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COMMON FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS

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Abstract. The purpose of this article is to give some common fixed points theorems for multivalued mappings which improve and generalize a result given by A. Latif and I. Beg in [1] and to present some common strict fixed points theorems.

Keywords: multivalued mapping, fixed point, strict fixed point, common fixed point, common strict fixed point.

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1 Introduction

Let X be a nonempty set. By $P(X)$ we shall understand the set of all nonempty subsets of X, i.e. $P(X) = \{ \ Y \mid \emptyset \neq Y \subseteq X \ \}$.

If (X, d) is a metric space we put

$$P_{cl}(X) = \{ \ Y \mid Y \in P(X) \text{ and } Y \text{ is a closed set } \}.$$

Let X be a nonempty set.

A *fixed point* of a multivalued mapping $T : X \to P(X)$ is a point $x \in X$ such that $x \in T(x)$. We denote by F_T the set of the fixed points of T.

A *strict fixed point* of a multivalued mapping $T : X \to P(X)$ is a point $x \in X$ such that $T(x) = \{x\}$. We denote by $(SF)_T$ the set of the strict fixed points of T.

Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of multivalued operators with nonempty values, i. e. $T_n : X \to P(X)$, for $n \in \mathbb{N}$. Then we denote by $ComFP(T)$ the set of the common fixed points of the multivalued operators T_n, for $n \in \mathbb{N}$, i. e.

$$ComFP(T) = \{ \ x \in X \mid x \in T_n(x), \text{ for all } n \in \mathbb{N} \ \} = \bigcap_{n\in\mathbb{N}} F_{T_n},$$

and by $ComSFP(T)$ the set of the common strict fixed points of the multivalued operators T_n, for $n \in \mathbb{N}$, i. e.

$$ComSFP(T) = \{ \ x \in X \mid T_n(x) = \{x\}, \text{ for all } n \in \mathbb{N} \ \} = \bigcap_{n\in\mathbb{N}} (SF)_{T_n}.$$

In [1], A. Latif and I. Beg gave the following theorem:

Theorem 1.1 *Let M be a nonempty closed subset of a complete metric space (X,d) and $(T_n)_{n\in\mathbb{N}}$ be a sequence of multivalued operators $T_n : M \to P_{cl}(M)$, for $n \in \mathbb{N}$. We suppose that there exists $h \in \mathbb{R}_+$, with $h < 1/2$ such that for any two operators T_i, T_j and for any $x \in M$, $u_x \in T_i(x)$ and for all $y \in M$, there exists $u_y \in T_j(y)$ with*

$$d(u_x, u_y) \leq h\ [d(x, u_x) + d(y, u_y)].$$

Then $(T_n)_{n\in\mathbb{N}}$ has a common fixed point.

Further on we shall give some results which improve and generalize this theorem.

2 Common fixed points

Theorem 2.1 *Let (X,d) be a metric space and $S,T : X \to P(X)$ be two multivalued operators. We suppose that at least one of the following conditions is satisfied:*

(i) *there exists $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a function with the property that $\varphi(0) = 0$ and such that for each $x \in X$, any $u_x \in S(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ so that we have*

$$d(u_x, u_y) \leq \varphi(d(x,y)). \tag{2.1}$$

(ii) *there exist $a_1, \ldots, a_5 \in \mathbb{R}_+$, with $a_3 + a_4 < 1$ such that for each $x \in X$, any $u_x \in S(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ so that we have*

$$d(u_x,u_y) \leq a_1\ d(x,y) + a_2\ d(x,u_x) + a_3\ d(y,u_y) + a_4\ d(x,u_y) + a_5\ d(y,u_x). \tag{2.2}$$

(iii) *there exists $a \in \mathbb{R}_+$, with $a < 1$ such that for each $x \in X$, any $u_x \in S(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ so that we have*

$$d(u_x,u_y) \leq a\ \max\{d(x,y), d(x,u_x), d(y,u_y), 1/2\ [d(x,u_y) + d(y,u_x)]\}. \tag{2.3}$$

(iv) *there exists $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ a function with the property that $\varphi(0,0,t,t,0) < t$, for all $t > 0$ and such that for each $x \in X$, any $u_x \in S(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ so that we have*

$$d(u_x,u_y) \leq \varphi(d(x,y), d(x,u_x), d(y,u_y), d(x,u_y), d(y,u_x)). \tag{2.4}$$

Then $F_S \subseteq F_T$.

Proof. We assume that the condition (i) is fulfilled. Let $x^* \in F_S$. Then $x^* \in S(x^*)$ and it follows that there exists $u \in T(x^*)$ such that $d(x^*, u) \leq \varphi(d(x^*, x^*)) = \varphi(0) = 0$. This implies that $u = x^*$. Therefore $x^* \in T(x^*)$ and we are able to write $F_S \subseteq F_T$.

We suppose now that the condition (ii) is verified. Let $x^* \in F_S$. So $x^* \in S(x^*)$ and there exists $u \in T(x^*)$ such that

$$d(x^*,u) \leq a_1\ d(x^*,x^*) + a_2\ d(x^*,x^*) + a_3\ d(x^*,u) + a_4\ d(x^*,u) + a_5\ d(x^*,x^*) =$$

$$= (a_3 + a_4)\ d(x^*, u).$$

From this we have that $u = x^*$ and therefore $x^* \in T(x^*)$, i. e. $x^* \in F_T$.

For the case when the condition (iii) is fulfilled, the demonstration is made similarly with the proof from the second case.

Finally, we assume that the condition (iv) is verified. Let $x^* \in F_S$. From $x^* \in S(x^*)$ we have that there exists $u \in T(x^*)$ such that

$$d(x^*, u) \leq \varphi(d(x^*, x^*),\ d(x^*, x^*),\ d(x^*, u),\ d(x^*, u),\ d(x^*, x^*)).$$

Introducing the notation $t = d(x^*, u)$ we obtain $t \leq \varphi(0, 0, t, t, 0)$. If we suppose that $t \neq 0$, then we reach the contradiction $t \leq \varphi(0, 0, t, t, 0) < t$. Thus $t = 0$, which means that $u = x^*$. It follows that $x^* \in T(x^*)$ and so $F_S \subseteq F_T$. ■

Remark 2.1 *If we take $a_2 = a_3 = a_4 = a_5 = 0$ in the condition (ii) of the Theorem* 2.1*, then we obtain a condition of contraction type.*

Remark 2.2 *If we take $a_1 = a_4 = a_5 = 0$ and $a_2 = a_3 = h$ $(0 \leq h < 1/2)$ in the condition (ii) of the Theorem* 2.1*, then we obtain a condition of Kannan type.*

Remark 2.3 *If we take $a_4 = a_5 = 0$ in the condition (ii) of the Theorem* 2.1*, then we obtain a condition of Reich type.*

Remark 2.4 *If in the Theorem* 2.1 *we ask that the pair of multivalued operators (T, S) to satisfy at least one similar condition with one of the conditions (ii)-(iv), then $F_S = F_T$.*

Let (X, d) be a metric space and $(T_n)_{n\in\mathbb{N}}$ be a sequence of multivalued operators $T_n : X \to P(X)$, for $n \in \mathbb{N}$.

If each pair of multivalued operators (T_0, T_n), for $n \in \mathbb{N}^*$, satisfies similar conditions as in the Theorem 2.1, then $F_{T_0} \subseteq F_{T_n}$, for all $n \in \mathbb{N}^*$, i. e. $F_{T_0} \subseteq \bigcap_{n\in\mathbb{N}^*} F_{T_n}$.

If each pair of multivalued operators (T_n, T_0), for $n \in \mathbb{N}^*$, satisfies similar conditions as in the Theorem 2.1, then $F_{T_n} \subseteq F_{T_0}$, for all $n \in \mathbb{N}^*$, i. e. $\bigcup_{n\in\mathbb{N}^*} F_{T_n} \subseteq F_{T_0}$.

If each pair of multivalued operators (T_0, T_n) and (T_n, T_0), for $n \in \mathbb{N}^*$, satisfies at least one similar condition with one of the conditions (ii)-(iv) of the Theorem 2.1, then $F_{T_n} = F_{T_0}$, for all $n \in \mathbb{N}^*$.

Let (X, d) be a metric space and $S, T : X \to P(X)$ be two multivalued operators. We consider the following conditions:

(i) there exists $\varphi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ a function such that for each $x \in X$, any $u_x \in S(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ so that we have

$$d(u_x, u_y) \leq \varphi_1(d(x, y)); \tag{2.5}$$

there exists $\varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ a function such that for each $x \in X$, any $u_x \in T(x)$ and for all $y \in X$, there exists $u_y \in S(y)$ so that we have

$$d(u_x, u_y) \leq \varphi_2(d(x, y)). \tag{2.6}$$

(*ii*) there exist $a_{11}, \ldots, a_{15} \in \mathbb{R}_+$ such that for each $x \in X$, any $u_x \in S(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ so that we have

$$d(u_x, u_y) \leq a_{11}\ d(x,y) + a_{12}\ d(x,u_x) + a_{13}\ d(y,u_y) + a_{14}\ d(x,u_y) + a_{15}\ d(y,u_x); \tag{2.7}$$

there exist $a_{21}, \ldots, a_{25} \in \mathbb{R}_+$ such that for each $x \in X$, any $u_x \in T(x)$ and for all $y \in X$, there exists $u_y \in S(y)$ so that we have

$$d(u_x, u_y) \leq a_{21}\ d(x,y) + a_{22}\ d(x,u_x) + a_{23}\ d(y,u_y) + a_{24}\ d(x,u_y) + a_{25}\ d(y,u_x). \tag{2.8}$$

(*iii*) there exists $a_1 \in \mathbb{R}_+$ such that for each $x \in X$, any $u_x \in S(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ so that we have

$$d(u_x, u_y) \leq a_1\ \max\{d(x,y), d(x,u_x), d(y,u_y), 1/2\ [d(x,u_y) + d(y,u_x)]\}; \tag{2.9}$$

there exists $a_2 \in \mathbb{R}_+$ such that for each $x \in X$, any $u_x \in T(x)$ and for all $y \in X$, there exists $u_y \in S(y)$ so that we have

$$d(u_x, u_y) \leq a_2\ \max\{d(x,y), d(x,u_x), d(y,u_y), 1/2\ [d(x,u_y) + d(y,u_x)]\}. \tag{2.10}$$

(*iv*) there exists $\varphi_1 : \mathbb{R}_+^5 \to \mathbb{R}_+$ a function such that for each $x \in X$, any $u_x \in S(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ so that we have

$$d(u_x, u_y) \leq \varphi_1(d(x,y), d(x,u_x), d(y,u_y), d(x,u_y), d(y,u_x)); \tag{2.11}$$

there exists $\varphi_2 : \mathbb{R}_+^5 \to \mathbb{R}_+$ a function such that for each $x \in X$, any $u_x \in T(x)$ and for all $y \in X$, there exists $u_y \in S(y)$ so that we have

$$d(u_x, u_y) \leq \varphi_2(d(x,y), d(x,u_x), d(y,u_y), d(x,u_y), d(y,u_x)). \tag{2.12}$$

Theorem 2.2 *Let* (X,d) *be a complete metric space and* $S, T : X \to P_{cl}(X)$ *be two multivalued operators. We suppose that at least one of the following conditions is satisfied*:

(*i*) *there exist* $a_{11}, \ldots, a_{15} \in \mathbb{R}_+$, *with* $a_{11} + a_{12} + a_{13} + 2a_{14} < 1$ *such that for each* $x \in X$, *any* $u_x \in S(x)$ *and for all* $y \in X$, *there exists* $u_y \in T(y)$ *so that the relation* (2.7) *holds*;

there exist $a_{21}, \ldots, a_{25} \in \mathbb{R}_+$, *with* $a_{21} + a_{22} + a_{23} + 2a_{24} < 1$ *such that for each* $x \in X$, *any* $u_x \in T(x)$ *and for all* $y \in X$, *there exists* $u_y \in S(y)$ *so that the relation* (2.8) *holds.*

(*ii*) *there exists* $a_1 \in \mathbb{R}_+$, *with* $a_1 < 1$ *such that for each* $x \in X$, *any* $u_x \in S(x)$ *and for all* $y \in X$, *there exists* $u_y \in T(y)$ *so that the relation* (2.9) *holds*;

there exists $a_2 \in \mathbb{R}_+$, *with* $a_2 < 1$ *such that for each* $x \in X$, *any* $u_x \in T(x)$ *and for all* $y \in X$, *there exists* $u_y \in S(y)$ *so that the relation* (2.10) *holds.*

Then $F_S = F_T \in P_{cl}(X)$.

Proof. Taking into account the Remark 2.4 we are able to write that $F_S = F_T$. We suppose that the condition (i) is fulfilled. Let $x_0 \in X$ and $x_1 \in S(x_0)$. It follows that there exists $x_2 \in T(x_1)$ such that

$$d(x_1, x_2) \leq l_1 \ d(x_0, x_1),$$

where $l_1 = (a_{11} + a_{12} + a_{14}) \ [1 - (a_{13} + a_{14})]^{-1} < 1$ and there exists $x_3 \in S(x_2)$ such that

$$d(x_2, x_3) \leq l_2 \ d(x_1, x_2),$$

where $l_2 = (a_{21} + a_{22} + a_{24}) \ [1 - (a_{23} + a_{24})]^{-1} < 1$. We put $l = \max \{l_1, l_2\} < 1$. Going on with this reasoning we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_0 \in X$, $x_{2n-1} \in S(x_{2n-2})$, $x_{2n} \in T(x_{2n-1})$ and with the following property

$$d(x_n, x_{n+1}) \leq l^n \ d(x_0, x_1),$$

for all $n \in \mathbb{N}^*$.

This inequality implies that $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence, because (X, d) is a complete metric space. Let $x^* = \lim_{n\to\infty} x_n$. From $x_{2n} \in T(x_{2n-1})$ we have that there exists $u_n \in S(x^*)$ such that

$$d(x_{2n}, u_n) \leq a_{21} \ d(x_{2n-1}, x^*) + a_{22} \ d(x_{2n-1}, x_{2n}) + a_{23} \ d(x^*, u_n) +$$

$$+ a_{24} \ d(x_{2n-1}, u_n) + a_{25} \ d(x^*, x_{2n}),$$

for all $n \in \mathbb{N}^*$. Using the triangle inequality we obtain

$$d(x^*, u_n) \leq [1 - (a_{23} + a_{24})]^{-1}[(a_{21} + a_{24}) \ d(x^*, x_{2n-1}) +$$

$$+ a_{22} \ d(x_{2n-1}, x_{2n}) + (1 + a_{25}) \ d(x^*, x_{2n})],$$

for all $n \in \mathbb{N}^*$. This implies that $d(x^*, u_n) \to 0$, as $n \to \infty$. Since $u_n \in S(x^*)$, for all $n \in \mathbb{N}^*$ and $S(x^*)$ is a closed set, it follows that $x^* \in S(x^*)$, i. e. $x^* \in F_S$. So $F_S = F_T \in P(X)$.

Let us prove now that F_S is a closed set. For this purpose let $x_n \in F_S = F_T$, for $n \in \mathbb{N}$, such that $x_n \to x^*$, as $n \to \infty$. From $x_n \in T(x_n)$ we have that there exists $u_n \in S(x^*)$ such that

$$d(x_n, u_n) \leq a_{21} \ d(x_n, x^*) + a_{22} \ d(x_n, x_n) + a_{23} \ d(x^*, u_n) + a_{24} \ d(x_n, u_n) + a_{25} \ d(x^*, x_n),$$

for all $n \in \mathbb{N}$. Using the triangle inequality we have

$$d(x^*, u_n) \leq [1 - (a_{23} + a_{24})]^{-1}(1 + a_{21} + a_{24} + a_{25}) \ d(x^*, x_n),$$

for all $n \in \mathbb{N}$. This implies that $d(x^*, u_n) \to 0$, as $n \to \infty$. Since $u_n \in S(x^*)$, for all $n \in \mathbb{N}$ and $S(x^*)$ is a closed set, it follows that $x^* \in S(x^*)$. Therefore $F_S = F_T \in P_{cl}(X)$.

For the case when the condition (ii) is fulfilled, the demonstration is made similarly with the proof from the first case. ■

Remark 2.5 *Let (X,d) be a complete metric space and $(T_n)_{n\in\mathbb{N}}$ be a sequence of multivalued operators $T_n : X \to P_{cl}(X)$, for $n \in \mathbb{N}$. If each pair of multivalued operators (T_0, T_n), for $n \in \mathbb{N}^*$, satisfies similar conditions as in the Theorem 2.2, then $F_{T_n} = F_{T_0} \in P_{cl}(X)$, for all $n \in \mathbb{N}^*$.*

3 Common strict fixed points

Theorem 3.1 *Let (X,d) be a metric space and $T : X \to P(X)$ be a multivalued operator. We suppose that at least one of the following conditions is satisfied:*

(i) *there exists $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a function with the property that $\varphi(t) < t$, for all $t > 0$ and such that for each $x \in X$, any $u_x \in T(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ so that the relation (2.1) holds.*

(ii) *there exist $a_1,\ldots,a_5 \in \mathbb{R}_+$, with $a_1+a_4+a_5 < 1$ such that for each $x \in X$, any $u_x \in T(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ so that the relation (2.2) holds.*

(iii) *there exists $a \in \mathbb{R}_+$, with $a < 1$ such that for each $x \in X$, any $u_x \in T(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ so that the relation (2.3) holds.*

(iv) *there exists $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ a function with the property that $\varphi(t,0,0,t,t) < t$, for all $t > 0$ and such that for each $x \in X$, any $u_x \in T(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ so that the relation (2.4) holds.*

If $(SF)_T \neq \emptyset$, then $F_T = (SF)_T = \{x^*\}$.

Proof. We assume that the condition (i) is fulfilled and we show in the first place that the strict fixed points set of the multivalued operator T is formed from a single element, i. e. $(SF)_T = \{x^*\}$. If we suppose that there exist $x^*, y^* \in (SF)_T$ such that $x^* \neq y^*$, then we reach the contradiction

$$d(x^*,y^*) \leq \varphi(d(x^*,y^*)) < d(x^*,y^*).$$

Therefore $y^* = x^*$ and $(SF)_T = \{x^*\}$. We shall prove now that any fixed point of the multivalued operator T coincides with the unique strict fixed point of T. Let $y^* \in F_T$. If we assume that $y^* \neq x^*$, then we reach the contradiction

$$d(y^*,x^*) \leq \varphi(d(y^*,x^*)) < d(y^*,x^*),$$

because $y^* \in T(y^*)$ and $T(x^*) = \{x^*\}$. So $y^* = x^*$ and we are able to write that $F_T \subseteq (SF)_T$. Thus $F_T = (SF)_T = \{x^*\}$.

We suppose now that the condition (ii) is verified. If we assume that there exist $x^*, y^* \in (SF)_T$, then for $T(x^*) = \{x^*\}$ and $T(y^*) = \{y^*\}$ we have

$$d(x^*,y^*) \leq a_1\ d(x^*,y^*) + a_2\ d(x^*,x^*) + a_3\ d(y^*,y^*) + a_4\ d(x^*,y^*) + a_5\ d(y^*,x^*) =$$

$$= (a_1 + a_4 + a_5)\ d(x^*, y^*).$$

This implies that $d(x^*, y^*) = 0$. So, if $(SF)_T \neq \emptyset$, then $(SF)_T = \{x^*\}$. Let $y^* \in F_T$. For $y^* \in T(y^*)$ and $T(x^*) = \{x^*\}$ we have

$$d(y^*, x^*) \leq a_1\ d(y^*, x^*) + a_2\ d(y^*, y^*) + a_3\ d(x^*, x^*) + a_4\ d(y^*, x^*) + a_5\ d(x^*, y^*) =$$

$$= (a_1 + a_4 + a_5)\ d(y^*, x^*).$$

It follows that $d(y^*, x^*) = 0$ and therefore $F_T \subseteq (SF)_T$. So $F_T = (SF)_T = \{x^*\}$.

For the case when the condition (iii) is fulfilled, the demonstration is made similarly with the proof from the second case and for the case when the condition (iv) is verified, the demonstration is made similarly with the proof from the first case. ■

Theorem 3.2 *Let (X, d) be a metric space and $S, T : X \to P(X)$ be two multivalued operators. We suppose that at least one of the following conditions is satisfied*:

(i) *there exist $a_{11}, \ldots, a_{15} \in \mathbb{R}_+$, with $a_{12} + a_{15} < 1$ and $a_{13} + a_{14} < 1$ such that for each $x \in X$, any $u_x \in S(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ so that the relation* (2.7) *holds*;

there exist $a_{21}, \ldots, a_{25} \in \mathbb{R}_+$, with $a_{22} + a_{25} < 1$ and $a_{23} + a_{24} < 1$ such that for each $x \in X$, any $u_x \in T(x)$ and for all $y \in X$, there exists $u_y \in S(y)$ so that the relation (2.8) *holds*;

$a_{11} + a_{14} + a_{15} < 1$ *or* $a_{21} + a_{24} + a_{25} < 1$.

(ii) *there exists $a_1 \in \mathbb{R}_+$, with $a_1 < 1$ such that for each $x \in X$, any $u_x \in S(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ so that the relation* (2.9) *holds*;

there exists $a_2 \in \mathbb{R}_+$, with $a_2 < 1$ such that for each $x \in X$, any $u_x \in T(x)$ and for all $y \in X$, there exists $u_y \in S(y)$ so that the relation (2.10) *holds.*

(iii) *there exists $\varphi_1 : \mathbb{R}^5_+ \to \mathbb{R}_+$ a function with the property that $\varphi_1(0, 0, t, t, 0) < t$ and $\varphi_1(0, t, 0, 0, t) < t$, for all $t > 0$ and such that for each $x \in X$, any $u_x \in S(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ so that the relation* (2.11) *holds*;

there exists $\varphi_2 : \mathbb{R}^5_+ \to \mathbb{R}_+$ a function with the property that $\varphi_2(0, 0, t, t, 0) < t$ and $\varphi_2(0, t, 0, 0, t) < t$, for all $t > 0$ and such that for each $x \in X$, any $u_x \in T(x)$ and for all $y \in X$, there exists $u_y \in S(y)$ so that the relation (2.12) *holds*;

$\min\ \{\ \varphi_1(t, 0, 0, t, t),\ \varphi_2(t, 0, 0, t, t)\ \} < t$, *for all* $t > 0$.

If $(SF)_S \cup (SF)_T \neq \emptyset$, then $F_S = F_T = (SF)_S = (SF)_T = \{x^*\}$.

Proof. Taking into account the Remark 2.4 we are able to write that $F_S = F_T$.

We assume that the condition (i) is fulfilled and we show that the strict fixed points set of S coincides with the strict fixed points set of T, i. e. $(SF)_S = (SF)_T$. Let $x^* \in (SF)_S \subseteq F_S = F_T$. If there exists $u \in T(x^*)$, then for $u \in T(x^*)$ and $S(x^*) = \{x^*\}$ we have

$$d(u, x^*) \leq a_{21}\ d(x^*, x^*) + a_{22}\ d(x^*, u) + a_{23}\ d(x^*, x^*) + a_{24}\ d(x^*, x^*) + a_{25}\ d(x^*, u) =$$

$$= (a_{22} + a_{25})\ d(u, x^*).$$

This implies that $d(u, x^*) = 0$. Hence $u = x^*$ and $T(x^*) = \{x^*\}$. So $(SF)_S \subseteq (SF)_T$. Analogously, we can show that $(SF)_T \subseteq (SF)_S$. Therefore $(SF)_S = (SF)_T$.

If we suppose that there exist $x^*, y^* \in (SF)_S = (SF)_T$ such that $x^* \neq y^*$, then we reach the contradiction

$$d(x^*, y^*) \leq \min\ \{\ a_{11} + a_{14} + a_{15},\ a_{21} + a_{24} + a_{25}\ \}\ d(x^*, y^*) < d(x^*, y^*).$$

Therefore $y^* = x^*$ and $(SF)_S = (SF)_T = \{x^*\}$.

Let $y^* \in F_S = F_T$. If we assume that $y^* \neq x^*$, then we reach the contradiction

$$d(y^*, x^*) \leq \min\ \{\ a_{11} + a_{14} + a_{15},\ a_{21} + a_{24} + a_{25}\ \}\ d(y^*, x^*) < d(y^*, x^*),$$

because $y^* \in S(y^*),\ T(x^*) = \{x^*\}$ and $y^* \in T(y^*),\ S(x^*) = \{x^*\}$. So $y^* = x^*$ and we are able to write that $F_S = F_T \subseteq (SF)_S = (SF)_T$.

Thus $F_S = F_T = (SF)_S = (SF)_T = \{x^*\}$.

For the case when is fulfilled the condition (*ii*) or the condition (*iii*), the proof is made similarly with the above. ■

Remark 3.1 *Let (X, d) be a metric space and $(T_n)_{n\in\mathbb{N}}$ be a sequence of multivalued operators $T_n : X \to P(X)$, for $n \in \mathbb{N}$. If each pair of multivalued operators (T_0, T_n), for $n \in \mathbb{N}^*$, satisfies similar conditions as in the Theorem* 3.2 *and $(SF)_{T_0} \neq \emptyset$, then $F_{T_n} = F_{T_0} = (SF)_{T_n} = (SF)_{T_0} = \{x^*\}$, for all $n \in \mathbb{N}^*$.*

Theorem 3.3 *Let (X, d) be a metric space and $(T_n)_{n\in\mathbb{N}}$ be a sequence of multivalued operators $T_n : X \to P(X)$, for $n \in \mathbb{N}$. We suppose that at least one of the following conditions is satisfied:*

(*i*) *there exists $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a function with the property that $\varphi(t) < t$, for all $t > 0$ and there exist $i, j \in \mathbb{N}$, with $i \neq j$ such that for each $x \in X$, any $u_x \in T_i(x)$ and for all $y \in X$, there exists $u_y \in T_j(y)$ so that the relation* (2.1) *holds.*

(*ii*) *there exist $a_1, \ldots, a_5 \in \mathbb{R}_+$, with $a_1 + a_4 + a_5 < 1$ and there exist $i, j \in \mathbb{N}$, with $i \neq j$ such that for each $x \in X$, any $u_x \in T_i(x)$ and for all $y \in X$, there exists $u_y \in T_j(y)$ so that the relation* (2.2) *holds.*

(*iii*) *there exists $a \in \mathbb{R}_+$, with $a < 1$ and there exist $i, j \in \mathbb{N}$, with $i \neq j$ such that for each $x \in X$, any $u_x \in T_i(x)$ and for all $y \in X$, there exists $u_y \in T_j(y)$ so that the relation* (2.3) *holds.*

(*iv*) *there exists $\varphi : \mathbb{R}^5_+ \to \mathbb{R}_+$ a function with the property that $\varphi(t, 0, 0, t, t) < t$, for all $t > 0$ and there exist $i, j \in \mathbb{N}$, with $i \neq j$ such that for each $x \in X$, any $u_x \in T_i(x)$ and for all $y \in X$, there exists $u_y \in T_j(y)$ so that the relation* (2.4) *holds.*

If $ComSFP(T) \neq \emptyset$, then $ComFP(T) = ComSFP(T) = \{x^*\}$.

Proof. We assume that the condition (i) is fulfilled and we show in the first place that the common strict fixed points set of the multivalued operators T_n, for $n \in \mathbb{N}$, is formed from a single element, i. e. $ComSFP(T) = \{x^*\}$. If we suppose that there exist $x^*, y^* \in ComSFP(T)$ such that $x^* \neq y^*$, then we reach the contradiction

$$d(x^*, y^*) \leq \varphi(d(x^*, y^*)) < d(x^*, y^*),$$

because $T_i(x^*) = \{x^*\}$ and $T_j(y^*) = \{y^*\}$. Therefore $y^* = x^*$ and $ComSFP(T) = \{x^*\}$. We shall prove now that any common fixed point of the multivalued operators T_n, for $n \in \mathbb{N}$, coincides with the unique common strict fixed point of the multivalued operators T_n, for $n \in \mathbb{N}$. Let $y^* \in ComFP(T)$. If we assume that $y^* \neq x^*$, then we reach the contradiction

$$d(y^*, x^*) \leq \varphi(d(y^*, x^*)) < d(y^*, x^*),$$

because $y^* \in T_i(y^*)$ and $T_j(x^*) = \{x^*\}$. So $y^* = x^*$ and we are able to write that $ComFP(T) \subseteq ComSFP(T)$. Thus $ComFP(T) = ComSFP(T) = \{x^*\}$.

We suppose now that the condition (ii) is verified. If we assume that there exist $x^*, y^* \in ComSFP(T)$, then for $T_i(x^*) = \{x^*\}$ and $T_j(y^*) = \{y^*\}$ we have

$$d(x^*, y^*) \leq a_1\ d(x^*, y^*) + a_2\ d(x^*, x^*) + a_3\ d(y^*, y^*) + a_4\ d(x^*, y^*) + a_5\ d(y^*, x^*) =$$

$$= (a_1 + a_4 + a_5)\ d(x^*, y^*).$$

This implies that $y^* = x^*$. So, if $ComSFP(T) \neq \emptyset$, then $ComSFP(T) = \{x^*\}$. Let $y^* \in ComFP(T)$. For $y^* \in T_i(y^*)$ and $T_j(x^*) = \{x^*\}$ we have

$$d(y^*, x^*) \leq a_1\ d(y^*, x^*) + a_2\ d(y^*, y^*) + a_3\ d(x^*, x^*) + a_4\ d(y^*, x^*) + a_5\ d(x^*, y^*) =$$

$$= (a_1 + a_4 + a_5)\ d(y^*, x^*).$$

It follows that $d(y^*, x^*) = 0$ and therefore $ComFP(T) \subseteq ComSFP(T)$.

So $ComFP(T) = ComSFP(T) = \{x^*\}$.

For the case when the condition (iii) is fulfilled, the demonstration is made similarly with the proof from the second case and for the case when the condition (iv) is verified, the demonstration is made similarly with the proof from the first case. ■

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