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DYNAMIC ITERATION METHODS FOR DIFFERENTIAL EQUATIONS WITH MIXED MODIFICATION OF THE ARGUMENT

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Abstract. In this paper we study dynamic iteration techniques for nonlinear differential equations with mixed modification of the argument. The dynamic iteration method generalizes the well known Picard iterations, improving significantly the convergence speed of the iterative process. It also have the advantage of decoupling the part containing the modified argument so the iteration steps consist in solving only ordinary differential equations. Error estimates are given, proving superlinear convergence for the dynamic iteration.

Keywords: dynamic iteration, mixed modified argument .
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1 Introduction

Consider the differential equation with a mixed modification of the argument

$$\begin{cases} x"(t) = f(t, x(t), x(t-\tau_1(t)), x(t+\tau_2(t))) \\ x(t) = \phi(t),\ t \in [t_\phi, 0] \\ x(t) = \psi(t),\ t \in [T, t_\psi], \end{cases} \tag{1.1}$$

where $0 \le \tau_1(t) \le t_\phi$, $0 \le \tau_2(t) \le t_\psi$, $t \in [0, T]$.
By taking

$$u(t) := \begin{cases} \phi(t), & t \in [t_\phi, 0] \\ \frac{\psi(T)-\phi(0)}{T} t - \phi(0), & t \in [0, T] \\ \psi(t), & t \in [T, t_\psi] \end{cases}$$

and

$$\overline{G}(t,s) := \begin{cases} 0, & t \in [t_\phi, 0] \\ G(t,s), & t \in [0, T] \\ 0 & t \in [T, t_\psi] \end{cases}$$

where

$$G(t,s)=\begin{cases}\frac{s(T-t)}{T}, & s\le t,\\ \frac{t(T-s)}{T}, & t\le s,\end{cases}\quad s,t\in[0,T],$$

the problem (1.1) can be written as a Fredholm integral equation of second species in the following way:

$$x(t)=u(t)+\int_0^T \overline{G}(t,s)f(s,x(s),x(s-\tau_1(s)),x(s+\tau_2(s)))ds \tag{1.2}$$

We express this integral equation in an operator form, by defining the integral operator $A:C[t_\phi,t_\psi]\to C[t_\phi,t_\psi]$

$$Ax(t):=u(t)+\int_0^T \overline{G}(t,s)f(s,x(s),x(s-\tau_1(s)),x(s+\tau_2(s)))ds$$

for each $t\in[0,T]$.

Problem (1.1) becomes now equivalent to the following fixed-point problem

$$x=Ax,\quad x\in C[t_\phi,t_\psi]. \tag{1.3}$$

We may use Picard-Banach theorem for establishing an existence and uniqueness result for the problem (1.1).

Theorem 1.1 *If there exists* $L>0$ *such that*

$$|f(t,u_1,v_1,w_1)-f(t,u_2,v_2,w_2)|\le L(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|) \tag{1.4}$$

$t\in[0,T]$, $u_1,v_1,w_1,u_2,v_2,w_2\in\mathbf{R}$ *and*

$$2L\sup_{t\in[0,T]}\int_0^T G(s,t)ds\le q<1 \tag{1.5}$$

then problem (1.1) *has a solution which is unique and may be obtained using succesive approximations, starting from any element of the space* $C[t_\phi,t_\psi]$.

Proof. The proof is straightforward, using the Picard-Banach fix-point theorem. □

Remark. A similar theorem can be given in a sphere from the space $C[t_\phi,t_\psi]$, but for the sake of simplicity, we will use Theorem 1.1.

Corollary 1.1 (**The Picard iteration**) *The Picard iteration* (PI) *defined by*

$$\begin{cases} x"_{k+1}(t)=f(t,x_k(t),x_k(t-\tau_1(t)),x_k(t+\tau_2(t))), & t\in[0,T]\\ x_0\in C[t_\phi,t_\psi] & \\ x_k|_{[t_\phi,0]}=\phi, & k\ge 0\\ x_k|_{[T,t_\psi]}=\psi, & k\ge 0\end{cases}$$

converges to the exact solution of problem (1.1).

Define the iteration error by $e_k = x - x_k$ for $k \geq 0$ and denote

$$a(t) := 2L \int_0^T G(s,t)ds.$$

Then, as a consequence of Theorem 1.1, the following error estimate holds

$$|e_{k+1}(t)| \leq a(t)^{n+1} |e_k(t)| \text{ for } t \in [0,T] \tag{1.6}$$

or, since $|a(t)| \leq 2LT$,

$$\|e_{k+1}\|_\infty \leq (2LT)^{k+1} \|e_0\|_\infty \tag{1.7}$$

2 The dynamic iteration

Consider the following iterative process, known as the dynamic iteration (DI) or the waveform relaxation (WR) method.

$$\begin{cases} x"_{k+1}(t) = f(t, x_{k+1}(t), x_k(t - \tau_1(t)), x_k(t + \tau_2(t))), & t \in [0,T] \\ x_{k+1}(0) = \phi(0) & \\ x_{k+1}(T) = \psi(T) & \\ x_0 \in C[t_\phi, t_\psi] & \\ x_k|_{[t_\phi, 0]} = \phi, & k \geq 0 \\ x_k|_{[T, t_\psi]} = \psi, & k \geq 0 \end{cases} \tag{2.8}$$

The iterative process (2.8) requires solving at each step a boundary value problem for a second order ordinary differential equation. We have hence decoupled the delay part of the problem from the main part. For solving the boundary value problem at each iteration step we may use a standard solver, provided with a continuous output. The resulting method is called numerical dynamic iteration (NDI) or numerical waveform relaxation (NWR).

3 Error estimates for the DI

The conditions in Theorem 1.1 insure existence and uniqueness of a solution for each step of the DI. Nevertheless, in order to get better error estimates, we will require two Lpischitz conditions for the function f in the following way:

$$\begin{aligned} &|f(t, u_1, v, w) - f(t, u_2, v, w)| \leq \lambda |u_1 - u_2| \\ &|f(t, u, v_1, w_1) - f(t, u, v_2, w_2)| \leq \nu(|v_1 - v_2| + |w_1 - w_2|) \end{aligned} \tag{3.9}$$

for any $t \in [a,b]$, $u, v, w, u_1, v_1, w_1, u_2, v_2, w_2 \in \mathbf{R}$.

Then Theorem 1.1 is still valid, by taking $L := \max\{\lambda, \nu\}$ and a unique solution for each step of the DI always exists. We now estimate the error.

$$|e_{k+1}(t)| = |x(t) - x_{k+1}(t)| = |Ax(t) - Ax_{k+1}(t)| \leq$$

$$
\begin{gathered}
\leq \left| \int_0^T G(t,s) f(s, x(s), x(s-\tau_1(s), x(s+\tau_2(s)))ds - \right. \\
\left. - \int_0^T G(t,s) f(s, x_{k+1}(s), x_k(s-\tau_1(s)), x_k(s+\tau_2(s)))ds \right| \leq \\
\leq \int_0^T G(t,s) \left| f(s, x(s), x(s-\tau_1(s)), x(s+\tau_2(s))) - \right. \\
\left. - f(s, x_{k+1}(s), x_k(s-\tau_1(s)), x_k(s+\tau_2(s))) \right| ds \leq \\
\leq \int_0^T G(t,s) \left| f(s, x(s), x(s-\tau_1(s)), x(s+\tau_2(s))) - \right. \\
\left. - f(s, x_{k+1}(s), x(s-\tau_1(s)), x(s+\tau_2(s))) \right| ds + \\
+ \int_0^T G(t,s) \left| f(s, x_{k+1}(s), x(s-\tau_1(s)), x(s+\tau_2(s))) - \right. \\
\left. - f(s, x_{k+1}(s), x_k(s-\tau_1(s)), x_k(s+\tau_2(s))) \right| ds \leq \\
\leq \int_0^T G(t,s) \left(\lambda |x(s) - x_{k+1}(s)| + \nu |x(s-\tau_1(s)) - x_k(s-\tau_1(s))| + \right. \\
\left. + \nu |x(s+\tau_2(s)) - x_k(s+\tau_2(s))| \right) ds \leq \\
\leq \lambda \int_0^T G(t,s) |e_{k+1}(s)| ds + \nu \int_0^T G(t,s) q(s) ds
\end{gathered}
$$

using the notation $q(s) := |e_k(s-\tau_1(s))| + |e_k(s+\tau_2(s))|$.
Taking into account that

$$q(s) \leq 2\|e_k\|_\infty,$$

the final estimate is

$$|e_{k+1}(t)| \leq \lambda \int_0^T G(t,s)|e_{k+1}(s)| + 2\nu\|e_k\|_\infty. \tag{3.10}$$

Consider now the following Fredholm linear integral equation

$$e(t) = \lambda \int_0^T G(t,s)e(s) + 2\nu\|e_k\|_\infty, \tag{3.11}$$

which is equivalent to the linear boundary value problem

$$\begin{cases} e''(t) + \lambda e(t) + \nu\|e_k\|_\infty = 0 \quad t \in [0,T] \\ e(0) = 0 \\ e(T) = 0 \end{cases} \tag{3.12}$$

The solution of this problem and also solution of (3.11) is obtained after simple computations:

$$e(t) = 2\frac{\nu}{\lambda}\left(\frac{\sin\sqrt{\lambda}(T-t) + \sin\sqrt{\lambda}t}{\sin\sqrt{\lambda}T} - 1\right)\|e_k\|_\infty \tag{3.13}$$

Remark. The solution of (3.11) given by (3.13) is always positive in the conditions of Theorem 1, namely

$$2\max\{\lambda,\nu\}\sup_{t\in[0,T]}\int_0^T G(s,t)ds \le q < 1. \tag{3.14}$$

Theorem 3.1 *The following estimate holds for the error of the dynamic iteration method applied to the problem* (1.1)*:*

$$\|e_{k+1}\|_\infty \le 2\frac{\nu}{\lambda}\left(\frac{2}{\sin\sqrt{\lambda}T} - 1\right)\|e_k\|_\infty \tag{3.15}$$

Proof. For the proof we will use an idea of Rus [3]. Consider the linear integral operator attached to the equation (3.11) $V : C[0,T] \to C[0,T]$ given by

$$Vx(t) := \lambda\int_0^T G(t,s)x(s) + 2\nu\|e_k\|_\infty$$

for any $t \in [0,T]$. Under condition (3.14) V is a contraction. Then, for any $x \in C[0,T]$

$$\lim_{n\to\infty} V^{(n)}x(t) = e(t), \quad t \in [0,T],$$

as e is the unique fixed point of V.

Above, we have used the notation $V^{(n)} := V \circ \cdots \circ V$ (n times).

Also, V is a monotone increasing operator. Using these facts and inequality (3.10) we have:

$$\begin{aligned} |e_{k+1}(t)| &\le V|e_{k+1}(t)| \le (V\circ V)|e_{k+1}(t)| \le \dots \\ &\le (V\circ\cdots\circ V)|e_{k+1}(t)| \le e(t) \end{aligned}$$

and using the expression of $e(t)$ given by (3.13) we obtain the error estimate

$$\begin{aligned} |e_{k+1}(t)| &\le 2\tfrac{\nu}{\lambda}\left(\tfrac{\sin\sqrt{\lambda}(T-t)+\sin\sqrt{\lambda}t}{\sin\sqrt{\lambda}T} - 1\right)\|e_k\|_\infty \\ &\le 2\tfrac{\nu}{\lambda}\left(\tfrac{2}{\sin\sqrt{\lambda}T} - 1\right)\|e_k\|_\infty \end{aligned}$$

for any $t \in [0,T]$. This implies

$$\|e_{k+1}\|_\infty \le 2\frac{\nu}{\lambda}\left(\frac{2}{\sin\sqrt{\lambda}T} - 1\right)\|e_k\|_\infty$$

and the theorem is proved. □

Final remarks and conclusions The error estimate in Theorem 3.1 proves superlinear convergence for the dynamic iteration. It is interesting to observe that big Lipschitz constants λ, which may affect error estimates for the Picard iteration, have no negative influence on the error estimate for the dynamic iteration. This aspect and the straightforward implementation make the dynamic iteration a promising tool in solving differential equations with mixed modification of the argument.

In this paper, for the sake of simplicity, we only treated the case of scalar equations. The extension to systems is a little bit more technical, but straightforward.

References

[1] **M.Bjorhus***On dynamic iteration for delay differential equations* BIT 34 (1994), 325-336

[2] **K.Burrage** *Parallel and Sequential Numerical Methods for Ordinary Differential Equations,* Oxford University Press, Oxford, 1995

[3] **I.A.Rus** *Picard operators and applications*, Seminar on Fixed Point Theory, "Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, Cluj-Napoca, Preprint Nr.3, 1996.