$$

where $v^+ = \max\{v, 0\}$, $v^- = \max\{-v, 0\}$. Assume that

$$f_0 \in W^{1,\infty}(0, T; L^2(\Omega)), \quad f_2 \in W^{1,\infty}(0, T; L^2(\Gamma_2)), \tag{3}$$

$$u_0 \in V, \quad \mu(u_0, v)_V + j(u_0, v) \ge (f(0), v)_V \quad \forall v \in V. \tag{4}$$

By Riesz's representation theorem, let the function $f : [0, T] \to V$ given by

$$(f(t), v)_V = \int_\Omega f_0(t)v\, dx + \int_{\Gamma_2} f_2(t)v\, da \quad \forall v \in V,\ t \in [0, T].$$

We are led to the following weak formulation of Problem $P_0$.

**Problem** $P_0'$**.** *Find a displacement field* $u : [0, T] \to V$ *such that*

$$
\left\{ \begin{array}{l} \mu(u(t), v - \dot{u}(t))_V + j(u(t), v) - j(u(t), \dot{u}(t)) \ge (f(t), v - \dot{u}(t))_V\\ \hfill \forall v \in V,\ \text{a.e. } t \in (0, T),\\ u(0) = u_0. \end{array}\right.
$$

We have the following result.

**Theorem 2.** [3] *Suppose that conditions (3) and (4) hold.*

*(i)′ Under the assumption (1), if in addition $L_1 + L_2 < \mu/C_0^2$, then there exists at least a solution u for Problem $P_0'$, which satisfies $u \in W^{1,\infty}(0,T;V)$.*

*(ii)′ If $g_1 : \Gamma_3 \to I\!R_-$ and $g_2 : \Gamma_3 \to I\!R_+$, $g_1, g_2 \in L^2(\Gamma_3)$, there exists a unique solution $u \in W^{1,\infty}(0,T;V)$ for Problem $P_0'$. Moreover, the mapping $(f, u_0) \longmapsto u$ is Lipschitzian from $W^{1,\infty}(0,T;V) \times V$ to $L^\infty(0,T;V)$.*

**Proof.** (Sketch) Note that conditions $(i_1)$-$(i_3)$ are satisfied for the data entering Problem $P_0'$ (see (3) and (4)). The hypotheses imposed on functions $g_1$, $g_2$ in part $(i)'$ of Theorem 2 imply that conditions $(j_1)$-$(j_6)$ are verified for the functional $j$ in (2) and allow the application of Theorem 1, $(i)$. The particular assumption in part $(ii)'$ shows directly that $(j_1)$-$(j_8)$ are satisfied, so parts $(ii)$ and $(iii)$ in Theorem 1 can be applied. □

## 4. A convergence result

In this section we see that in a particular situation for $g_1, g_2$ the solution of Problem $P_0'$ (that is the weak solution of the elastic Problem $P_0$) can be obtained as a limit of weak solutions of viscoelastic problems. For each $\theta > 0$ consider the following problem.

**Problem $P_\theta$.** *Find a displacement field $u_\theta : [0,T] \to V$ such that*

$$\begin{cases} \theta(\dot u_\theta(t), v - \dot u_\theta(t))_V + \mu(u_\theta(t), v - \dot u_\theta(t))_V + j(\int_0^t |\dot u_\theta(s)|ds, v) \\ -j(\int_0^t |\dot u_\theta(s)|ds, \dot u_\theta(t)) \geq (f(t), v - \dot u_\theta(t))_V \quad \forall v \in V,\ a.e.\ t \in (0,T), \\ u_\theta(0) = u_0. \end{cases}$$

Problem $P_\theta$ arises as the variational formulation of a mechanical problem which is analogous to Problem $P_0$ with two differences: the law is viscoelastic, that is $\boldsymbol{\sigma} = 2\theta\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \lambda(\text{tr}\ \boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u})$, and in the boundary condition on $\Gamma_3 \times (0,T)$ one sets $g := g_2 = -g_1$ and $|u|$ is replaced by $\int_0^t |\dot u(s)|ds$. The existence and uniqueness of the solution $u_\theta \in W^{1,\infty}(0,T;V)$ to Problem $P_\theta$ is provided in [2] and hold under the following hypotheses: assumption (1) becomes a corresponding condition for $g$, (3) is replaced by $f_0 \in L^\infty(0,T;L^2(\Omega))$, $f_2 \in L^\infty(0,T;L^2(\Gamma_2))$ and in place of (4) we take $u_0 \in V$. The argument relies on Banach's fixed point theorem in the space $L^\infty(0,T;V)$.

Our convergence result is now stated. If $g \in L^\infty(\Gamma_3)$ one obtains the corresponding result in [2].

**Theorem 3.** *Assume $g : \Gamma_3 \to I\!R_+$, $g \in L^2(\Gamma_3)$, (3) and (4) verified. Then $u_\theta \to u$ in $C([0,T];V)$ as $\theta \to 0^+$, where $u_\theta, u$ are the solutions to Problems $P_\theta, P_0'$, respectively, for $j : V \to I\!R$, $j(v) = \int_{\Gamma_3} g|v|da$, $\forall v \in V$.*

**Proof.** (Sketch) Using the fact that $u_\theta, u \in W^{1,\infty}(0,T;V)$ are the unique solutions to Problems $P_\theta, P_0'$, respectively, implies

$$\mu \frac{d}{dt} \|u(t) - u_\theta(t)\|_V^2 \leq 2\theta(\dot u_\theta(t), \dot u(t) - \dot u_\theta(t))_V \qquad \text{a.e. } t \in (0,T).$$

By integration we are led to

$$\mu \|u_\theta(s) - u(s)\|_V^2 \leq \frac{\theta}{2} \int_0^T \|\dot{u}(t)\|_V^2 dt \quad \forall s \in [0, T],$$

which yields the result. □

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