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MULTIPLE SOLUTIONS FOR NEUMANN PROBLEM WITH P-LAPLACIAN

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Abstract. In this paper we prove that the Neumann problem with p-Laplacean:

$$(\mathcal{P})\begin{cases} -\Delta_p u + |u|^{p-2}u = f(x,u), \text{ in}\,\Omega \\ |\bigtriangledown u|^{p-2}\frac{\partial u}{\partial n} = 0, \text{ on } \partial\Omega \end{cases}$$

has an unbounded sequence of solutions in $W^{1,p}(\Omega)$,$1 < p < \infty$, using a multiple version of the "Mountain Pass" theorem.

1. Introduction and preliminary results

Let Ω be an open bounded subset in $\mathbf{R}^N$, $N\geq 2$, with smooth boundary, $1 < p < \infty$, $f : \Omega \text{ x } \mathbf{R} \to \mathbf{R}$ be a Caratheodory function which satisfies the growth condition:

$$|f(x,s)| \leq c(|s|^{q-1}+1), a.e.x \in \Omega, (\forall)s \in \mathbf{R}, \tag{1}$$

where $c \geq 0$ is constant, $1 < q < p^* = \begin{cases} \frac{Np}{N-p}, \text{ if } p < N \\ +\infty, \text{ if } p \geq N \end{cases}$.

We consider the Neumann problem $(\mathcal{P})$, where Δ_p is the p-Laplacian operator defined by

$$\Delta_p \text{u} = \text{div}(\text{—}\nabla\text{u—}^{p-2}\nabla\text{u}) \text{ and } \frac{\partial u}{\partial n} = \nabla u \cdot n$$

We shall use the standard notation:

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{\partial u}{\partial x_i} \in L^p(\Omega), i \in \overline{1,N} \right\}$$

equipped with the norm

$$||u||_{1,p}^p = ||u||_{0,p}^p + \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{0,p}^p$$

where $\|\cdot\|_{0,p}$ is the usual norm on $L^p(\Omega)$.

We define a new equivalent norm on the space $W^{1,p}(\Omega)$:

$$|||u|||_{1,p}^p = ||u||_{0,p}^p + ||\nabla u||_{0,p}^p = \int\limits_\Omega |u|^p + \int\limits_\Omega \left(\sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \right)^2 \right)^{p/2}$$

Then the space $(W^{1,p}(\Omega), |||\cdot|||_{1,p})$ is separable, reflexive and uniformly convex Banach space.

The dual norm on $(W^{1,p}(\Omega), |||\cdot|||_{1,p})^*$ is denoted by $|||\cdot|||_{*}$..

The operator -Δ_p may be seen acting from $W^{1,p}(\Omega)$ into $(W^{1,p}(\Omega))^*$ by

$$< -\Delta_p u, v >= \int\limits_\Omega |\nabla u|^{p-2} \nabla u \nabla v, \text{for all } u, v \in W^{1,p}(\Omega)$$

Definition 1. A function u $\in W^{1,p}(\Omega)$ is said to be a solution for the problem $(\mathcal{P})$ iff

$$\int\limits_\Omega |\nabla u|^{p-2} \nabla u \nabla v + \int\limits_\Omega |u|^{p-2} uv = \int\limits_\Omega f(x,u)v, \text{ for all } v \in W^{1,p}(\Omega)$$

If $u \in W^{1,p}(\Omega)$ and $\Delta_p u \in L^{p'}(\Omega)$ we can speak about $|\nabla u|^{p-2} \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega}$ and $|\nabla u|^{p-2} \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} \in W^{-\frac{1}{p'},p'}(\partial\Omega)$ (see e.g.[6]).

Let $\Psi : L^q(\Omega) \to \mathbf{R}$ be defined by

$$\Psi(u) = \int\limits_\Omega F(x,u), \text{ where } F(x,s) = \int_0^s f(x,\tau) d\tau$$

The function F is Caratheodory and

$$|F(x,s)| \leq c_1(|s|^q + 1), \text{ a.e. } x \in \Omega, (\forall)\ s \in \mathbf{R} \tag{2}$$

where $c_1 \geq 0$ is constant.

The functional Ψ is continuously Frechet differentiable on $L^q(\Omega)$ and Ψ'(u)=N$_f$(u), for all $u \in L^q(\Omega)$, where N_f is the Nemytskii operator of f:

$$N_f(u)(x) = f(x, u(x)), \text{ a.e. } x \in \Omega$$

Let $\varphi : [0,\infty) \to \mathbf{R}$ be a normalization function defined by $\varphi(t) = t^{p-1}$ and

$$J_\varphi : W^{1,p}(\Omega) \to \mathcal{P}((W^{1,p}(\Omega))^*)$$

be the duality mapping corresponding to φ.

Then $J_\varphi u = \partial\phi(u)$ for all $u \in W^{1,p}(\Omega)$ (see [5]) where

$$\phi(u) = \int\limits_0^{|||u|||_{1,p}} \varphi(t) dt = \frac{1}{p} |||u|||_{1,p}^p$$

and $\partial\phi$ is the subdifferential of φ in the sense of convex analysis.

The functional ϕ is convex continuously Frechet differentiable on $W^{1,p}(\Omega)$ and ϕ'$(u) = -\Delta_p u + |u|^{p-2}u$, for all $u \in W^{1,p}(\Omega)$.

So J_φ is single valued and

$$J_\varphi u = \phi'(u) = -\Delta_p u + |u|^{p-2}u, \text{ for all } u \in W^{1,p}(\Omega).$$

Then the Euler-Lagrange functional $\mathcal{F} : W^{1,p}(\Omega) \to \mathbf{R}$,

$$\mathcal{F}(u) = \phi(u) - \varphi(u) = \frac{1}{p}|||u|||^p_{1,p} - \int_\Omega F(x,u) \text{ is } C^1 \text{ in } W^{1,p}(\Omega)$$

and

$$\mathcal{F}'(u) = \phi'(u) - \varphi'(u) = -\Delta_p u + |u|^{p-2}u - N_f(u).$$

If $u \in W^{1,p}(\Omega)$ is a critical point for $\mathcal{F}$, that is $\mathcal{F}\,'(u) = 0$, then $\Delta_p u + |u|^{p-2}u = N_f(u)$ and consequently u is solution for the problem $(\mathcal{P})$.

In order to show that the functional $\mathcal{F}$ has an unbounded sequence of critical points we use a multiple version of the "Mountain Pass" theorem (see e.g. Theorem 9.12 in [7]).

Theorem 1.1. Let X be an infinite dimensional real Banach space and let $f \in C^1(X, \mathbf{R})$ be even, satisfy (P.S.) condition. Suppose $f(0) = 0$ and :

(i) there are constants $\rho, \alpha > 0$ such that $f\big|_{||x||=\rho} \geq \alpha$.

(ii) for each finite dimensional subspace X_1 of X the set $\{x \in X : f(x) \geq 0\}$ is bounded. Then f possesses an unbounded sequence of critical values.

We recall that the functional $f \in C^1(X, \mathbf{R})$ satisfies the Palais-Smale condition (P.S.) if for every sequence $(u_n) \subset X$ with $(f(u_n))$ bounded and $f'(u_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence.

Since $W^{1,p}(\Omega)$ is uniformly convex and J_φ is single valued then J_φ satisfies the(S_+)condition: if $u_n \rightharpoonup u$ (weakly in $W^{1,p}(\Omega)$) and

$$\lim_{n\to\infty} sup < J_\varphi u_n, u_n - u > \leq 0, \text{ then } u_n \to u \text{ (see e.g. [5], Proposition2).}$$

2. Existence result

We need the following result:

Proposition 2.1. Suppose the Caratheodory function $f : \Omega \text{ x } \mathbf{R} \to \mathbf{R}$ satisfies:

(i) the growth condition (1)

(ii) there are numbers $\theta > p$ and $s_0 > 0$ such that

$$0 \leq \theta F(x,s) \leq sf(x,s), \text{ for a.e. } x \in \Omega \text{ , } (\forall) \ |s| \geq s_0. \tag{3}$$

Then, if X_1 is a finite dimensional subspace of $W^{1,p}(\Omega)$ the set $S = \{u \in X_1 : \mathcal{F}(u) \geq 0\}$ is bounded in $W^{1,p}(\Omega)$.

Proof. From (3) there is $\gamma \in L^\infty(\Omega)$, $\gamma > 0$ on Ω (see[5]), such that

$$F(x,s) \geq \gamma(x)|s|^\theta, \text{ a.e. } x \in \Omega,, \ (\forall) \ |s| \geq s_0. \tag{4}$$

For $u \in W^{1,p}(\Omega)$ let us denote

$$\Omega_1(u) = \{x \in \Omega : |u(x)| \geq s_0\}, \Omega_2(u) = \Omega \setminus \Omega_1(u) \ .$$

By (2) we have

$$\left|\int\limits_{\Omega_2(u)} F(x,u)\right| \le \int\limits_{\Omega_2(u)} |F(x,u)| \le \int\limits_{\Omega_2(u)} c_1(|u|^q+1) \le c_1\int\limits_{\Omega} s_0^q + \int\limits_{\Omega} c_1 = c_1(s_0^q+1)vol\ \Omega = k_1$$

and using (4) we have

$$\begin{aligned}\mathcal{F}(u) &= \frac{1}{p}|||u|||_{1,p} - \int\limits_{\Omega_1(u)} F(x,u) - \int\limits_{\Omega_2(u)} F(x,u) \le \\ &\le \frac{1}{p}|||u|||_{1,p}^p - \int\limits_{\Omega_1(u)} \gamma(x)|u|^\theta + k_1 = \\ &= \frac{1}{p}|||u|||_{1,p}^p - \int\limits_{\Omega} \gamma(x)|u|^\theta + \int\limits_{\Omega_2(u)} \gamma(x)|u|^\theta + k_1 \le \\ &\le \frac{1}{p}|||u|||_{1,p}^p - \int\limits_{\Omega} \gamma(x)|u|^\theta + k_2\end{aligned} \tag{5}$$

where $k_2 = ||\gamma||_\infty s_0^2 vol\ \Omega + k_1$.

The functional $||\cdot||_\gamma : W^{1,p}(\Omega) \to \mathbf{R}$, defined by

$$||u||_\gamma = \left(\int\limits_{\Omega} \gamma(x)|u|^\theta\right)^{\frac{1}{\theta}} \text{ is a norm on } W^{1,p}(\Omega).$$

On the finite dimensional subspace X_1 the norms $|||\cdot|||_{1,p}$ and $||\cdot||_\gamma$ being equivalent, there is a constant $\tilde{k} = \tilde{k}(X_1) > 0$ such that$|||u|||_{1,p} \le \tilde{k}\left(\int\limits_{\Omega}\gamma(x)|u|^\theta\right)^{\frac{1}{\theta}}$ for all $u \in X_1$.

Consequently, by (5), on X_1 it holds:

$$\begin{aligned}\mathcal{F}(u) &\le \frac{1}{p}\tilde{k}^p\left(\int\limits_{\Omega}\gamma(x)|u|^\theta\right)^{\frac{1}{\theta}} - \int\limits_{\Omega}\gamma(x)|u|^\theta + k_2 = \\ &= \frac{1}{p}\tilde{k}^p||u||_\gamma^p - ||u||_\gamma^\theta + k_2.\end{aligned}$$

Therefore $\frac{1}{p}\tilde{k}^p||u||_\gamma^p - ||u||_\gamma^\theta + k_2 \ge 0$ fol all $u \in S$ and since $\theta > p$ we conclude that S is bounded in $W^{1,p}(\Omega)$ (in the norme $||\cdot||_\gamma$ and so in the norme $|||\cdot|||_{1,p}$).

Now, we can state

Theorem 2.1. Suppose the Caratheodory functions $f : \Omega\text{x}\mathbf{R} \to \mathbf{R}$ is odd in the second argument : $f(x,s) = -f(x,-s)$ and satisfies:

(i) there is $q \in (1,p^*)$ such that

$|f(x,s)| \le\ c(|s|^{q-1}+1)$, a.e. $x \in \Omega$, $(\forall)\ s \in \mathbf{R}$

(ii) $\limsup_{s\to 0} \frac{f(x,s)}{|s|^{p-2}s} < \lambda_1$ uniformly with a.e. $x \in \Omega$,
where $\lambda_1 = \inf \left\{ \frac{|||v|||^p_{1,p}}{||v||^p_{0,p}} : v \in W^{1,p}(\Omega), v \neq 0 \right\}$.
(iii) there are constants $\theta > p$ and $s_0 > 0$ such that
$0 < \theta F(x,s) \leq \; sf(x,s)$ for a.e. $x \in \Omega, (\forall)\ |s| \geq s_0$
Then the problem $(\mathcal{P})$ has an unbounded sequence of solutions.

Proof. It's enough to show that $\mathcal{F}$ has an unbounded sequence of critical points in $W^{1,p}(\Omega)$.

For this we shall use the theorem 1.1.

Clearly $\mathcal{F}(0) = 0$ and $\mathcal{F}$ is even since f is odd.

By (i), (ii) and (iii) and proposition 2.1 it results that $\mathcal{F}$ satisfies the (PS) condition and hypothesis (i) and (ii) of the theorem 1.1.

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