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ON A CLASS OF FUNCTIONAL-INTEGRAL EQUATIONS

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Abstract. Using the weakly Picard operator technique, we study the data dependence of the solutions set for a functional-integral equation.

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1 Introduction

In this paper we study the following functional-integral equation

$$x(t) = F\left(t, x(a), \int_a^b H(t, s, x(s))\,ds, \int_a^t K(t, s, x(s))\,ds\right), t \in [a, b]$$

using the weakly Picard operators techniques.

2 Basic results from weakly Picard operators theory

Let (X, d) be a metric space and $A : X \to X$. In this paper we shall use the following notations :

$$P(X) := \{Y \subset X \mid Y \neq \emptyset\},$$

$$F_A := \{x \in X \mid A(x) = x\}$$

$$I(A) := \{Y \in P(X) \mid A(Y) \subset Y\} \ ,$$

$$A^{n+1} := A \circ A^n,$$

$$A^0 := 1_X \ , \quad A^1 := A, \quad n \in N$$

Definition 2.1 (Rus [1], [2]). The operator $A : X \to X$ is said to be a weakly Picard operator (briefly w.P.o.) if the sequence

$$(A^n(x))_{n \in N}$$

converges, for all x∈X, and the limit (which may depend on x) is a fixed point of A.

Definition 2.2 (Rus [1], [2]) If A is w.P.o, then we consider the operator A^∞ defined by

$$A^\infty : X \to X,\ A^\infty := \lim_{n\to\infty} A^n(x).$$

We remark that

$$A^\infty(X) = F_A.$$

Definition 2.3 (Rus [1], [2]) If A is w.P.o. and $F_A = \{x^*\}$, then by definition the operator A is a Picard operator.

Remark 2.1 If A is a Picard operator, then

$$F_{A^n} = F_A = \{x^*\}, \quad n \in N^*$$

Remark 2.2 If A is w.P.o., then

$$F_{A^n} = F_A \neq \emptyset, \quad n \in N$$

Remark 2.3 For some examples and properties of Picard operators and w.P.o. see:[2], [3], [4], [5].

We have:

Theorem 2.1$(see[4])$. Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an operator.

Let $x, y \in X$ such that $x < y$, $x \leq A(x)$, $y \geq A(y)$. We suppose that

(i) A is w.P.o.;

(ii) A is monotone increasing.

Then

(a) x ≤ A^∞(x) ≤ A^∞(x) ≤ y ;

(b) A^∞(x) is the minimal fixed point of A in [x,y] and A^∞(y) is the maximal fixed point of A in [x,y].

3 A class of w.P.o. and stability of the fixed point set

Definition 3.1. (see [5]) An operator $A : X \to X$ is c-w.P.o. if A is w.P.o. and

$$d(x, A^\infty(x)) \leq cd(x, A(x)),\ \forall\ x \in X.$$

Example 3.1. If (X,d) is a complete metric space and the operator $A : X \to X$ is an a-contraction, then A is c-w.P.o. with c=$(1\text{-a})^{-1}$.

Example 3.2. Let (X,d) be a complete metric space and $A : X \to X$. We suppose that there exists a∈[0,1[such that

$$d\left(A^2(x), A(x)\right) \leq ad(x, A(x)),\ \forall\ x \in X.$$

Then A is c-w.P.o. with c=$(1\text{-a})^{-1}$.

We have

Theorem 3.1. ([2], [4]) Let (X,d) be a metric space and $A_i : X \to X$, i=1,2. We suppose that

(i) the operator A_i are c_i-w.P.o., i=1,2;

(ii) there exists $\eta > 0$ such that

$$d\left(A_1(x), A_2(x)\right) \le \eta,\ \forall\ x \in X.$$

Then

$$H\left(A_1^\infty(X), A_2^\infty(x)\right) \le \ \max(c_1, c_2).$$

Theorem 3.2. ([2], [4]) Let (X,d) be a metric space and $A : X \to X$ an operator.The operator A is w.P.o.

(c-w.P.o.) if and only if there exists a partition of X

$$X = \bigcup_{\lambda \in \Lambda} X_\lambda$$

such that

(a) $X_\lambda \in$ I(A), $\lambda \in \Lambda$;

(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ is a Picard (c-Picard) operator for all $\lambda \in \Lambda$.

4 Data dependence of the solutions set

Let X be a Banach space. Let $H \in C\left([a,b] \times [a,b] \times X, X\right)$, $K \in C\left([a,b] \times [a,b] \times X, X\right)$ and $F \in C([a,b] \times X^3, X)$. We consider the following integral equation

$$x\left(t\right) = F\left(t, x\left(a\right), \int_a^b H\left(t,s,x\left(s\right)\right) ds, \int_a^t K\left(t,s,x\left(s\right)\right) ds\right), t \in [a,b] \tag{4.1}$$

We consider the Banach space C([a,b],X) with Cebâşev norm

$$||\ x\ ||_c := \max_{t \in [a,b]} ||\ x(t)\ ||$$

We suppose that:

(c_1) there exists $l_1 > 0$ such that

$$||\ H(t,s,u) - H(t,s,v)|| \le l_1 ||u - v||,\ \forall t \in [a,b]\,,\ \forall u, v \in X$$

(c_2) there exists $l_2 > 0$ such that

$$||\ K(t,s,u) - K(t,s,v)|| \le l_2 ||u - v||,\ \forall t \in [a,b]\,,\ \forall u, v \in X$$

(c_3) there exists $l_3 > 0$, $l_4 > 0$ such that

$$||\ F(t,u,v_1,w_1) - F(t,u,v_1,w_2)|| \le l_3 ||v_1 - v_2|| + l_4 ||w_1 - w_2||,$$

$\forall t \in [a,b]\,,\ \forall u, v_i, w_i \in X,\ i = 1,2$

(c_4) $(l_3 l_1 + l_4 l_2)(b-a) < 1$

(c_5) $F\left(a, \alpha, \int_a^b H(a,s,x(s)ds, 0\right) = \alpha,\ \forall x \in C([a,b]\,, X)$

Theorem 4.1 Consider the equation (4.1) and suppose that conditions (c_1)-(c_5) are satisfied. If S⊂C(I,X) is the solutions set of this equations, then

$$cardS\ = cardX.$$

Proof. Consider the operator $A : C([a,b]\,, X) \to C([a,b]\,, X)$, defined by

$$A(x)(t) := F\left(t, x\,(a)\,, \int_a^b H\,(t,s,x\,(s))\,ds, \int_a^t K\,(t,s,x\,(s))\,ds\right)$$

Let $\alpha \in X$ and

$$X_\alpha := \{x \in C\,([a,b]\,, X)\ \mid x\,(a) = \alpha\}$$

then

$$C([a,b]\,, X) = \bigcup_{\alpha \in X} X_\alpha$$

From (c_5) we have that X_α ∈I(A).

Let

$$A_\alpha := A|_{X_\alpha} : X_\alpha \to X_\alpha$$

From (c_1)-(c_5) it follows that

$$||A_\alpha(x) - A_\alpha(y)||_c \le (l_3 l_1 + l_4 l_2)(b-a)||x-y||_{c,}, \quad x, y \in C([a,b]\,, X),\ \alpha \in X.$$

From (c_4) we have

$$A_\alpha : (X_\alpha, ||\ .\ ||_c) \to (X_\alpha, ||\ .\ ||_c).$$

Moreover A_α is c-Picard operator with

$$c = [1 - (l_3 l_1 + l_4 l_2)(b-a)]^{-1}.$$

From the Theorem 3.2 we have that the operator A is c-w.P.o. It is clear that:

(a) cardS=cardX ;

(b) $||A^2$(x)-A(x)$||\le (l_3 l_1 + l_4 l_2)(b-a)||x - A(x)||, forall x \in C([a,b]\,, X).$

5 Comparison theorems

Theorem 5.1 Consider the equations

$$(5.1) \qquad x\,(t) = F\left(t, x\,(a)\,, \int_a^b H_1\,(t,s,x\,(s))\,ds, \int_a^t K_1\,(t,s,x\,(s))\,ds\right), t \in [a,b]$$

$$x(t) = F\left(t, x(a), \int_a^b H_2(t, s, x(s))\, ds, \int_a^t K_2(t, s, x(s))\, ds\right), t \in [a, b] \tag{5.2}$$

under the conditions (c_1)-(c_5).

Let S_1 the solutions set of the equation (5.1) and S_2 the solutions set of the equation (5.2). We suppose that there exists η_1,$\eta_2 > 0$ such that

$$\|\, H_1(t, u, v) - H_2(t, u, v)\| \leq \eta_1,\ \forall t \in [a, b],\ \forall u, v \in X$$

$$\|\, K_1(t, u, v) - K_2(t, u, v)\| \leq \eta_2,\ \forall t \in [a, b],\ \forall u, v \in X$$

Then

$$H(S_1, S_2) \leq (\eta_1 l_3 + \eta_2 l_4) \max\{c_1, c_2\},$$

Proof. Consider the operators $A_1, A_2 : C([a, b], X) \to C([a, b], X)$, defined by

$$A_1(x)(t) = F\left(t, x(a), \int_a^b H_1(t, s, x(s))\, ds, \int_a^t K_1(t, s, x(s))\, ds\right)$$

$$A_2(x)(t) = F\left(t, x(a), \int_a^b H_2(t, s, x(s))\, ds, \int_a^t K_2(t, s, x(s))\, ds\right)$$

From the Theorems 4.1 we have A_1 is c_1-w.P.o. and A_2 is c_2-w.P.o..

From the conditions (c_1)-(c_5) and the assumptions we have

$$\|A_1(x) - A_2(x)\|_c \leq (\eta_1 l_3 + \eta_2 l_4)(b - a)$$

so the proof follows from the Theorem 4.1.

Theorem 5.2 Let X be ordered Banach space.Consider the equations (5.1) under the following conditions:

(i) the conditions (c_1)-(c_5);

(ii) the operators F(t,·, ·, ·), H(t,·,·), K(t,·,·) are monoton increasing.

Let x and y be two solutions of the equations (5.1).If $x(a) \leq y(a)$, then $x(t) \leq y(t)$ for all t∈[a,b].

Proof. Let X_α be as in the proof of the theorem 4.1. Then x ∈ $X_{x(a)}$ and y ∈ $X_{y(a)}$

Moreover

$$x = A^\infty(x_1),\ \forall\ x_1 \in X_{x(a)};$$

$$y = A^\infty(y_1),\ \forall\ y_1 \in X_{y(a)};$$

If u∈X then we denote by ũ the operator ũ : $C([a, b], X) \to C([a, b], X)$, defined by ũ(t)=u, t ∈[a,b]

It is clear that $\widetilde{x(a)}$ ∈$X_{x(a)}$, $\widetilde{y(a)}$ ∈$X_{y(a)}$, and $\widetilde{x(a)} \leq \widetilde{y(a)}$ .By the monotony of the operator A^∞(see the theorem 2.1) we have that $A^\infty(\widetilde{x(a)}) \leq A^\infty(\widetilde{y(a)})$, i.e.,$x \leq y$.

Theorem 5.3 Consider the equations (5.1) and (5.2) under the conditions of the theorem 5.2. We suppose that $H_1 \leq H_2$ and $K_1 \leq K_2$.Let $x_1 \in C([a, b], X)$ be a solution of the equations (5.1) and $x_2 \in C([a,b], X)$ be a solution of the equations (5.2). If $x_1(a) \leq x_2(a)$, then $x_1 \leq x_2$.

Proof. Let A_1, A_2 be the operator A from the proof of the theorem 4.1. The operators A_i i=1,2 are monotone increasing w.P.o. , and $A_1 \leq A_2$. From these conditions we have that $A_1^\infty \leq A_2^\infty$, $\widetilde{x_1(a)} \leq \widetilde{x_2(a)}$, $x_1 \in X_{x_1(a)}$, $x_2 \in X_{x_2(a)}$. So $x_1 = A_1^\infty(\widetilde{x_1(a)}) \leq A_2^\infty(\widetilde{x_2(a)}) = x_2$.

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