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MAPPINGS WITH THE INTERSECTION PROPERTY AND A MEASURE OF NONCOMPACTNESS ON SEPARABLE BANACH SPACES

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Abstract. The aim of this note is to present examples of mappings with the intersection property. Also we shall give an explicite definition of an abstract measure of noncompactness on a separable Banach space and we shall prove that Hausdorff's noncompactness measure is a mapping with the intersection property.

Keywords: measure of noncompactness, (θ, a)-contraction mapping, separable Banach space
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1 Mappings with the intersection property

Notations. Let X be a nonempty set and
$P(X) = \{M|\ M \subset X\}$,
$P_b(X) = \{M \subset X|\ M \text{ is bounded}\}$,
$P_{b,cl}(X) = \{M \subset X|\ M \text{ is bounded and closed}\}$,
$P_{b,cl,cv}(X) = \{M \subset X|\ M \text{ is bounded, closed, convex}\}$,
$P_{cp} = \{M \subset X|\ M \text{ is compact}\}$.

Definition 1.1. ([4]) Let X be a nonempty set, $Z \subset P(X)$, Z is nonempty. A mapping $\theta : Z \to R$, has the intersection property if $Y_{n+1} \subset Y_n$, $n \in N$ and $\lim\limits_{n\to\infty} \theta(Y_n) = 0$ implies that $Y_\infty := \bigcap\limits_{n\in N} Y_n$ is nonempty, $Y_\infty \in Z$ and $\theta(Y_\infty) = 0$. Let us illustrate this notion by

Example 1.1. [5] Let $(X, |\cdot|)$ be a Banach space. A mapping $\alpha : P_b(X) \to R_+$ is said to be a Danes-Pasicki noncompactness measure if:
(i) $\alpha(A) = 0$ implies $\overline{A} \in P_{cp}(X)$;
(ii) $A \subset B$ implies $\alpha(A) \leq \alpha(B)$;
(iii) $\alpha(A \cup \{x\}) = \alpha(A)$ for all $A \in P_b(X)$ and $x \in X$.
We denote such a mapping by α_{DP}.

Theorem 1.1. *α_{DP} has the intersection property.*

Proof. Let $Z = P_{b,cl}(X)$, $\theta = \alpha_{DP}$, $Y_{n+1} \subset Y_n$, $Y_n \in Z$, $n \in N$, $\lim\limits_{n\to\infty} \theta(Y_n) = 0$ and $a_n \in Y_n$, $n \in N$. Let $B_n = \{a_n, a_{n+1}, \dots\}$. Obviously $B_{n+1} \subset B_n$ for every

$n \in N$. Since $a_{n+p} \in Y_{n+p} \subset Y_n$ for $p \geq 1$ we have $B_n \subset Y_n$, $n \in N$ and since $Y_n \in P_{b,cl}(X)$ we obtain $B_n \in P_b(X)$ for every $n \in N$. From (iii) we have

$$\theta(B_0) = \theta(B_1) = \cdots = \theta(B_n). \tag{1}$$

From (ii) we obtain $0 \leq \theta(B_n) \leq \theta(Y_n)$ and since $\lim\limits_{n\to\infty} \theta(Y_n) = 0$, we have $\lim\limits_{n\to\infty} \theta(B_n) = 0$. This relation and (1) implies $\theta(B_0) = 0$. Using (i) we obtain $\overline{B_0} \in P_{cp}(X)$. Let $a^* \in \overline{B_0}$. Then, there exists $a_{n_i} \in B_0$, $a_{n_i} \to a^*$ as $n_i \to \infty$. We have $a_{n_i} \in Y_{n_i}$ and $Y_{n_i} \in P_{cl}(X)$. This implies $a^* \in Y_{n_i}$ for $n_i \geq n^*$, $n^* \in N$. But $Y_{n_i} \subset Y_{n^*}$ for $n_i \geq n^*$. It follows that $a^* \in Y_n$ for every $n \in N$, so $a^* \in Y_\infty$ and Y_∞ is nonempty. From $Y_\infty \subset Y_n$, we obtain $\theta \leq \theta(Y_\infty) \leq \theta(Y_n)$. Since $\lim\limits_{n\to\infty} \theta(Y_n) = 0$ we have $\theta(Y_\infty) = 0$. Now $Y_n \in P_{b,cl}(X)$ for every $n \in N$, implies $Y_\infty = \bigcap\limits_{n\in N} Y_n \in P_{b,cl}(X)$, so α_{DP} has the intersection property.

The next lemma gives us an other example of mapping with the intersection property.

Lemma 1.2. *Let X be a Banach space, $Y \subset P(X)$, Y is nonempty and $\theta_1, \theta_2 : Y \to R_+$, $\varphi : R_+^2 \to R_+$. Let $\theta : Y \to R_+$ be given by $\theta(A) = \varphi(\theta_1(A), \theta_2(A))$ for $A \in Y$. Suppose that:*

(i) $\varphi \in C(R_+^2, R_+)$;

(ii) $Z_{\varphi(\cdot,x)} = \{0\}$ for evert $x \in R_+$;

(iii) θ_1 and θ_2 are increasing mappings and θ_1 has the intersection property.

Here $Z_{\varphi(\cdot,x)} = \{y \in R_+ |\ \varphi(y, x) = 0\}$.

Then $\theta = \varphi(\theta_1, \theta_2)$ has the intersection property.

Proof. Let $A_n \in Y$, $A_{n+1} \subset A_n$ for $n \in N$, $\theta(A_n) \to 0$, $n \to \infty$ $(*)$. Since θ_1 and θ_2 are increasing, we have $0 \leq \theta_i(A_{n+1}) \leq \theta_i(A_n) \leq \theta_i(A_0)$, $i = \overline{1,2}$. This implies that the sequences $(\theta_1(A_n))_n$ and $(\theta_2(A_n))_n$ are decreasing and bounded. Then, there exists $\lim\limits_{n\to\infty} \theta_1(A_n) = a_1$ and $\lim\limits_{n\to\infty} \theta_2(A_n) = a_2$. Now, using $(*)$, we have $\lim\limits_{n\to\infty} \varphi(\theta_1(A_n), \theta_2(A_n)) = 0$ and since φ is continuous, we obtain $\varphi(a_1, a_2) = 0$. This together with (ii) implies $a_1 = 0$, so $\theta_1(A_n) \to 0$, $n \to \infty$. Since θ_1 has the intersection property it follows that $A_\infty := \bigcap\limits_{n\in N} A_n$ is nonempty, $A_\infty \in Y$ and $\theta_1(A_\infty) = 0$. Then, from (ii) we derive $\theta(A_\infty) = \varphi(\theta_1(A_\infty), \theta_2(A_\infty)) = \varphi(0, \theta_2(A_\infty)) = 0$. Thus the proof is complete.

Let X be a Banach space and $(X, S(X), M)$ be the fixed point structure (see [5]), where $S(X) = P_{b,cl,cv}(X)$ and $M(Z) = \{f : Z \to Z |\ f$ is continous$\}$ for each $Z \in S(X)$. Consider $Y = S(X)$. Let $\eta : P(X) \to P(X)$, $\eta(A) = \overline{A}$ for every $A \in P(X)$. Obviously $\eta(Y) = Y$. We consider a mapping $\theta : Y \to \mathbb{R}_+$ as in the previous lemma. We have:

(1) η is a closure operator, $S(X) = \eta(Y) = Y$, $\theta(\eta(A)) = \theta(A)$ for $A \in Y$ since $\eta(A) = A$ for $A \in Y$.

(2) $F_\eta \cap Z_\theta \subset S(X)$ since $A \in F_\eta \cap Z_\theta$ implies $A \in F_\eta \Rightarrow \eta(A) = A \Rightarrow \overline{A} = A \Rightarrow A \in S(X)$.

From (1) and (2) it follows that (θ, η) is a pair compatible with the fixed point structure. Let $Z \in \eta(Y)$ and $f \in M(Z)$. According to Lemma 1.2, we have:

(i') $\theta/_{\eta(Y)}$ has the intersection property.

In addition suppose:

(ii') f is (θ, a)-contraction.

Theorem 1.3. [4] Suppose (i'), (ii') and above fixed point structures. Then:

a) $I(f) \cap S(X)$ is nonempty;

b) F_f is nonempty;

c) if $F_f \in Y$, then $\theta(F_f) = 0$.

Here $I_f(G) = \{G \subset Z |\ f(G) \subset G\}$.

2 A measure of noncompactness on separable Banach space

Let E be a Banach space, $P_{b,c}(E) = \{Y \subset E| \ Y \text{ is nonempty, bounded and countable}\}$, $d(\cdot,\cdot) : E \times E \to R_+$ be the metric induced by the norm of E, and $d(x, M) = \inf\limits_{y \in M} d(x, y)$, $M \subset E$.

Definition 2.1. We say that a mapping $\beta : P_{b,c}(E) \to R_+$ is an abstract measure of noncompactness on E if:

(C_1) $A \subset B$ implies $\beta(A) \leq \beta(B)$, for every $A, B \in P_{b,c}(E)$;

(C_2) $\beta(A) = \beta(\overline{A})$ for every $A \in P_{b,c}(E)$;

(C_3) $\beta(coA) = \beta(A)$ for every $A \in P_{b,c}(E)$;

(C_4) $\beta(\{b\} \cup A) = \beta(A)$ for every $b \in E$ and $A \in P_{b,c}(E)$;

(C_5) $\beta(A) = 0$ if and only if A is relatively compact.

Theorem 2.1. *Let E be a separable Banach space. Assume $E = \overline{\bigcup\limits_{n \in N} E_n}$, where $(E_n)_{n \in \mathbb{N}}$ is an increasing sequence of finite dimensional subspaces of E. For any bounded countable set $A = \{x_m|\ m \in N\} \subset E$ let*

$$\beta(A) = \lim_{n \to \infty} \overline{\lim_{m \to \infty}} d(x_m, E_n).$$

Then β is a measure of noncompactness on E in the sense of Definition 2.1.

Proof. (C_1) Let $A, B \in P_{b,c}(E)$. From the definition of upper limit, we have

$$\overline{\lim_{m \to \infty}} d(x_m, E_n) \leq \overline{\lim_{m \to \infty}} (y_m, E_n)$$

for $x_m \in A$ and $y_m \in B$. This implies that

$$\lim_{n \to \infty} \overline{\lim_{m \to \infty}} d(x_m, E_n) \leq \lim_{n \to \infty} \overline{\lim_{m \to \infty}} d(y_m, E_n)$$

for $x_m \in A$ and $y_m \in B$. It follows that $\beta(A) \leq \beta(B)$.

(C_2) We have $A \subset \overline{A}$. Then, by (C_1),

$$\beta(A) \leq \beta(\overline{A}) \tag{1}$$

We have

$$\overline{A} = \{x_m |\ \exists\ y_p^m \in A,\ \exists\ p_0 \in N,\ \forall\ p \geq p_0,\ \forall\ \varepsilon > 0,\ d(y_p^m, x_m) < \varepsilon\} \qquad (*)$$

Let $\varepsilon > 0$ and $p \geq m$. We have

$$\beta(\overline{A}) = \lim_{n\to\infty} \overline{\lim_{m\to\infty}} d(x_m, E_n) \leq \lim_{n\to\infty} \overline{\lim_{m\to\infty}} d(x_m, y_p^m) + \lim_{n\to\infty} \overline{\lim_{m\to\infty}} d(y_p^m, E_n).$$

But $p \geq m$ and $y_p^m \in A$. This implies (from $(*)$)

$$\lim_{n\to\infty} \overline{\lim_{m\to\infty}} d(x_m, y_p^m) \leq \varepsilon$$

and

$$\lim_{n\to\infty} \overline{\lim_{m\to\infty}} d(y_p^m, E_n) = \beta(A).$$

That is $\beta(\overline{A}) \leq \beta(A) + \varepsilon$, for every $\varepsilon > 0$. Letting $\varepsilon \to 0$, we obtain $\beta(\overline{A}) \leq \beta(A)$ and from (1), one has $\beta(A) = \beta(\overline{A})$.

(C_3) Let $(coA)_Q = \{(1 - a_q)a_m + a_q b_m |\ a_m, b_m \in A,\ b_m \in A,\ a_q \in [0, 1] \cap Q\}$. Obviously, $A \subset (coA)_Q$ and from (C_1),

$$\beta(A) \leq \beta((coA)_{\mathbb{Q}}). \qquad (2)$$

Now

$$\beta((coA)_Q) = \lim_{n\to\infty} \overline{\lim_{m\to\infty}} d((1 - a_q)a_m + a_q b_m, E_n) \leq$$

$$\leq \lim_{n\to\infty} \overline{\lim_{m\to\infty}} d((1 - a_q)a_m, E_n) + \lim_{n\to\infty} \overline{\lim_{m\to\infty}} d(a_q b_m, E_n) =$$

$$= \lim_{n\to\infty} \overline{\lim_{m\to\infty}} d((1 - a_q)a_m, (1 - a_q)E_n) + \lim_{n\to\infty} \overline{\lim_{m\to\infty}} d(a_q b_m, a_q E_n) =$$

$$= (1 - a_q) \lim_{n\to\infty} \overline{\lim_{m\to\infty}} d(a_m, E_n) + a_q \lim_{n\to\infty} \overline{\lim_{m\to\infty}} d(b_m, E_n) =$$

$$= (1 - a_q)\beta(A) + a_q\beta(A) = \beta(A).$$

This and (2) implies

$$\beta(A) = \beta((coA)_Q) \qquad (3)$$

Since $\overline{Q} = R$, we have $\overline{(coA)_Q} = coA$. From (3) and (C_2) we obtain

$$\beta(coA) = \beta(\overline{(coA)_Q}) = \beta((coA)_Q) = \beta(A).$$

That is (C_3).

(C_4) We have $b \in E$ and $E = \overline{\bigcup_{n\in N} E_n}$. Then there exists $(y_p)_{p\in\mathbb{N}} \in \bigcup_{n\in N} E_n$ with $d(y_p, b) < \varepsilon$, for every $p \geq p_0$ and every $\varepsilon > 0$. Since $E_n \subset E_{n+1}$ for any $n \in N$, we obtain that there exists $n_0 \in N$ such that for every $n \geq n_0$ there is an $y_p \in E_n$ with $d(y_p, b) < \varepsilon$, for $p \geq p_0$ $\quad (**)$.

According to $(**)$ we have

$$\beta(\{b\}) = \lim_{n\to\infty} \overline{\lim_{m\to\infty}} d(b, E_n) = \lim_{n\to\infty} d(b, E_n) \leq$$

$$\leq \lim_{n\to\infty} d(b, y_p) + \lim_{n\to\infty} d(y_p, E_n) \leq \varepsilon + 0 = \varepsilon,$$

for every $\varepsilon > 0$. That is $\beta(\{b\}) = 0$. It follows (C_4).

(C_5) Let be $\beta(A) = 0$. We have

$$\lim_{n\to\infty} \overline{\lim_{m\to\infty}} d(x_m, E_n) = 0.$$

Then there exists $n_0 \in N$ such that for every $n \geq n_0$ and $\varepsilon > 0$,

$$\overline{\lim_{m\to\infty}} d(x_m, E_n) < \varepsilon.$$

It follows that for every $n \geq n_0$, $\varepsilon > 0$ and $x_m \in A$, there exists $y_m \in E_n$ with $d(x_m, y_m) < \varepsilon$. Therefore, for every $n \geq n_0$, $\{y_m\} \subset E_n$ is an ε-net for A (4).

Let $z_n \in E_n$ be fixed. We have

$$d(z_n, y_m) \leq d(y_m, x_m) + d(x_m, z_n) < \varepsilon + d(x_m, z_n), \quad x_m \in A,$$

A is bounded. These imply that $d(x_m, z_n) \leq c_n$. We obtain $d(z_n, y_m) \leq \varepsilon + c_n$. Since z_n is fixed we have that $\{y_m\}$ is bounded. Then $\{y_m\}$ is relatively compact and, from (4) and Hausdorff's theorem (see [3]), we have that A is relatively compact.

Now let A be a relatively compact set. From Hausdorff's theorem, for every $\varepsilon > 0$, there exists a finite net R_ε for A. This implies that for every $x_m \in A$, there exists $y_m \in R_\varepsilon$, with $d(x_m, y_m) < \varepsilon$ for every $\varepsilon > 0$. We have

$$\beta(A) = \lim_{n\to\infty} \overline{\lim_{m\to\infty}} d(x_m, E_n) \leq$$

$$\leq \lim_{n\to\infty} \overline{\lim_{m\to\infty}} d(x_m, y_m) + \lim_{n\to\infty} \overline{\lim_{m\to\infty}} d(y_m, E_n) \leq \varepsilon + \beta(R_\varepsilon).$$

But R_ε is a finite net and from (C_4) we have $\beta(R_\varepsilon) = 0$. Then $\beta(A) \leq \varepsilon$ for every $\varepsilon > 0$. Letting $\varepsilon \to 0$, we have $\beta(A) = 0$. The theorem is completely demonstrated.

Remarks. 1) The above definition of β has the advantage that it is given in terms of countable bounded sets. Such sets occur in fixed point theorems of Mönch type (see [2]). Recall that originally the Hausdorff measure of noncompactness is defined for all bounded sets of a Banach space.

2) We can replace (C_5) with

(C_5') $\beta(A) = 0$ implies A is relatively compact. Notice (C_5') is more general than (C_5).

3) The conditions $(C_1), (C_4), (C_5')$ and Theorem 1.1 shows that $\beta : P_{b,c}(E) \to R_+$,

$$\beta(A) = \lim_{n\to\infty} \overline{\lim_{m\to\infty}} d(x_m, E_n)$$

where $x_m \in A$, $E = \overline{\bigcup_{m\in A} E_n}$, $E_n \subset E_{n+1}$, $n \in N$, $(E_n)_{n\in\mathbb{N}}$ is an increasing sequence of finite dimensional Banach subspaces, is a mapping with the intersection property.

Theorem 2.2. *Let E be a Banach space, $E = \overline{\bigcup_{n\in N} E_n}$, where $(E_n)_{n\in\mathbb{N}}$ is an increasing sequence of finite dimensional subspaces of E. Let $f : E \to E$ be such that there exists $L > 0$ with $d(f(u), f(v)) \leq Ld(u, v)$, for every $u, v \in E$. Then $\beta(f(M)) \leq L\beta(M)$, for every $M \in P_{b,c}(E)$.*

Proof. We have

$$\beta(f(M)) := \lim_{n\to\infty} \overline{\lim_{m\to\infty}} d(f(x_m), E_m)$$

Since $E_n \subset E_{n+1}$ for $n \in N$ we have that there exists $x_0 \in M$ such that $x_0 \in E_n$,

$$f(x_0) \in E_n \text{ for } n \geq n_0, \text{ for some one } n_0 \in N. \tag{4}$$

We have

$$d(f(x_m), E_n) \leq d(f(x_m), f(x_0)) + d(f(x_0), E_n) \leq Ld(x_m, x_0) + d(f(x_0), E_n) \leq$$

$$\leq L[d(x_m, E_n) + d(x_0, E_n)] + d(f(x_0), E_n) \tag{5}$$

From (4) we have

$$\lim_{n\to\infty} \overline{\lim_{m\to\infty}} d(x_0, E_n) = \lim_{n\to\infty} d(x_0, E_n) = 0$$

and

$$\lim_{n\to\infty} \overline{\lim_{m\to\infty}} d(f(x_0), E_n) = \lim_{n\to\infty} d(f(x_0), E_n) = 0$$

and for $n \to \infty$ in (5) we obtain

$$\lim_{n\to\infty} \overline{\lim_{m\to\infty}} d(f(x_m), E_n) \leq \lim_{n\to\infty} \overline{\lim_{m\to\infty}} d(x_m, E_n).$$

That is $\beta(f(M)) \leq L\beta(M)$.

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