

## A MÖNCH TYPE GENERALIZATION OF THE EILENBERG-MONTGOMERY FIXED POINT THEOREM

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**Abstract.** We establish a Mönch type generalization of the Eilenberg-Montgomery fixed point theorem for multi-valued maps with acyclic values.

**Key words:** Fixed point, acyclic set, multi-valued map

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In [1] Eilenberg and Montgomery established the following extension of the Bohnenblust-Karlin fixed point theorem for multi-valued maps.

**Theorem 1** *Let  $\Xi$  be an acyclic, absolute neighborhood retract,  $\Theta$  a compact metric space,  $r : \Theta \rightarrow \Xi$  a continuous single valued map and  $T : \Xi \rightarrow 2^\Theta \setminus \{\emptyset\}$  an upper semicontinuous map such that all the sets  $T(x)$  are acyclic for  $x \in \Xi$ . Then the combined multi-valued map  $rT : \Xi \rightarrow 2^\Xi$  has a fixed point.*

An extension of this theorem for condensing (noncompact) maps  $rT$  is due to Fitzpatrick and Petryshyn [2].

The aim of this note is to prove the following generalization of the Eilenberg-Montgomery theorem.

**Theorem 2** *Let  $D$  be a closed, convex subset of a Banach space  $X$ ,  $Y$  a metric space,  $T : D \rightarrow 2^Y \setminus \{\emptyset\}$  a map with acyclic values, and  $r : Y \rightarrow D$  continuous. Assume  $\text{graph}(T)$  is closed,  $T$  maps compact sets into relatively compact sets and that for some  $x_0 \in D$  one has*

$$\left. \begin{array}{l} M \subset D, M = \text{conv}(\{x_0\} \cup rT(M)) \\ \text{and } \overline{M} = \overline{C} \text{ with } C \subset M, C \text{ countable} \end{array} \right\} \implies \overline{M} \text{ compact.} \quad (1)$$

*Then there exists  $x \in D$  with  $x \in rT(x)$ .*

**Proof.** First note since  $r$  is continuous, the map  $N := rT$  also has a closed graph and maps compact sets into relatively compact sets.

Following the steps (a) and (b) of the proof of Theorem 3.1 in [4], we find a convex set  $M \subset D$  with  $M = \text{conv}(\{x_0\} \cup rT(M))$  and  $K := \overline{M}$  compact. Next, instead of steps (c)-(d) of the above mentioned proof, we follow:

(c\*) Proof of inclusion  $rT(K) \subset K$ . Let  $\varepsilon > 0$  be fixed. According to Theorem 1.2.23 in [3],  $rT$  is upper semicontinuous. Consequently, for each  $x \in M$  there exists an open neighborhood  $V_x$  of  $x$  such that  $rT(y) \subset rT(x) + B_\varepsilon(0)$  for all  $y \in V_x$ . Since for  $x \in M$ , one has  $rT(x) \subset K$ , it follows that  $rT(y) \subset K_\varepsilon := K + B_\varepsilon(0)$  for every  $y \in V_x$ .  $M$  being dense in  $K$ , we have that  $\{V_x : x \in M\}$  is a cover of  $K$ . Consequently,  $rT(K) \subset K_\varepsilon$ . Hence  $rT(K) \subset \bigcap_{\varepsilon > 0} K_\varepsilon = K$ .

(d\*) Application of the Eilenberg-Montgomery theorem. Since every compact and convex subset of a Banach space is an absolute neighborhood retract and is acyclic, we may apply Theorem 1 to:  $\Xi := K$  and  $\Theta := T(K)$ . ■

*Remarks.* (a) Under the assumptions of Theorem 2 the values of  $T$  are nonempty, compact and acyclic.

(b) Theorem 2 is in particular true if  $T : D \rightarrow 2^Y$  has nonempty, compact and acyclic values and  $T$  is upper semicontinuous.

The next result is the continuation type version of Theorem 2.

**Theorem 3** *Let  $K$  be a closed, convex subset of a Banach space  $X$ ,  $U$  a convex, relatively open subset of  $K$ ,  $Y$  a metric space,  $T : \overline{U} \rightarrow 2^Y \setminus \{\emptyset\}$  with acyclic values and  $r : Y \rightarrow K$  continuous. Assume  $\text{graph}(T)$  is closed,  $T$  maps compact sets into relatively compact sets and that for some  $x_0 \in U$ , the following two conditions are satisfied:*

$$\left. \begin{array}{l} M \subset \overline{U}, M \subset \text{conv}(\{x_0\} \cup rT(M)) \\ \text{and } \overline{M} = \overline{C} \text{ with } C \subset M, C \text{ countable} \end{array} \right\} \implies \overline{M} \text{ compact}; \quad (2)$$

$$x \notin (1 - \lambda)x_0 + \lambda rT(x) \quad (3)$$

for all  $x \in \overline{U} \setminus U$ ,  $\lambda \in (0, 1)$ . Then there exists  $x \in \overline{U}$  with  $x \in rT(x)$ .

**Proof.** Let  $D = \overline{\text{conv}}(\{x_0\} \cup rT(\overline{U}))$ . Clearly,  $x_0 \in D \subset K$ . Let  $P : K \rightarrow \overline{U}$  be given by

$$P(x) = \begin{cases} x & \text{for } x \in \overline{U} \\ \bar{x} & \text{for } x \in K \setminus \overline{U} \end{cases}$$

Here  $\bar{x} = (1 - \lambda)x_0 + \lambda x \in \overline{U} \setminus U$ ,  $\lambda \in (0, 1)$ . It is easy to see that  $P$  is continuous.

Let  $\tilde{T} : D \rightarrow 2^Y$ , be given by  $\tilde{T}(x) = T(P(x))$ . Clearly its values are nonempty and acyclic. Also, it is easily seen that  $\text{graph}(\tilde{T})$  is closed and  $\tilde{T}$  maps compact sets into relatively compact sets. Next we check (1) for  $r\tilde{T}$ . Let  $M \subset D$  be such that  $M = \text{conv}(\{x_0\} \cup r\tilde{T}(M))$  and  $\overline{M} = \overline{C}$  for some countable set  $C \subset M$ . Since

$$\begin{aligned} P(M) &\subset \text{conv}(\{x_0\} \cup M) \subset \text{conv}(\{x_0\} \cup r\tilde{T}(M)) \\ &= \text{conv}(\{x_0\} \cup rTP(M)), \end{aligned} \quad (4)$$

$$\overline{P(M)} = \overline{P(C)}, \quad P(C) \subset P(M) \quad \text{and} \quad P(C) \text{ is countable,}$$

from (2) we deduce that  $P(M)$  is relatively compact. Then  $r\tilde{T}(M) = rTP(M)$  is relatively compact and Mazur's lemma implies that  $\overline{M}$  is compact. Thus (1) holds for  $r\tilde{T}$ .

Now we apply Theorem 2 to deduce that there exists an  $x \in D \subset \overline{U}$  with  $x \in r\tilde{T}(x)$ . We claim that  $x \in D \cap \overline{U}$ . Assume the contrary, that is  $x \in D \setminus \overline{U}$ . Then  $x \in rT(\bar{x})$ , where  $\bar{x} = (1 - \lambda)x_0 + \lambda x \in \overline{U} \setminus U$ ,  $\lambda \in (0, 1)$ . Then  $x = (1/\lambda)\bar{x} + (1 - 1/\lambda)x_0 \in rT(\bar{x})$ . Hence  $\bar{x} \in (1 - \lambda)x_0 + \lambda rT(\bar{x})$ , which contradicts (3). Thus  $x \in D \cap \overline{U}$  and so  $x \in rT(x)$ . ■

## References

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