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ABOUT DIFFERENCE-DIFFERENTIAL EQUATIONS WHICH APPEAR IN NUMBER THEORY

Antal Bege

Babeş-Bolyai University
Cluj-Napoca, Romania
E-mail: abege@math.ubbcluj.ro

Abstract. In this paper we prove, that the solution of the equation

$$(Q_n(t)\,p(t))' = kp(t) - kp(t+\alpha)$$

is unique if the solution that have normalized polynomial-like boundary condition.

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1. Introduction

In the sieve theories there occur a pair of difference-differential equations with retarded arguments. These functions appear in several asymptotic formulas.
For example let $P_1(n)$ the largest prime factor of n and $P_2(n)$ the second largest prime factor of n. Then we have (Wheller [4]):

$$\sum_{\substack{1 \le n \le x \\ P_2(n) \le (P_1(n))^{\frac{1}{u}}}} 1 = e^{\gamma} p(u)\, x + O\left(\frac{x}{\log x}\right)$$

where γ denotes the Euler's constant and $p(u)$ the Dickman function, which satisfies the following difference-differential equation:

$$up'(u) + p(u-1) = 0 \qquad (u > 1)$$

or

$$(up(u))' = p(u) - p(u-1)$$

In [1], [2], [3] H. G. Diamond, H. Halberstram, H. E. Richert and G. Tenenbaum obtained some properties of these functions.
In this paper we have studied the following generalized difference-differential equations:

$$\left(Q_n\left(t\right)p\left(t\right)\right)' = kp\left(t\right) - kp\left(t+\alpha\right)$$

where $Q_n\left(t\right)$ is a polinom of degree n with positive coefficients and k, α are a positive constants.
We prove that the solution of this equation unique (**Theorem 2.2**) and positive (**Theorem 2.3**) under some conditions.

2. Main results

Theorem 2.1.
Let $Q_n\left(t\right)$ be a polynom of degree n with positive coefficients and k, α positive constants.
If $p \in C^1\left(\left(o, \infty\right)\right)$ satisfies

$$\left(Q_n\left(t\right)p\left(t\right)\right)' = kp\left(t\right) - kp\left(t+\alpha\right) \tag{1}$$

and

$$p\left(t\right) \sim t^a \tag{2}$$

then $a = -n$.
Proof
Integrating the difference-differencial equations we have:

$$Q_n\left(t\right)p\left(t\right) + k\int\limits_t^{t+\alpha} p\left(s\right)\ ds = c$$

where c is some constant. By (2) we have:

$$t^{n+a} + kt^a \longrightarrow c, \qquad t \longrightarrow \infty.$$

It follows that $a + n = 0$ or $a = -n$.
This argument shows that $c = 1$, so we have a following integral equation:

$$Q_n\left(t\right)p\left(t\right) + k\int\limits_t^{t+\alpha} p\left(s\right)\ ds = 1, \qquad t > 0. \tag{3}$$

Now we show the uniqueness of p.

Theorem 2.2.
For each $k > 0$ there exists at most one function $p \in C^1((0,\infty))$ which satisfies (1) *and* (2).
Proof
Let $\overline{p} := p_1 - p_2$ where p_1 and p_2 satisfy (1) and (2) for the same value of k. Each of p_1 and p_2 satisfies (3) and hence

$$Q_n(t)\,\overline{p}(t) = -k \int_t^{t+\alpha} \overline{p}(s)\; ds$$

Also by (2) we have

$$Q_n(t)\,\overline{p}(t) \longrightarrow 0$$

or

$$\overline{p}(t) = o(t^n).$$

Suppose that

$$|\overline{p}(t)| < M \cdot t^{-n}$$

for some positive number M and $t \geq t_0$. On this range we have:

$$|Q_n(t)\,\overline{p}(t)| \leq k \cdot M \int_t^{t+\alpha} s^{-n}\; ds < k \cdot M \cdot t^{-n}$$

and so

$$t^n\,|\overline{p}(t)| < k \cdot M \cdot t^{-n}$$

$$|\overline{p}(t)| < k \cdot M \cdot t^{-2n} \leq \tfrac{1}{2} M \cdot t^{-n}$$

if $u_0 \geq \max\left\{u_0, (2k)^{\frac{1}{n}}\right\}$.
It follows in same way that

$$|\overline{p}(t)| < \left(\frac{1}{2}\right)^l \cdot M \cdot t^{-n} \qquad \forall l \geq 1$$

and if $l \longmapsto \infty$ we have $|\overline{p}(t)| = 0$ for all t.

Theorem 2.3
If $p(t) > 0 \qquad \forall t \in (0,1]$ and $Q_n'(0) > k$ the solution $p \in C^1((0,\infty))$ of (1) *and* (2) *satisfies the following properties:*

$$p(t) > 0 \qquad \forall t > 0$$

$$p'(t) < 0 \qquad \forall t > 0.$$

Proof
If exist, let $\tau := \sup\{t \mid p(t) \le 0\}$.
If τ is finite, then $\tau > 1$ since p is continuous and satisfies $p(t) > 0$ for $0 \le t \le 1$.We have $p(t) > 0$ for $t > \tau$ and by (1) we can then write

$$Q_n(t)\,p'(t) = -kp(t+\alpha) - (Q_n'(t) - k)\,p(t) \tag{4}$$

By the $Q_n'(0) > k$ implies, that the right hand side negative. This shows that $p'(t) < 0$ or the function p is strict decreasing :

$$p(t) < p(\tau) \le 0 \qquad \forall t > \tau$$

which contradiction with (2).
If τ is infinite, exist $(t_k)_{k\ge1}$such that $t_k \longrightarrow \infty$ if $k \to \infty$ and

$$p(t_k) \le 0 \qquad \forall k \ge 1.$$

This implies that

$$(t_k)^n\,p(t_k) \le 0$$

which contradiction with (2).
This shows that τ not exist and $p(t) > 0 \quad \forall t \in [0,\infty)$.
The $p'(t) < 0$ follows by (4).

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