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FIBER PICARD OPERATORS

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Abstract. The purpose of this article is to give a fiber Picard operator theorem in generalized metric spaces.

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1 Introduction

In this paper we consider the following notion of generalized metric space, given by A.Branciari [1]:

Definition 1.1 *Let X be a set and $d : X \times X \to \mathbb{R}_+$ a mapping such that:*
(i) for all $x, y \in X \quad d(x, y) = 0 \Leftrightarrow x = y$;
(ii) for all $x, y \in X \quad d(x, y) = d(y, x)$;
(iii) for all $x, y \in X$ and for all $z, w \in X$, each of them different from x and y, one has

$$d(x, y) \leq d(x, z) + d(z, w) + d(w, y).$$

Then we will say that (X, d) is a generalized metric space (g.m.s.).

Remark 1.1 . *The related definitions of balls, Cauchy sequences and completeness of a g.m.s. are the same as in the standard metric spaces. In [1] Branciari generalized the Banach contraction mapping principle for such g.m.s.*

Our purpose is to give a fiber Picard operator theorem (see I. A. Rus [2] and [3]) in g.m.s.

2 Main results

Definition 2.1 *Let (X,d) be a g.s.m. An operator $A : X \to X$ is Picard operator if there exists $x^* \in X$ such that $F_A = \{x^*\}$ and the sequence $(A^n(x))_{n \in \mathbb{N}}$ converges to x^* for all $x \in X$.*

Definition 2.2 *Let (X,d) be a g.s.m. An operator $A : X \to X$ is weakly Picard operator if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$ and the limit $x^*(x)$ is a fixed point of A.*
For a weakly Picard operator A, we will consider the following operator:

$$A^\infty : X \to X, \quad A^\infty(x) := \lim_{n\to\infty} A^n(x) = x^*(x)$$

Lemma 2.1 *Let (X, d) be a complete g.m.s. and $A_n, A : X \to X$, $n \in \mathbb{N}$ some operators. We suppose that:*
(i) the sequence $(A_n)_{n\in\mathbb{N}}$ pointwise converges to A;
(ii) there exists $a \in]0, 1[$ such that the operators A_n and A, $n \in \mathbb{N}$, are a-contractions.
Then the sequence $(A_n \circ A_{n-1} \circ ... \circ A_0)_{n\in\mathbb{N}}$ pointwise converges to A^∞.

Proof. Since A is a contraction in g.m.s., it follows from Theorem 2.1 in [1] that A has a unique fixed point; let it be x^*. Then $A^\infty(x) = x^*$, for all $x \in X$.
Let $x \in X$. We have:

$$d(A_nA_{n-1}...A_0(x), x^*) \le d(A_nA_{n-1}...A_0(x), A_nA_{n-1}...A_0(x^*))+$$
$$+d(A_nA_{n-1}...A_0(x^*), A_nA_{n-1}...A_1(x^*)) + d(A_nA_{n-1}...A_1(x^*), x^*) \le$$
$$\le a^{n+1}d(x, x^*) + a^n d(A_0(x^*), x^*) + d(A_nA_{n-1}...A_1(x^*), x^*)$$

We have two cases:
I. n=2k. Then:

$$d(A_{2k}A_{2k-1}...A_0(x), x^*) \le a^{2k+1}d(x, x^*) + \sum_{i=0}^{2k} a^{2k-i}d(A_i(x^*), x^*)-$$
$$-[a^{2k-1}d(A_1(x^*), x^*) + a^{2k}d(A_0(x^*), x^*)] + d(A_{2k}A_{2k-1}(x^*), x^*)$$

where the sum tends to 0, by Cauchy Lemma, and the others terms also tend to zero when k tends to infinity.
II. n=2k+1. Then:

$$d(A_{2k+1}A_{2k}...A_0(x), x^*) \le a^{2k+2}d(x, x^*) + \sum_{i=0}^{2k+1} a^{2k+1-i}d(A_i(x^*), x^*) \xrightarrow[k\to\infty]{} 0$$

Lemma 2.2 *Let (X, d) and (Y, ρ) be two g.m.s., $(x_n)_{n\in\mathbb{N}} \subset X$, $x^* \in X$ and $f : X \times Y \to Y$ an operator. We suppose that:*
(i) $x_n \xrightarrow[n\to\infty]{} x^*$;
(ii) the operator $f(\cdot, y) : X \to Y$ is continuous for all $x \in X$;
(iii) there exists $a \in]0, 1[$ such that the operator $f(x, \cdot) : Y \to Y$ is an a-contraction for all $x \in X$;
(iv) (Y, ρ) is a complete g.s.m.
Then the sequence defined by

$$y_{n+1} = f(x_n, y_n), \quad y_1 = y, \quad n \in \mathbb{N}$$

converges to y^*, the unique fixed point of $f(x^*, \cdot)$, for all $y \in Y$.

Proof. We take $A_n, A : Y \to Y;\ A_n(y) = f(x_n, y),\ A(y) = f(x^*, y)$ and the proof follows from Lemma 1.1.
The main result is:

Theorem 2.1 *Let (X, d) and (Y, ρ) be two complete g.m.s.,*
$B : X \to X,\ C : X \times Y \to Y$ two operators such that:
(i) $B : X \to X$ is a Picard operator;
(ii) the operator $C(\cdot, y) : X \to Y$ is continuous for all $x \in X$;
(iii) there exists $a \in]0, 1[$ such that the operator $C(x, \cdot) : Y \to Y$ is an a-contraction for all $x \in X$.
Then

$$A : X \times Y \to X \times Y, A(x, y) := (B(x), C(x, y))$$

is a Picard operator with the unique fixed point (x^*, y^*), where x^* is the fixed point of B and y^* is the fixed point of $C(x^*, \cdot)$.

Proof. We have $A^n(x, y) = (B^n(x), CA^{n-1}(x, y))$ and taking the sequences
$x_n := B^n(x) \xrightarrow[n\to\infty]{} x^*\ \forall x \in X$ and $y_n := CA^{n-1}(x, y)\ \forall (x, y) \in X \times Y$
it follows that $y_{n+1} = C(x_n, y_n)$.
We are in the hypothesis of Lemma 1.1., so
$y_n \xrightarrow[n\to\infty]{} y^*$ and $A^n(x, y) \xrightarrow[n\to\infty]{} (x^*, y^*)$.

The concept of g.m.s. can be generalized in the following way:

Definition 2.3 *Let X be a set, $\nu \in \mathbb{N}$ and $d : X \times X \to \mathbb{R}_+$ a mapping such that:*
(i) and (ii) from Definition 1. ;
(iii) for all $x, y \in X$ and for all distinct $z_i \in X,\ i = \overline{1, \nu}$, each of them different from x and y, $z_0 := x,\ z_{\nu+1} := y$, one has

$$d(x, y) \leq \sum_{i=0}^{\nu} d(z_i, z_{i+1}).$$

Then we will say that (X, d) is a generalized metric space of order ν (ν-g.m.s.).

Remark. The Banach mapping contraction principle and the fiber Picard operator theorem in ν-g.m.s. are similar with the corresponding ones in g.m.s.
The proof of Lemma 1.1. in ν-g.m.s. is similar with the proof in g.m.s. We will have:

$$d(A_nA_{n-1}...A_0(x), x^*) \leq d(A_nA_{n-1}...A_0(x), A_nA_{n-1}...A_0(x^*))+$$

$$+d(A_nA_{n-1}...A_0(x^*), A_nA_{n-1}...A_1(x^*)) + ...+$$

$$+d(A_nA_{n-1}...A_{\nu-2}(x^*), A_nA_{n-1}...A_{\nu-1}(x^*)) + d(A_nA_{n-1}...A_{\nu-1}(x^*), x^*) \leq$$

$$\leq a^{n+1}d(x, x^*) + \sum_{i=0}^{\nu-2} a^{n-i}d(A_i(x^*), x^*) + d(A_nA_{n-1}...A_{\nu-1}(x^*), x^*)$$

There are ν cases, so we take $n = \nu k + l,\ k \in \mathbb{N},\ l = \overline{0, \nu - 1}$

$$d(A_{\nu k+l}...A_0(x), x^*) \leq a^{\nu k+l+1} d(x, x^*) + \sum_{i=0}^{\nu k+l} a^{\nu k+l-i} d(A_i(x^*), x^*) -$$

$$- \sum_{i=\nu k-2}^{\nu k+l} a^{\nu k+l-i} d(A_i(x^*), x^*) + d(A_{\nu k+l}...A_{\nu k-1}(x^*), x^*) \xrightarrow[k\to\infty]{} 0.$$

References

[1] A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, Publ. Math. Debrecen 57, 1-2(2000), 31-37.

[2] I. A. Rus, Fiber Picard operators and applications, Studia Univ. Babeş-Bolyai, seria Mathematica, 44(1999), 89-98.

[3] I. A. Rus, A fibre generalized contraction theorem and applications, Mathematica, 41(1999), 85-90.