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ON SOME GRONWALL INEQUALITIES

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Abstract. The purpose of this note is to obtain an upper bound for the solutions of the Bernoulli integral inequation via Picard operator technique.

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1 Introduction

In the paper [2] N. Lungu has obtained an upper bound for the solutions of the Bernoulli-integral inequation:

$$y(x) \leq c + \int_a^x [P(s)\, y(s) + Q(s)\, y^{\alpha}(s)]\, ds \qquad (1) \qquad (1)$$

where $y \in C([a,b], R_+); x \in [a,b]; \alpha \geq 0, \alpha \neq 0, \alpha \neq 1; c \geq 0$; $P, Q \in C[a,b]$.

The aim of this paper is to obtain an similary results using the Picard operators techniques.

We shall use the following abstract lemma of Rus (see [1], [3], [4] and [5]).

Theorem 1. Let $(X, d, \leq)$ be an ordered metric space. Let $A : X \to X$ be such that:

(i) A is monotone increasing ;
(ii) A is a Picard operator $(F_A = \{x_A^*\})$

Then (a) $x \leq A(x) \Rightarrow x \leq x_A^*$;
(b) $x \geq A(x) \Rightarrow x \geq x_A^*$.

2 Main results

Let us consider the problem:

$$y(x) = c + \int_a^x [P(s)\, y(s) + Q(s)\, y^{\alpha}(s)]\, ds \qquad (2) \qquad (2)$$

Theorem 2. Let us suppose that:

(i) $y \in C([a,b], R_+)$; $P, Q \in C[a,b]$;
(ii) $P(x) \geq 0$, $Q(x) \geq 0$ for all $x \in [a,b]$;
(iii) $[(R+c)M_1 + (R+c)^\alpha M_2](b-a) \leq R$ where M_1, M_2, R be such that $|P(x)| \leq M_1$, $|Q(x)| \leq M_2$ and
$y \in \overline{B}(c,R) \subset (C([a,b], R_+), \|.\|_C) \Longrightarrow y(x) \in R_+$ for all $x \in [a,b]$.

Then there exists a unique solution $y^* \in \overline{B}(c,R)$ for the integral equation (2) and if $y \in C([a,b], R_+)$ is a solution of (1) then

$$y(x) \leq e^{\int_a^x P(s)ds} \left[c + (1-\alpha) \int_a^x Q(t) e^{(\alpha-1)\int_a^t P(s)ds} dt \right]^{\frac{1}{1-\alpha}}$$

for all $x \in [a,b]$.

Proof. Let us consider the operator $A : \overline{B}(c,R) \to C([a,b], R_+)$

$$A(y)(x) = c + \int_a^x [P(s)y(s) + Q(s)y^\alpha(s)]\,ds$$

From (iii) we have that $A\left(\overline{B}(c,R)\right) \subset \overline{B}(c,R)$.

Let $A : \overline{B}(c,R) \to \overline{B}(c,R)$. Then the problem (2) becomes

$$y = A(y) \tag{3}$$

and the problem (1) becomes

$$y \leq A(y) \tag{4}$$

Let $\|\cdot\|_B$ be a Bielecki norm and denote by d_B the induced metric.

For $y, z \in C([a,b], R_+)$ we consider the order relation $\leq$ such that $y \leq z$ if $y(x) \leq z(x)$ for all x.

From $(i) - (iii)$ it follows that

$$\|Ay - Az\|_B \leq \frac{M_1 + \alpha M_2 (R+c)^{\alpha-1}}{\gamma} \|y - z\|_B ,$$

for all $y, z \in \overline{B}(c,R)$, $\gamma \geq 0, \gamma \neq 0$.

Let us consider γ such that $\dfrac{M_1 + \alpha M_2 (R+c)^{\alpha-1}}{\gamma} < 11$. Then A is a contraction mapping on the complete metric space $\left(\overline{B}(c,R), d_B\right)$. By the Banach contraction mapping theorem, A is Picard and let us denote by y^* its unique fixed points.

$$y^*(x) = e^{\int_a^x P(s)ds} \left[c + (1-\alpha) \int_a^x Q(t) e^{(\alpha-1)\int_a^t P(s)ds} dt \right]^{\frac{1}{1-\alpha}} .$$

Using assumption (ii) , it is easy to see that A is monotone increasing.

In conclusion, the operator A verifies the hypothesis of Theorem 1, so, if $y \leq A(y)$ then $y \leq y^*$.So, the theorem has been proved.

References

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