Seminar on Fixed Point Theory Cluj-Napoca, Volume 3, 2002, 3-12
http://www.math.ubbcluj.ro/~nodeacj/journal.htm

PLANE ORBITS FOR SYNGE'S ELECTROMAGNETIC TWO BODY PROBLEM (I) A ZERO OF A PROPER MAPPING

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Abstract. The main purpose of the present paper is to formulate Kepler problem for two charged particles as a consequence of Synge's equation of motion. We show also an existence of circular orbits.
Keywords: Synge's equations, plane orbit.
AMS Subject Classification: 70F05.

1. Introduction

At the very beginning of 20-th century conventional physics was completely unable to account for the observed existence of stable atoms whose electrons remain at great distance from their respective nuclei (cf. for instance [1]). N. Bohr (1918) in his atom model postulated discrete stationary states even though this violates the classical electrodynamics. Irrespective of the development of quantum mechanics J. L. Synge [2] (cf. also [3]) formulated two-body problem of classical electrodynamics using Lienard-Wiechert retarded potentials (cf. [4]). So, for the first time, he took into account the finite velocity of propagation of interaction in the equations of motion - the basic assumption of Einstein special relativity theory. Synge's result was based on previous ones due to W. Pauli [5], who succeeded to express the Lorentz pondermotive force in a relativistic form. Not until 1963 Driver [6] recognized the one-dimensional case of the two-body problem as a system of functional differential equations with delays depending on the unknown trajectories. It turned out that from the point of view of modern theory of functional differential equations (cf. [7], [8]) Synge's equations (3-dimensional case) form a nonlinear system of neutral type with respect to unknown velocities. By fixed point approach sufficient conditions for the existence and uniqueness of escape trajectories have been formulated (cf. [9],[10]).

The paper consists of six sections. Section 2 is devoted to the equations of motion namely the Synge's equations. They are 8 in number while the unknown trajectories are 6 in number. First it is shown that 2 of equations are implied by the rest ones. So we have to consider a system of 6 equations for 6 unknown functions. In section 3 equations of motion for two-dimensional case in polar coordinates are given. Section

4 treats one-dimensional case. The result obtained confirms those of R.D.Driver [6]. Section 5 is devoted to two-dimensional case. The system of equations of motion is presented as a second order one. It is easy to check it has circle orbits, that is, $\rho = const$ with constant angular velocity $\dot{\varphi} = const$. This system is presented in equivalent form as a first order one.

2. J.L. Synge's equations of motion

As in [9] we denote by $x^{(p)} = (x_1^{(p)}(t), x_2^{(p)}(t), x_3^{(p)}(t), x_4^{(p)}(t) = ict)(p = 1,2)(i^2 = -1)$ the space-time coordinates of the moving particles, by m_p - their proper masses, by e_p - their charges, c - the speed of light. The coordinates of the velocity vectors are $u^{(p)} = (u_1^{(p)}(t), u_2^{(p)}(t), u_3^{(p)}(t))(p = 1,2)$. The coordinates of the unit tangent vectors to the world-lines are (cf. [2], [3]):

$$\lambda_\alpha^{(p)} = \frac{\gamma_p u_\alpha^{(p)}(t)}{c} = \frac{u_\alpha^{(p)}(t)}{\Delta_p}(\alpha = 1,2,3), \lambda_4^{(p)} = i\gamma_p = \frac{ic}{\Delta_p} \tag{1}$$

where $\gamma_p = (1 - \frac{1}{c^2}\sum_{\alpha=1}^{3}[u_\alpha^{(p)}(t)]^2)^{-\frac{1}{2}}, \Delta_p = (c^2 - \sum_{\alpha=1}^{3}[u_\alpha^{(p)}(t)]^2)^{\frac{1}{2}}$. It follows $\gamma_p = c/\Delta_p$.

By $< ., . >_4$ we denote the scalar product in the Minkowski space, while by $< ., . >$ - the scalar product in 3-dimensional Euclidean subspace. The equations of motion modelling the interaction of two moving charged particles are the following (cf. [2], [3]):

$$m_p \frac{d\lambda_r^{(p)}}{ds_p} = \frac{e_p}{c^2} F_{rn}^{(p)} \lambda_n^{(p)} (r = 1,2,3,4) \tag{2}$$

where the elements of proper time are $ds_p = \frac{c}{\gamma_p} dt = \Delta_p dt (p = 1,2)$. Recall that in (2) there is a summation in $n(n = 1,2,3)$. The elements $F_{rn}^{(p)}$ of the electromagnetic tensors are derived by the retarded Lienard-Wiechert potentials $A_r^{(p)} = -\frac{e_p \lambda_r^{(p)}}{\langle \lambda^{(p)}, \xi^{(pq)} \rangle_4} (r = 1,2,3,4)$, that is, $F_{rn}^{(p)} = \frac{\partial A_n^{(p)}}{\partial x_r^{(p)}} - \frac{\partial A_r^{(p)}}{\partial x_n^{(p)}}$. By $\xi^{(pq)}$ we denote the isotropic vectors (cf. [9], [10])

$$\xi^{(pq)} = (x_1^{(p)}(t) - x_1^{(q)}(t - \tau_{pq}(t)), x_2^{(p)}(t) - x_2^{(q)}(t - \tau_{pq}(t)), x_3^{(p)}(t) - x_3^{(q)}(t - \tau_{pq}(t)), ic\tau_{pq}(t))$$

where $\langle \xi^{(p,q)}, \xi^{(p,q)} \rangle_4 = 0$ or

$$\tau_{pq}(t) = \frac{1}{c}\left(\sum_{\beta=1}^{3}[x_\beta^{(p)}(t) - x_\beta^{(q)}(t - \tau_{pq}(t))]^2\right)^{\frac{1}{2}}, ((pq) = (12), (21)). \tag{3_{pq}}$$

Calculating $F_{rn}^{(p)}$ as in [9], we write equations from (2) in the form:

$$\frac{d\lambda_\alpha^{(p)}}{ds_p} = \frac{Q_p}{c^2}\left\{\frac{\xi_\alpha^{(pq)}\langle \lambda^{(p)}, \lambda^{(q)} \rangle_4 - \lambda_\alpha^{(q)} \langle \lambda^{(p)}, \xi^{(pq)} \rangle_4}{\langle \lambda^{(q)}, \xi^{(pq)} \rangle_4^3}\left[1 + \left\langle \xi^{(pq)}, \frac{d\lambda^{(q)}}{ds_q} \right\rangle_4\right] + \right.$$

$$+\frac{1}{\langle\lambda^{(q)},\xi^{(pq)}\rangle_4^2}\left[\langle\lambda^{(p)},\xi^{(pq)}\rangle_4\frac{d\lambda_\alpha^{(q)}}{ds_q}-\left\langle\xi^{(pq)},\frac{d\lambda^{(q)}}{ds_q}\right\rangle_4\xi_\alpha^{(pq)}\right]\Bigg\}\,(\alpha=1,2,3) \qquad (4.\alpha)$$

$$\frac{d\lambda_4^{(p)}}{ds_p}=\frac{Q_p}{c^2}\left\{\frac{\xi_4^{(pq)}\langle\lambda^{(p)},\lambda^{(q)}\rangle_4-\lambda_4^{(q)}\langle\lambda^{(p)},\xi^{(pq)}\rangle_4}{\langle\lambda^{(q)},\xi^{(pq)}\rangle_4^3}\left[1+\left\langle\xi^{(pq)},\frac{d\lambda^{(q)}}{ds_q}\right\rangle_4\right]+\right.$$

$$\left.+\frac{1}{\langle\lambda^{(q)},\xi^{(pq)}\rangle_4^2}\left[\langle\lambda^{(p)},\xi^{(pq)}\rangle_4\frac{d\lambda_4^{(q)}}{ds_q}-\left\langle\xi^{(pq)},\frac{d\lambda^{(q)}}{ds_q}\right\rangle_4\xi_4^{(pq)}\right]\right\} \qquad (4.4)$$

where $Q_p=e_1e_2/m_p(p=1,2)$. Further on we denote $u^{(q)}\equiv u^{(q)}(t-\tau_{pq})$,

$$\lambda^{(q)}=(\gamma_{pq}u_1^{(q)}/c,\gamma_{pq}u_2^{(q)}/c,\gamma_{pq}u_3^{(q)}/c,i\gamma_{pq})=(u_1^{(q)}/\Delta_{pq},u_2^{(q)}/\Delta_{pq},u_3^{(q)}/\Delta_{pq},ic/\Delta_{pq})$$

where $\gamma_{pq}=\left(1-\frac{1}{c^2}\sum_{\alpha=1}^{3}[u_\alpha^{(q)}(t-\tau_{pq}(t)]^2\right)^{-\frac{1}{2}}$, $\Delta_{pq}=\left(c^2-\sum_{\alpha=1}^{3}[u_\alpha^{(q)}(t-\tau_{pq}(t))]^2\right)^{\frac{1}{2}}$

and $\frac{d\lambda_\alpha^{(p)}}{ds_p}=\frac{d(\frac{\gamma_p}{c}u_\alpha^{(p)})}{\frac{c}{\gamma_p}dt}=\frac{d(\frac{u_\alpha^{(p)}}{\Delta_p})}{\Delta_p dt}=\frac{1}{\Delta_p^2}\dot{u}_\alpha^{(p)}+\frac{u_\alpha^{(p)}}{\Delta_p^4}\langle u^{(p)},\dot{u}^{(p)}\rangle(\alpha=1,2,3)$

$\frac{d\lambda_4^{(p)}}{ds_p}=\frac{d(i\gamma_p)}{\frac{c}{\gamma_p}dt}=\frac{icd(\frac{1}{\Delta_p})}{\Delta_p dt}=\frac{ic}{\Delta_p^4}\langle u^{(p)},\dot{u}^{(p)}\rangle$, where the dot means a differentiation in t.

In order to calculate $\frac{d\lambda_\alpha}{ds_q}$ we need the derivative $\frac{dt}{dt_{pq}}\equiv D_{pq}$ which should be calculated from the relation

$t-t_{pq}=\frac{1}{c}\left(\sum_{\alpha=1}^{3}[x_\alpha^{(p)}(t)-x_\alpha^{(q)}(t_{pq})]^2\right)^{\frac{1}{2}}$ $(t_{pq}<t; t-\tau_{pq}(t)=t_{pq}$ by assumption).

So we have $\frac{dt}{dt_{pq}}-1=\frac{\sum_{\alpha=1}^{3}[x_\alpha^{(p)}(t)-x_\alpha^{(q)}(t_{pq})][u_\alpha^{(p)}(t)\frac{dt}{dt_{pq}}-u_\alpha^{(q)}(t_{pq})]}{c\left(\sum_{\alpha=1}^{3}[x_\alpha^{(p)}(t)-x_\alpha^{(q)}(t_{pq})]^2\right)^{\frac{1}{2}}}$.

Since ($3pq$) has a unique solution (cf. [9], [10]) we can solve the above equation with respect to D_{pq}:

$D_{pq}=\frac{c^2\tau_{pq}-\langle\xi^{pq},u^{(q)}\rangle_4}{c^2\tau_{pq}-\langle\xi^{pq},u^{(p)}\rangle_4}$. We have also $\frac{d}{ds_p}=\frac{d}{\Delta_p dt}$.

Then $\frac{d}{ds_q}=\frac{1}{\Delta_{pq}}\frac{d}{dt_{pq}}=\frac{1}{\Delta_{pq}}\frac{d}{dt_{pq}}\frac{d}{dt}=\frac{D_{pq}}{\Delta_{pq}}\frac{d}{dt}$;

$$\frac{d\lambda_\alpha^{(p)}}{ds_q}=\frac{d(\frac{\gamma_{pq}}{c}u_\alpha^{(q)})}{\frac{c}{\gamma_{pq}}dt_{pq}}=\frac{d\left(\frac{u_\alpha^{(q)}}{\Delta_{pq}}\right)}{\Delta_{pq}dt_{pq}}=D_{pq}\frac{d\left(\frac{u_\alpha^{(q)}}{\Delta_{pq}}\right)}{\Delta_{pq}dt_{pq}}=$$

$$=D_{pq}\left[\dot{u}_\alpha^{(q)}\frac{1}{\Delta_{pq}^2}+\frac{u_\alpha^{(q)}}{\Delta_{pq}^4}\langle u^{(q)},\dot{u}^{(q)}\rangle\right](\alpha=1,2,3);$$

$$\frac{d\lambda_4^{(q)}}{ds_q} = \frac{icD_{pq}}{\Delta_{pq}^4}\langle u^{(q)}, \dot{u}^{(q)}\rangle; \langle \lambda^{(p)}\lambda^{(q)}\rangle_4 = \frac{\langle u^{(p)}, u^{(q)}\rangle - c^2}{\Delta_p \Delta_{pq}};$$

$$\langle \lambda^{(p)}, \xi^{(pq)}\rangle_4 = \frac{\langle u^{(p)}, \xi^{(pq)}\rangle - c^2\tau_{pq}}{\Delta_p}; \langle \lambda^{(q)}, \xi^{(pq)}\rangle_4 = \frac{\langle u^{(q)}, \xi^{(pq)}\rangle - c^2\tau_{pq}}{\Delta_{pq}};$$

$$\langle \xi^{(pq)}, \frac{d\lambda^q}{ds_q}\rangle_4 = D_{pq}\left[\frac{1}{\Delta_{pq}^2}\langle \xi^{(pq)}, \dot{u}^{(q)}\rangle + \frac{\langle \xi^{(pq)}, u^q\rangle - c^2\tau_{pq}}{\Delta_{pq}^4}\langle u^{(q)}, \dot{u}^{(q)}\rangle\right];$$

$$\langle \lambda^{(p)}, \frac{d\lambda^q}{ds_q}\rangle_4 = \frac{D_{pq}}{\Delta_p\Delta_{pq}^2}\left[\langle u^{(p)}, \dot{u}^{(q)}\rangle + \frac{\langle u^{(p)}, u^q\rangle - c^2}{\Delta_{pq}^2}\langle u^{(q)}, \dot{u}^{(q)}\rangle\right].$$

We note that in the above expressions $\xi^{(pq)}$ is 4-dimensional vector in the left-hand sides, while in the right-hand sides $\xi^{(pq)}$ is 3-dimensional part of the first three coordinates.

Replacing the above expressions in (4.α) and (4.4) and performing some obvious transformations we obtain for $(pq) = (12), (21), \alpha = 1, 2, 3:$

$$\frac{1}{\Delta_p}\dot{u}_\alpha^{(p)} + \frac{u_\alpha^{(p)}}{\Delta_p^3}\langle u^{(p)}, \dot{u}^{(p)}\rangle = \frac{Q_p}{c^2}\left\{\frac{[c^2 - \langle u^{(p)}, u^{(q)}\rangle]\xi_\alpha^{(pq)} - [c^2\tau_{pq} - \langle u^{(p)}, \xi^{(pq)}\rangle]u_\alpha^q}{[c^2\tau_{pq} - \langle u^{(q)}, \xi^{(pq)}\rangle]^3}\cdot\right.$$

$$\frac{\Delta_{pq}^4 + D_{pq}\left[\Delta_{pq}^2\langle \xi^{(pq)}, \dot{u}^{(q)}\rangle + (\langle \xi^{(pq)}, u^{(q)}\rangle - c^2\tau_{pq})\langle u^{(q)}, \dot{u}^{(q)}\rangle\right]}{\Delta_{pq}^2} + \qquad (5_{p\alpha})$$

$$+D_{pq}\frac{[\langle u^{(p)}, \xi^{(pq)}\rangle - c^2\tau_{pq}][\dot{u}_\alpha^{(q)} + u_\alpha^q\langle u^{(q)}, \dot{u}^{(q)}\rangle/\Delta_{pq}^2]}{[c^2\tau_{pq} - \langle u^{(q)}, \xi^{(pq)}\rangle]^2} -$$

$$\left. -D_{pq}\frac{[\langle u^{(p)}, \dot{u}^{(q)}\rangle + (\langle u^{(p)}, u^{(q)}\rangle - c^2)/\Delta_{pq}^2]\langle u^{(q)}, \dot{u}^{(q)}\rangle\xi^{(pq)}}{[c^2\tau_{pq} - \langle u^{(q)}, \xi^{(pq)}\rangle]^2}\right\},$$

$$\frac{1}{\Delta_p^3}\langle u^{(p)}, \dot{u}^{(p)}\rangle = \frac{Q_p}{c^2}\left\{\frac{\langle u^{(p)}, \xi^{(pq)}\rangle - \tau_{pq}\langle u^{(p)}, u^{(q)}\rangle}{[c^2\tau_{pq} - \langle u^{(q)}, \xi^{(pq)}\rangle]^3}\right.$$

$$\left[\Delta_{pq}^2 + D_{pq}(\langle \xi^{(pq)}, \dot{u}^{(q)}\rangle + (\langle \xi^{(pq)}, u^{(q)}\rangle - c^2\tau_{pq})\langle u^{(q)}, \dot{u}^{(q)}\rangle/\Delta_{pq}^2)\right] + \qquad (5_{p4})$$

$$+D_{pq}\frac{\langle u^{(p)}, \xi^{(pq)}\rangle\langle u^{(q)}, \dot{u}^{(q)}\rangle/\Delta_{pq}^2 - \tau_{pq}\langle u^{(p)}, \dot{u}^{(q)}\rangle - \tau_{pq}\langle u^{(p)}, u^{(q)}\rangle\langle u^{(q)}, \dot{u}^{(q)}\rangle/\Delta_{pq}^2}{[c^2\tau_{pq} - \langle u^{(q)}, \xi^{(pq)}\rangle]^2}$$

One can prove (as in [10]) that (5_{p4}) is a consequence of $(5_{p\alpha})$. Indeed, multiplying $(5_{p\alpha})$ by $u_\alpha^{(p)}$, summing up in α and dividing into c^2 we obtain (5_{p4}). Therefore we can consider a system consisting of the $1^{st}, 2^{nd}, 3^{rd}, 5^{th}, 6^{th}$ and 7^{th} equations. The last equations form a nonlinear functional differential system of neutral type (cf. [7], [8]) with respect to the unknown velocities. The delays τ_{pq} depend on the unknown trajectories by the relations (3_{pq}).

Let us formulate the initial value problem for $(5_{p\alpha})$ in the following way: to find unknown velocities $u_\alpha^{(p)}(t)(p = 1, 2; \alpha = 1, 2, 3)$ for $t \geq 0$ satisfying equations $(6_{1\alpha})$, $(6_{2\alpha})$ of motion (written in details below):

$$\frac{1}{\Delta_1}\dot{u}_\alpha^{(1)} + \frac{u_\alpha^{(1)}}{\Delta_1^3}\langle u^{(1)}, \dot{u}^{(1)}\rangle = \frac{Q_1}{c^2}\left\{\frac{[c^2 - \langle u^{(1)}, u^{(2)}\rangle]\xi^{(12)} - [c^2\tau_{12} - \langle u^{(1)}, \xi^{(12)}\rangle]u_\alpha^{(2)}}{[c^2\tau_{12} - \langle u^{(2)}, \xi^{(12)}\rangle]^3}\right.$$

$$\cdot[\Delta_{12}^4 + D_{12}\Delta_{12}^2\langle \xi_{(12)}, \dot{u}^{(2)}\rangle + (\langle \xi_{(12)}, u^{(2)}\rangle - c^2\tau_{12})\langle u^{(2)}, \dot{u}^{(2)}\rangle]/\Delta_{12}^2 + \qquad (6_{1\alpha})$$

$$+D_{12}\frac{(\langle u^{(1)},\xi^{(12)}\rangle - c^2\tau_{12})\dot{u}^{(2)}_\alpha - \langle u^{(1)},\dot{u}^{(2)}\rangle\xi^{(12)}_\alpha + (\langle u^{(1)},\xi^{(12)}\rangle - c^2\tau_{12})u^{(2)}_\alpha\langle u^{(2)},\dot{u}^{(2)}\rangle/\Delta^2_{12}}{[c^2\tau_{12}-\langle u^{(2)},\xi^{(12)}\rangle]^2}$$
$$\left.+D_{12}\frac{(c^2-\langle u^{(1)},u^{(2)}\rangle)\xi^{(12)}_\alpha\langle u^{(2)},\dot{u}^{(2)}\rangle/\Delta^2_{12}}{[c^2\tau_{12}-\langle u^{(2)},\xi^{(12)}\rangle]^2}\right\}$$

Recall that in the above equations $u^{(1)}=u^{(1)}(t), u^{(2)}=u^{(2)}(t-\tau_{12})$. We also have

$$\frac{1}{\Delta_2}\dot{u}^{(2)}_\alpha + \frac{u^{(2)}_\alpha}{\Delta^3_2}\langle u^2,\dot{u}^{(2)}\rangle = \frac{Q_1}{c^2}\left\{\frac{[c^2-\langle u^2,u^{(1)}\rangle]\xi^{(21)} - c^2\tau_{21} - \langle u^{(2)},\xi^{(21)}\rangle]u^{(1)}_\alpha}{[c^2\tau_{21}-\langle u^{(1)},\xi^{(21)}\rangle]^3}\right..$$

$$[\Delta^4_{21} + D_{21}\Delta^2_{21}\langle\xi_{(21)},\dot{u}^{(1)}\rangle + (\langle\xi_{(21)},u^{(1)}\rangle - c^2\tau_{21})\langle u^{(1)},\dot{u}^{(1)}\rangle]/\Delta^2_{21} + \qquad (6_{2\alpha})$$
$$+D_{21}\frac{(\langle u^{(2)},\xi^{(21)}\rangle - c^2\tau_{21})\dot{u}^{(1)}_\alpha - \langle u^{(2)},\dot{u}^{(1)}\rangle\xi^{(21)}_\alpha + (\langle u^{(2)},\xi^{(21)}\rangle - c^2\tau_{21})u^{(1)}_\alpha\langle u^{(1)},\dot{u}^{(1)}\rangle/\Delta^2_{21}}{[c^2\tau_{21}-\langle u^{(1)},\xi^{(21)}\rangle]^2}$$
$$\left.+D_{21}\frac{(c^2-\langle u^{(2)},u^{(1)}\rangle)\xi^{(21)}_\alpha\langle u^{(1)},\dot{u}^{(1)}\rangle/\Delta^2_{21}}{[c^2\tau_{21}-\langle u^{(1)},\xi^{(21)}\rangle]^2}\right\}$$

Recall that in the above equations $u^{(2)}=u^{(2)}(t), u^{(1)}=u^{(1)}(t-\tau_{21})$. We note the delay functions $\tau_{pq}(t)$ satisfy functional equations (3_{pq}) for $t\in R^1$. For $t\le 0$ $u^{(p)}_\alpha(t)$ are prescribed functions $u^{-(p)}_\alpha(t)$, i.e.

$$u^{(p)}_\alpha(t) = u^{-(p)}_\alpha(t), t\le 0, \text{ where } u^{-(p)}_\alpha(t) = \frac{d\overline{x}^{(p)}_\alpha(t)}{dt}, t\le 0 \qquad (\overline{6}_{\alpha p})$$

This means that for prescribed trajectories $(\overline{x}^{(1)}_1(t),\overline{x}^{(1)}_2(t),\overline{x}^{(1)}_3(t)),(\overline{x}^{(2)}_1(t),\overline{x}^{(2)}_2(t),\overline{x}^{(2)}_3(t))$ for $t\le 0$ one has to find trajectories, satisfying the above system of equations for $t>0$. (We recall, $x^{(p)}_\alpha(t) = x^{(p)}_{\alpha 0} + \int_0^t u^{(p)}_\alpha(s)ds$ where $x^{(p)}_{\alpha 0}$ are the coordinates of the initial positions).

3. Equation of motion in polar coordinates

In what follows we consider plane motion in Ox_2x_3 coordinate plane for equations $(6_{p\alpha}),(6_{pq}),(\overline{6}_{\alpha p}), p=1,2;\alpha=1,2,3;(pq)=(12),(21)$. We suppose that the first particle P_1 is fixed at the origin $O(0,0,0)$, that is, $P_1:\left|\begin{array}{l} x^{(1)}_1(t)=0\\ x^{(1)}_2(t)=0, t\in(-\infty,\infty)\\ x^{(1)}_3(t)=0\end{array}\right.$.

It follows by necessity $\left|\begin{array}{l} x^{-(1)}_1(t)=0\\ x^{-(1)}_2(t)=0\\ x^{-(1)}_3(t)=0\end{array}\right.$. Passing to the polar coordinates we can put

$$P_2:\left|\begin{array}{l} x^{(2)}_1(t)=0\\ x^{(2)}_2(t)=\rho(t)\cos\varphi(t)\\ x^{(2)}_3(t)=\rho(t)\sin\varphi(t)\end{array}\right. \qquad \text{where } \rho(t)>0.$$

For the velocities and accelerations of the particles we obtain

$$\left|\begin{array}{l} u^{(1)}_1(t)=0\\ u^{(1)}_2(t)=0\\ u^{(1)}_3(t)=0\end{array}\right. \quad \left|\begin{array}{l} w^{(1)}_1(t)=0\\ w^{(1)}_2(t)=0\\ w^{(1)}_3(t)=0\end{array}\right. \quad \left|\begin{array}{l} u^{(2)}_1(t)=0\\ u^{(2)}_2(t)=\dot\rho(t)\cos\varphi(t)-\rho(t)\dot\varphi(t)\sin\varphi(t)\\ u^{(3)}_3(t)=\dot\rho(t)\sin\varphi(t)+\rho(t)\dot\varphi(t)\cos\varphi(t)\end{array}\right.$$

$$\left| \begin{array}{l} w_1^{(2)}(t)=0 \\ w_2^{(2)}(t)=[\ddot\rho(t)-\rho(t)\dot\varphi^2(t)]\cos\varphi(t)-[2\dot\rho(t)\dot\varphi(t)+\rho(t)\ddot\varphi(t)]\sin\varphi(t) \\ w_3^{(3)}(t)=[\ddot\rho(t)-\rho(t)\dot\varphi^2(t)]\sin\varphi(t)+[2\dot\rho(t)\dot\varphi(t)+\rho(t)\ddot\varphi(t)]\cos\varphi(t) \end{array}\right.$$

Then for $(pq)=(12)$ we have
$\Delta_1=c,\quad \Delta_{12}=\sqrt{c^2-\dot\rho^2-\rho^2\dot\varphi^2},$
$\langle u^{(1)},\dot u^{(1)}\rangle=0,\quad \langle u^{(1)},u^{(2)}\rangle=0,\quad \langle u^{(1)},\xi^{(12)}\rangle=0,\quad \langle u^{(2)},\xi^{(12)}\rangle=-\rho\dot\rho^2,$
$\xi^{(12)}(0,-\rho\cos\varphi,-\rho\sin\varphi).$

Recall that in the above equations $u^{(1)}=u^{(1)}(t),u^{(2)}=u^{(2)}(t-\tau_{12}(t))$ and the argument of $\rho,\dot\rho,\ddot\rho,\varphi,\dot\varphi,\ddot\varphi$ is $t-\tau_{12}(t)$.

We know from [6] that $\tau_{12}=\sqrt{\langle\xi^{(12)},\xi^{(12)}\rangle}/c$ or in polar coordinates $\tau_{12}(t)=\rho(t-\tau_{12})/c$. The last equation has a unique solution provided $|\dot\rho|\le\bar c<c$ for some constant $\bar c>0$ (cf. [9]). Since $D_{12}=\dfrac{c\sqrt{\langle\xi^{(12)},\xi^{(12)}\rangle}-\langle\xi^{(12)},u^{(2)}\rangle}{c\sqrt{\langle\xi^{(12)},\xi^{(12)}\rangle}-\langle\xi^{(12)},u^{(1)}\rangle}=\dfrac{c^2\tau_{12}+\rho\dot\rho}{c^2\tau_{12}}=\dfrac{c+\dot\rho}{\rho},$
$\langle u^{(1)},\dot u^{(2)}\rangle=0,\langle\xi^{(12)},\dot u^{(2)}\rangle=\rho^2\dot\varphi^2-\rho\ddot\rho,$
$\langle u^{(2)},\dot u^{(2)}\rangle=\dot\rho\ddot\rho+\rho\dot\rho\dot\varphi^2+\rho^2\dot\varphi\ddot\varphi$
then replacing the above expressions in $(6_{1\alpha})$ we obtain

$$\frac{c^2\xi_\alpha^{(12)}-c^{12}\tau_{12}u_\alpha^2}{[c^2\tau_{12}+\rho\dot\rho]^3}\left[\Delta_{12}^2+D_{12}(\rho^2\dot\varphi^2-\rho\ddot\rho)+\frac{D_{12}(-\rho\dot\rho-c^2\tau_{12})}{\Delta_{12}^2}\langle u^{(2)},\dot u^{(2)}\rangle\right]+$$
$$+D_{12}\frac{-c^{12}\tau_{12}[\dot u_\alpha^{(2)}+u_\alpha^{(2)}\langle u^{(2)},\dot u^{(2)}\rangle/\Delta_{12}^2]+c^2\langle u^{(2)},\dot u^{(2)}\rangle\xi_\alpha^{(12)}/\Delta_{12}^2}{[c^2\tau_{12}+\rho\dot\rho]^2}=0,$$
$$\tau_{12}D_{12}(c^2\tau_{12}+\rho\dot\rho)\dot u_\alpha^{(2)}+\rho D_{12}(\xi_\alpha^{(12)}-\tau_{12}u_\alpha^{(2)})\ddot\rho=(\Delta_{12}^2+\rho^2D_{12}\dot\varphi^2)(\xi_\alpha^{(12)}-\tau_{12}u_\alpha^{(2)}).$$

For $\alpha=1$ we obtain the identity $0=0$. For $\alpha=2,3$ we obtain the system
$$[(c+\dot\rho)^2\cos\varphi+(c+\dot\rho)M]\ddot\rho-(c+\dot\rho)^2\rho\sin\varphi\ddot\varphi=(c+\dot\rho)^2(\rho\dot\varphi^2\cos\varphi+2\dot\rho\dot\varphi\sin\varphi)+MP_{12}$$
$$[(c+\dot\rho)^2\sin\varphi+(c+\dot\rho)N]\ddot\rho+(c+\dot\rho)^2\rho\cos\varphi\ddot\varphi=(c+\dot\rho)^2(\rho\dot\varphi^2\sin\varphi-2\dot\rho\dot\varphi\cos\varphi)-NP_{12}$$
where $M=-c\cos\varphi-\dot\rho\cos\varphi+\rho\dot\varphi\sin\varphi,N=c\sin\varphi+\dot\rho\cos\varphi+rho\dot\varphi\cos\varphi,P_{12}=\Delta_{12}^2+\rho^2\dot\varphi^2D_{12}$. We assume there is no collision for $t\le0$, i.e. $\rho(t-\tau_{12}(t))\ne0$. Therefore the above system has no solution because its determinant
$\delta=\begin{vmatrix}(c+\dot\rho)^2\cos\varphi+(c+\dot\rho)M & -(c+\dot\rho)^2\rho\sin\varphi\\ (c+\dot\rho)^2\sin\varphi-(c+\dot\rho)N & (c+\dot\rho)^2\rho\cos\varphi\end{vmatrix}=0$, while in view of $|\dot\rho|\le\bar c<c$
$\delta_1=\begin{vmatrix}(c+\dot\rho)^2(\rho\dot\varphi\cos\varphi+2\dot\rho\dot\varphi\sin\varphi)+MP_{12} & -(c+\dot\rho)^2\rho\sin\varphi\\ (c+\dot\rho)^2(\rho\dot\varphi\sin\varphi-2\dot\rho\dot\varphi\cos\varphi)-NP_{12} & (c+\dot\rho)^2\rho\cos\varphi\end{vmatrix}=-\rho(c+\dot\rho)^3\Delta_{12}^2\ne0.$

So we have to consider only the second group of equations namely $(6_{2\alpha})$ $(\alpha=1,2,3)$. Since
$\xi^{(21)}(0,\rho(t)\cos\varphi(t),\rho(t)\sin\varphi(t));\quad \langle u^{(1)},\dot u^{(1)}\rangle=0;\quad \langle u^{(2)},\dot u^{(1)}\rangle=0;$
$\langle\xi^{(21)},\dot u^{(1)}\rangle=0,\ \langle u^{(2)},\dot u^{(2)}\rangle=\dot\rho\ddot\rho+\rho\dot\rho\dot\varphi^2+\rho^2\dot\varphi\ddot\varphi;\quad \langle u^{(2)},u^{(1)}\rangle=0;$
$\langle u^{(2)},\xi^{(21)}\rangle=-\rho\dot\rho;\quad \langle u^{(1)},\xi^{(21)}\rangle=0;\ \tau_{21}(t)=\dfrac{\rho(t)}{c};\quad \Delta_2=\sqrt{c^2-\dot\rho^2-\rho^2\dot\varphi^2};$
$\Delta_{21}=c;\quad D_{21}=\frac{c}{c+\dot\rho}$ for $(6_{2\alpha})(\alpha=1,2,3)$ we obtain:

$$\Delta_2^2\dot u_\alpha^{(2)}+u_\alpha^{(2)}<u^{(2)},\dot u^{(2)}>=Q_2\Delta_2^3\xi_\alpha^{(21)}/c\rho^3.$$

For $\alpha = 1$ we obtain the identity $0 = 0$. For $\alpha = 2, 3$ we obtain the following system:
$[\Delta_2^2 \cos\varphi + \dot\rho(\dot\rho\cos\varphi - \rho\dot\varphi\sin\varphi)]\ddot\rho + [(\dot\rho\cos\varphi - \rho\dot\varphi\sin\varphi)\rho^2\dot\varphi - \Delta_2^2\rho\sin\varphi]\ddot\varphi =$
$= \Delta_2^2\rho\dot\varphi^2\cos\varphi + 2\Delta_2^2\dot\rho\dot\varphi\sin\varphi + (\rho\dot\varphi\sin\varphi - \dot\rho\cos\varphi)\rho\dot\rho\dot\varphi^2 + \dfrac{Q_2\Delta_2^3\cos\varphi}{c\rho^2},$
$[\Delta_2^2 \sin\varphi + \dot\rho(\dot\rho\sin\varphi + \rho\dot\varphi\cos\varphi)]\ddot\rho + [(\dot\rho\sin\varphi + \rho\dot\varphi\cos\varphi)\rho^2\dot\varphi + \Delta_2^2\rho\cos\varphi]\ddot\varphi =$
$= \Delta_2^2\rho\dot\varphi^2\sin\varphi - 2\Delta_2^2\dot\rho\dot\varphi\cos\varphi - (\rho\dot\varphi\cos\varphi + \dot\rho\sin\varphi)\rho\dot\rho\dot\varphi^2 + \dfrac{Q_2\Delta_2^3\sin\varphi}{c\rho^2}.$

The above system can be solved with respect to $\ddot\rho, \ddot\varphi$ because

$$\delta = \begin{vmatrix} \Delta_2^2 \cos\varphi + \dot\rho(\dot\rho\cos\varphi - \rho\dot\varphi\sin\varphi) & (\dot\rho\cos\varphi - \rho\dot\varphi\sin\varphi)\rho^2\dot\varphi - \Delta_2^2\rho\sin\varphi \\ \Delta_2^2 \sin\varphi + \dot\rho(\dot\rho\sin\varphi + \rho\dot\varphi\cos\varphi) & (\dot\rho\sin\varphi + \rho\dot\varphi\cos\varphi)\rho^2\dot\varphi + \Delta_2^2\rho\cos\varphi \end{vmatrix} = c^2\rho\Delta_2^2 \neq 0.$$

$\delta_1 = \Delta_2^2 c^2\rho^2\dot\varphi^2 + \dfrac{Q_2\Delta_2^3(c^2 - \dot\rho^2)}{c\rho}$ and $\delta_2 = -2\Delta_2^2\dot\rho\dot\varphi(c^2 + \dfrac{Q_2\Delta_2}{2c\rho})$

Then we have

$$\ddot\rho = \rho\dot\varphi^2 + \frac{Q_2\Delta_2(c^2 - \dot\rho^2)}{\rho^2 c^3} \text{ and } \ddot\varphi = -\frac{2\dot\rho\dot\varphi}{\rho} - \frac{Q_2\dot\rho\dot\varphi\Delta_2}{c^3\rho^2} \tag{7}$$

Therefore we consider just equations (7) for $t \geq 0$.

4. One dimensional case

Here we consider the motion of two charged particles on a straight line $\dot\varphi = 0$. Then the second equation from (7) becomes the identity and the first one $\ddot\rho = Q_2(c^2 - \dot\rho^2)^{\frac{3}{2}}/c^3\rho^2$, where $\rho = \rho(t)$ and $t \geq 0$. Assume that the particles have opposite signs. Therefore $-Q_2 > 0$. Denote by $A = -\dfrac{Q_2}{c^3} > 0$. Then as usually we set $\dot\rho = z, \ddot\rho = z\dfrac{dz}{d\rho}$. So we obtain $z\dfrac{dz}{d\rho} = -A\dfrac{(c^2 - z^2)^{\frac{3}{2}}}{\rho^2}$, $(c^2 - \dot\rho^2)^{-\frac{1}{2}} = \dfrac{A}{\rho} + D$, where $D = (c^2 - \dot\rho_0^2)^{-\frac{1}{2}} + \dfrac{Q_2}{c^3\rho_0}$ and $\rho_0 = \rho(0), \dot\rho_0 = \dot\rho(0)$ are initial conditions.Further on we have $\displaystyle\int \frac{D\rho + A}{\sqrt{(c^2D^2 - 1)\rho^2 + 2c^2DA\rho + c^2A^2}}d\rho = \pm t + E$, where E is a constant.

Introduce a new variable η by Euler substitution and putting $B = c^2D^2 - 1$ we obtain $(B\rho^2 + 2c^2DA\rho + c^2A^2)^{\frac{1}{2}} = \rho\eta + cA$. Hence $2A\displaystyle\int \frac{-\eta^2 + 2cD\eta - B - 2}{(\eta^2 - B^2)}d\eta = \pm t + E$.

Consider the case $B > 0$, that is, $c^2\left[(c^2 - \dot\rho_0^2)^{-\frac{1}{2}} + \dfrac{Q_2}{c^3\rho_0}\right]^2 - 1 > 0$. The last inequality is satisfied for suitably chosen $\rho_0, \dot\rho_0$. It is easy to formulate conditions implying the above inequality. Indeed, since $(c^2 - \dot\rho_0^2)^{-\frac{1}{2}} > 1/c$ then $D > \dfrac{1}{c} + \dfrac{Q_2}{c^3\rho_0} = \dfrac{1}{c}\dfrac{\rho_0c^2 + Q_2}{c^2\rho_0} > 0$ because for the hydrogen atom $Q_2 = e_1^2/m_2, e_1 = 1,6.10^{-19}q, m_2 = 9.10^{31}kg, c = 3.10^8 m/s$ which yields $\rho_0c^2 + Q_2 =, 28.10^{-11}.9.10^{16} - 2,84.10^{-8} > 0$.

Then $B > 0 \Leftrightarrow cD > 1$ or $c(c^2 - \dot\rho_0^2)^{-\frac{1}{2}} + \dfrac{Q_2}{c^2\rho_0} > 0 \Leftrightarrow c(c^2 - \dot\rho_0^2)^{-\frac{1}{2}} > 1 - \dfrac{Q_2}{c^2\rho_0} > 1$.

Denote by $q = 1 - \frac{Q_2}{c^2\rho_0}$. It follows $c > \dot\rho_0 > c\sqrt{q^2-1}/q$ Therefore $B > 0$ and then we have

$$2A\int \frac{-\eta^2 + 2cD\eta - B - 2}{(\eta^2 - B^2)} d\eta =$$

$$2A\left(-\frac{A_1}{\eta - \sqrt{B}} + A_2 \ln|\eta - \sqrt{B}| - \frac{A_3}{\eta + \sqrt{B}} + A_4 \ln|\eta + \sqrt{B}|\right) = \pm t + E \qquad (8)$$

where $A_1 = (-B + cD\sqrt{B} - 1)/2B, A_2 = 1/2B\sqrt{B}, A_3 = -(B + cD\sqrt{B} + 1)/2B, A_4 = -1/2B\sqrt{B}$.

It is easy to verify that $-A_1 > 0$ and $A_2 > 0$. Since $\eta = (\sqrt{B\rho^2 + 2c^2DA\rho + c^2A^2} - cA)/\rho$ then we have $\eta - \sqrt{B} = (\sqrt{B\rho^2 + 2c^2DA\rho + c^2A^2} - cA - \sqrt{B}\rho)/\rho$. For $t \to \pm\infty$ the right-hand side of (8) should tend to $\pm\infty$. This is possible if $\eta - \sqrt{B} \to 0$. Obviously the last relation is implied by $\rho(t) \to \infty$ as $t \to \infty$. On the other hand the differential equation $\ddot\rho(t) = \frac{Q_2}{c^3}\frac{[c^2 - \dot\rho^2(t)]^{\frac{3}{2}}}{\rho^2(t)}$ shows that if $\lim_{t\to t_0} \rho(t) = 0$ for some $t_0 > 0$ it follows by necessity $\lim_{t\to t_0} \dot\rho(t) = c$ because $\ddot\rho(t)$ should be bounded. So we obtain a confirmation of the results from [6].

It remains to consider the case $B > 0$ or $\dot\rho_0 < \frac{c\sqrt{q^2-1}}{q}$. Put $B_1 = -B > 0$ and then (8) becomes

$$2A\int \frac{-\eta^2 + 2cD\eta - B_1 - 2}{(\eta^2 - B_1^2)} d\eta = 2A(-B_1^{-\frac{3}{2}} arctg(\frac{\eta}{\sqrt{B_1}}) + \frac{(B_1 - 1)\eta}{B_1(\eta^2 + B^1)} - \frac{cD}{\eta^2 + B^1}) = \pm t + const.$$

Obviously the left-hand side of the last equality remains bounded while the right hand side is unbounded for any values of $t \to \infty$.

5. Two-dimensional case

This section is devoted to the investigation of two-dimensional case of two-body problem. First we consider the system of equations of motion already derived in a previous section, namely

$$\ddot\rho(t) = \rho(t)\dot\varphi^2(t) + \frac{Q_2}{c^3}\frac{[c^2 - \dot\rho^2(t)]\sqrt{c^2 - \dot\rho^2(t) - \rho^2(t)\dot\varphi^2(t)}}{\rho^2(t)} \qquad (9.1)$$

$$\ddot\varphi = -\frac{2\dot\rho(t)\dot\varphi(t)}{\rho(t)}\left[1 + \frac{Q_2\sqrt{c^2 - \dot\rho^2(t) - \rho^2(t)\dot\varphi^2(t)}}{2c^3\rho(t)}\right] \qquad (9.2)$$

for $t > 0$ and initial conditions $\rho(0) = \rho_0, \dot\rho(0) = \dot\rho_0, \varphi(0) = \varphi_0, \dot\varphi(0) = \dot\varphi_0$.

Let us put $\rho = const$ which implies $\dot\rho(t) = \ddot\rho(t) = 0$. Then (9.1) and (9.2) become

$$\rho(t)\dot\varphi^2(t) + \frac{Q_2\sqrt{c^2 - \rho^2(t)\dot\varphi^2(t)}}{c\rho^2(t)} = 0, \ddot\varphi(t) = 0. \qquad (10)$$

The second equation of (10) yields $\varphi(t) = \dot\varphi_0 t + \varphi_0$. Without loss of generality one can assume $\varphi_0 = 0$. Since $\nu = \rho\dot\varphi$ then for the linear velocity ν we obtain the following equation

$$c\rho\nu^2 + Q_2\sqrt{c^2 - \nu^2} = 0.$$

This equation obviously has a positive solution $\nu^2 = \dfrac{-Q_2^2 + \sqrt{Q_2^4 + 4c^4\rho^2 Q_2^2}}{2c^2\rho^2}$ since $Q_2 = e^2/m < 0$. But $Q_2 = -(1,6.10^{-19})^2/9.10^{-31} \approx -2,84.10^{-8}$ and then $\nu^2 \approx |Q_2|/\rho = e^2/m\rho$ which consides with known results (cf.[14]).

In the next paper we prove more general existence result.

References

[1] BORN M. Atomic Physics, 8th ed., Blacky & Sons Ltd., Glasgow, 1969.

[2] SYNGE J. L. On the electromagnetic two-body problem. Proc. Roy. Soc. (London), ser. A, 177 (1940), 118-139.

[3] SYNGE J. L. Classical Dynamics, Springer Verlag, 1960.

[4] ROHRLICH F. Classical Charged Particles. Addison, Wesley Publ. Company, Inc. Reading Massachusetts, 1965.

[5] PAULI W. Relativitatstheory. Encycl. Der Math. Wissenschaften. B.5, H.S. Ard. 19, 1921.

[6] DRIVER R. D. A two-body problem of classical electrodynamics: the one-dimensional case. Ann. Physics, v.21, N1 (1963), 122-142.

[7] HALE J.K., S.M. VERDUYN-LUNEL Functional Differential Equations, Springer Verlag, 1993.

[8] KAMENSKII G. A., A. L. SKUBACHEVSKII, Linear boundary value problems for differential difference equations, Moscow, 1992 (in Russian)

[9] ANGELOV V. G. On the Synge equations in three- dimensional two-body problem of classical electrodynamics, J. Math. Anal. Appl. v.151, N2 (1990), 488-511.

[10] ANGELOV V. G. Escape trajectories of J. L. Synge equations. J. Non Anal. RWA, ser. B, v.1 (2000), 189-204.

[11] ANGELOV V. G. Fixed point theorem in uniform spaces and applications. Czechoslovak Math. J., v.37 (112), (1987), 19-39.

[12] WEISSINGER J. Zur Theorie und Anwendungen der Iterationsverfahrens. Math. Nachr., v.8 (1952), 193-212.

[13] BIELICKI A. Bull. Acad. Polon. Sci. v.4, N5 (1956).

[14] BEISER A. Perspectives of modern physics. Mc Graw Hill, Book Company, New York.