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ON AN INTEGRAL EQUATION WITH DEVIATING ARGUMENT

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Abstract. Continuous and differentiability dependence of the solution for a class of integral equations with deviating argument is proved.

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1 Introduction

In paper [1] C. Avramescu investigates the following integral equation:

$$x(t) = \int_{-\infty}^{\infty} K(t,s,x(s))F(s,x)ds + f(t) \tag{1}$$

We shall generalize the equation (1) and we shall study it by means of the theory of Picard operators. A generalization of it, is the following equation:

$$x(t) = \int_{-t}^{t} K(t,s,x(s))ds + g(t), \quad t \in [-T,T], \quad T > 0 \tag{2}$$

We shall obtain a result concerning the existence and uniqueness of the solution by Contraction Principle (see [2]), and also a result concerning the differentiability of the solution with respect to a parameter.

2. In what follows we shall study the equation (2), where $K \in C([-T,T] \times [-T,T] \times \mathbb{R})$ and $g \in C([-T,T])$.

We have:

Theorem 1. *Suppose that:*

(i) there exists a function $a : \mathbb{R} \to \mathbb{R}_+^*$ *such that*

$$|K(t,s,u) - K(t,s,v)| \leq a(s)|u - v|,$$

for all $t, s \in [-T,T]$, $u, v \in \mathbb{R}$.

(ii) $q := \int_{-T}^{T} a(s)ds < 1$.

Then:

a) the equation (2) has a unique solution x^* *in* $C([-T,T])$
b) for all $x_0 \in X$, *the sequence* $(x_n)_{n\in\mathbb{N}}$ *defined by:*

$$x_{n+1}(t) = \int_{-t}^{t} K(t,s,x_n(s))ds + g(t), \quad n \in \mathbb{N}$$

converges uniformly to x^*.

Proof. Let us consider the operator

$$B : C([-T,T]) \to C([-T,T]), \quad B(x)(t) = \int_{-t}^{t} K(t,s,x(s))ds + g(t).$$

We have

$$|B(x)(t) - B(y)(t)| \leq \int_{-t}^{t} |K(t,s,x(s)) - K(t,s,y(s))|ds \leq$$

$$\leq \int_{-t}^{t} a(s)|x(s) - y(s)|ds \leq q\|x - y\|_C$$

for all $x, y \in C([-T,T])$ and $t \in [-T,T]$.

Then $\|B(x) - B(y)\|_C \leq q\|x - y\|_C$ which shows that B is a contraction. The proof follows from the Contraction Principle. □

3. Convenient choosing of the space - comprising the parameter in definition domain of the operator - allows us to establish the continuous dependence of the solution with respect to the parameters. Hence we shall consider the following equation:

$$x(t;\lambda) = \int_{-t}^{t} K(t,s,x(s;\lambda);\lambda)ds + g(t), \quad t \in [-T,T], \quad T > 0 \tag{3}$$

where $K \in C([-T,T] \times [-T,T] \times \mathbb{R} \times J)$, $\lambda \in J$ and $J \subset \mathbb{R}$ a compact interval.

As above we can prove the following

Theorem 2. *Suppose that*

(i) there exists a function $a : \mathbb{R} \to \mathbb{R}_+^*$ *such that:*

$$|K(t,s,u;\lambda) - K(t,s,v;\lambda)| \leq a(s)|u - v|, \ \textit{for all}\ t,s \in [-T,T],\ u,v \in \mathbb{R},\ \lambda \in J.$$

(ii) $q := \displaystyle\int_{-T}^{T} a(s)ds < 1.$

Then:

(a) the equation (3) has a unique solution x^* *in* $C([-T,T] \times J) := X$
(b) for all $x_0 \in X$, *the sequence* $(x_n)_{n\in\mathbb{N}}$ *defined by:*

$$x_{n+1}(t;\lambda) = \int_{-t}^{t} K(t,s,x_n(s;\lambda);\lambda)ds + g(t), \quad n \in \mathbb{N}$$

converges uniformly to x^*.

Remark 1. In the assumption of Theorem 2, for each fixed λ, the equation (3) has a solution in $C([-T,T]\times J)$ depending continuously on λ.

4. In this section we shall need the following result (see [3], [4], [5]).

Theorem 3. (of fiber contraction principle) *Let be (X,d) be a metric space, (Y,ρ) a complete metric space and $A : X\times Y \to X\times Y$ - an operator.*

Suppose that:

(i) $A\in C(X\times Y, X\times Y)$;

(ii) $A(x,y) = (B(x), C(x,y))$, for all $x\in X$, $y\in Y$;

(iii) $B : X\to X$ is a Picard operator;

(iv) There exists a number $\alpha\in]0,1[$ such that:

$$\rho(C(x,y),C(x,z)) \le \alpha\rho(x,y),\ \textit{for all } x\in X \textit{ and } y,z\in Y.$$

Then A is a Picard operator.

We shall consider the equation (3) and we have:

Theorem 4. *Suppose that:*

(i) there exists a function $a:\mathbb{R}\to\mathbb{R}_+^*$ such that:

$$|K(t,s,u;\lambda) - K(t,s,v;\lambda)| \le a(s)|u-v|,\ \textit{for all } t,s\in[-T,T],\ u,v\in\mathbb{R},\ \lambda\in J.$$

(ii) $q := \int_{-T}^{T} a(s)ds < 1$

(iii) $K(t,s,\cdot;\cdot)\in C^1(\mathbb{R}\times J)$.

Then $x^*(t;\cdot)\in C^1(J)$.

Proof. Let us prove that there exists $\dfrac{\partial x^*}{\partial\lambda}$ and $\dfrac{\partial x^*}{\partial\lambda}\in C([-T,T]\times J)$.

If we suppose that there exists $\dfrac{\partial x^*}{\partial\lambda}$, then from relation (3) we have:

$$\frac{\partial x^*}{\partial\lambda}(t;\lambda) = \int_{-t}^{t} \frac{\partial K(t,s,x^*(s;\lambda);\lambda)}{\partial x^*}\cdot\frac{\partial x^*(s;\lambda)}{\partial\lambda}ds + \int_{-t}^{t}\frac{\partial K(t,s,x^*(s;\lambda);\lambda)}{\partial\lambda}ds.$$

This relation suggest us to consider the following operator:

$$C: X\times X\to X,\quad C(x,y)(t;\lambda) = \tag{4}$$

$$= \int_{-t}^{t}\frac{\partial K(t,s,x(s;\lambda);\lambda)}{\partial x}y(s;\lambda)ds + \int_{-t}^{t}\frac{\partial K(t,s,x(s;\lambda);\lambda)}{\partial\lambda}ds$$

where $X = C([-T,T]\times J)$.

Then $A: X\times X\to X\times X$, $A(x,y) = (B(x), C(x,y))$, for all $x,y\in X$, where $B: X\to X$,

$$B(x)(t;\lambda) = \int_{-t}^{t} K(t,s,x(s;\lambda);\lambda)ds + g(t).$$

From relation (i) we remark that:

$$\left|\frac{\partial K(t,s,u;\lambda)}{\partial x}\right| \le a(s),\ \text{for all } t,s\in[-T,T],\ u\in\mathbb{R},\ \lambda\in J. \tag{5}$$

From relation (4) and (5) results that $C(x,\cdot)$ is a q-contraction. Let us take $F_{C(x^*,\cdot)} = y^*$. Then this satisfies the assumptions of Theorem 3. By this Theorem, results that the operator A is a Picard operator and the sequences $x_{n+1} = B(x_n)$, $y_{n+1} = C(x_n, y_n)$ converges uniformly (with respect to $t \in [-T,T]$, $\lambda \in J$) to $(x^*, y^*) \in F_A$, for all $x_0, y_0 \in X$.

If we take $x_0, y_0 \in X$ such that $y_0 = \dfrac{\partial x_0}{\partial \lambda}$, then $y_1 = \dfrac{\partial x_1}{\partial \lambda}$.

By induction, we prove that $y_n = \dfrac{\partial x_n}{\partial \lambda}$. Thus

$$x_n \overset{unif}{\longrightarrow} x^* \text{ as } n \to \infty,$$

$$\frac{\partial x_n}{\partial x^*} \overset{unif}{\longrightarrow} y^* \text{ as } n \to \infty.$$

These imply that there exists $\dfrac{\partial x^*}{\partial \lambda}$ and $\dfrac{\partial x^*}{\partial \lambda} = y^*$ (see [4]).

The proof is complete. □

5. We are able to generalize this theorem, considering the following integral equation:

$$x(t;\lambda) = \int_{-t}^{t} K(t,s,x(s;\lambda);\lambda)ds + g(t,x(t;\lambda);\lambda), \quad t \in [-T,T],\ T > 0 \tag{6}$$

where $K \in C([-T,T] \times [-T,T] \times \mathbb{R} \times J)$, $g \in C([-T,T] \times \mathbb{R} \times J)$, $\lambda \in J$ and $J \subset \mathbb{R}$ is a compact interval.

We have:

Theorem 5. *Suppose that:*

(i) there exists a function $a : \mathbb{R} \to \mathbb{R}_+^*$ *such that:*

$$|K(t,s,u;\lambda) - K(t,s,v;\lambda)| \le a(s)|u-v|, \textit{ for all } t,s \in [-T,T],\ u,v \in \mathbb{R},\ \lambda \in J.$$

(ii) there exists $L_g > 0$ *such that*

$$|g(t,u;\lambda) - g(t,v;\lambda)| \le L_g|u-v|, \textit{ for all } t \in [-T,T],\ u,v \in \mathbb{R},\ \lambda \in J.$$

(iii) $q + L_g < 1$.

Then:

(a) the equation (6) has a unique solution x^* *in* X.

(b) for all $x_0 \in X$, *the sequence* $(x_n)_{n\in\mathbb{N}}$ *defined by:*

$$x_{n+1}(t;\lambda) = \int_{-t}^{t} K(t,s,x_n,(s;\lambda);\lambda)ds + g(t,x_n(t;\lambda);\lambda)$$

converges uniformly to x^*.

(c) if $K(t,s,\cdot;\cdot) \in C^1(\mathbb{R} \times J)$ *and* $g(t,\cdot;\cdot) \in C^1(\mathbb{R} \times J)$, *then* $x^*(t;\cdot) \in C^1(J)$.

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