Seminar on Fixed Point Theory Cluj-Napoca, **Volume 1, 2000, 63-68**
http://www.math.ubbcluj.ro/~nodeacj/journal.htm

FIXED POINT THEOREMS FOR MULTIVALUED EXPANSIVE OPERATORS

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Abstract. The main goal of this paper is to study the existence and data dependence of the fixed points for a class of generalized expansive multifunctions.

Keywords: expansive multifunction, fixed point, data dependence
AMS Subject Classification: 47H10

1 Introduction

In some papers from the 80's, some fixed point theorems for expansive-type single-valued mappings are proved (see [1], [2], [6] etc.). The main purpose of this note is to extend the above mentioned result to the multivalued case. The date dependence of the fixed points set is also studied.

Let (X, d) be a metric space. Throughout the paper we use the following symbols:
$P(X) := \{Y \subset X|\ Y \neq \emptyset\}$,
$P_p(X) := \{Y \in P(X)|\ Y$ has the property "p"$\}$, where "p" could be: cl = closed, cp = compact, b = bounded, etc.

We consider now the following functionals:
$D : P(X) \times P(X) \to \mathbb{R}_+,\quad D(A, B) = \inf\{d(a, b)|\ a \in A,\ b \in B\}$;
$\delta : P_b(X) \times P_b(X) \to \mathbb{R}_+,\quad \delta(A, B) = \sup\{d(a, b)|\ a \in A,\ b \in B\}$;
$\rho : P_b(X) \times P_b(X) \to \mathbb{R}_+,\quad \rho(A, B) = \sup\{D(a, B)|\ a \in A\}$;
$H : P_b(X) \times P_b(X) \to \mathbb{R}_+,\quad H(A, B) = \max\{\rho(A, B), \rho(B, A)\}$.

It is well known that $(P_{b,cl}(X), H)$ is a metric space and if (X, d) is complete then $(P_{b,cl}(X), H)$ is complete too. Also, the following properties are true (see [3]).

Lemma 1.1. *Let (X, d) be a metric space, $A, B \in P(X)$ and $\eta > 0$. If*
i) for each $a \in A$ there is $b \in B$ such that $d(a, b) \leq \eta$ and
ii) for each $b \in B$ there is $a \in A$ such that $d(a, b) \leq \eta$.
Then $H(A, B) \leq \eta$.

Lemma 1.2. *Let (X, d) be a metric space, $A, B \in P(X)$ and $q \in \mathbb{R}$, $q > 1$. Then for every $a \in A$ there exists $b \in B$ such that $d(a, b) \leq qH(A, B)$.*

Let $T : X \to P(X)$ be a multivalued operator. Then $x^* \in X$ is a fixed point for T if $x^* \in T(x^*)$. The set of all fixed points will be denoted by F_T. If $x^* \in X$ has the property $\{x^*\} = T(x^*)$, then x^* is said to be a strict fixed point for T and the symbol $(SF)_T$ denotes the strict fixed points set of T. The multivalued operator T is surjective if and only if $T(X) := \bigcup\limits_{x\in X} T(x) = X$.

2 Main results

The first main result of the paper is the following existence theorems for a class of expansive multivalued operator.

Theorem 2.1. *Let (X,d) be a complete metric space and $T : X \to P_{b,cl}(X)$ be a surjective multivalued operator. If there exist $a, b, c \in \mathbb{R}_+$ ($b < 1$ and $a + b + c > 1$) such that*

$$d(y_1, y_2) \geq ad(x_1, x_2) + bd(x_1, y_1) + cd(x_2, y_2),\ \textit{for each } x_i \in X \tag{1}$$

and each $y_i \in T(x_i)$, $i \in \{1, 2\}$, with $y_1 \neq y_2$.

Then $F_T \neq \emptyset$.

Moreover, if $a > 1$ then $F_T = \{x^*\}$.

Proof. Let $x_0 \in X$ be arbitrarily. Because T is surjective we can find $x_1 \in X$ such that $x_0 \in T(x_1)$. Using the same argument, we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_{n-1} \in T(x_n)$ for each $n \in \mathbb{N}$, $n \geq 1$.

If there exists $m \in \mathbb{N}$, $m \geq 1$ such that $x_m = x_{m-1}$ then $x_m \in F_T$ and the proof is complete. Let us suppose now that $x_n \neq x_{n-1}$, for each $n \in \mathbb{N}$, $n \geq 1$.

From (1) we deduce:

$$d(x_{n-1}, x_n) \geq ad(x_n, x_{n+1}) + bd(x_n, x_{n-1}) + cd(x_{n+1}, x_n),\ \text{for } n \geq 1$$

and hence

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n),\ \text{where } k = \frac{1-b}{a+c} < 1.$$

Obviously, we get that

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1)$$

and after a simple computation we obtain

$$d(x_n, x_{n+m}) \leq \frac{k^n}{1-k} d(x_0, x_1).$$

It follows that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and hence convergent in the complete metric space (X,d). Let $x^* = \lim\limits_{n\to\infty} x_n$. We will prove that $x^* \in F_T$. For this purpose, let us consider an element $y^* \in T^{-1}(x^*)$. We have successively:

$$d(x_n, x^*) \geq ad(x_{n+1}, y^*) + bd(x_{n+1}, x_n) + cd(y^*, x^*),\ \text{for } n \geq 1.$$

Taking $n \to \infty$ we obtain $0 \geq (a+c)d(y^*, x^*)$ and hence $x^* = y^*$, proving that $x^* \in T(x^*)$.

For the uniqueness of the fixed point, let us suppose, by contradiction, that there exists x_1^* and x_2^* two distinct fixed points for T. Then

$$d(x_1^*, x_2^*) \geq ad(x_1^*, x_2^*) + bd(x_1^*, x_1^*) + cd(x_2^*, x_2^*) = ad(x_1^*, x_2^*).$$

Because $a > 1$ we get the desired contradiction. Hence $F_T = \{x^*\}$. □

Example. Let $X = [0, 4]$ and $d : X \times X \to \mathbb{R}$ the metric given by the following formula:

$$d(t_1, t_2) = \begin{cases} 1, & \text{if} \quad (t_1, t_2) \in [0, 2) \times [0, 2) \cup [2, 4] \times [0, 1) \cup [0, 1) \times [2, 4] \\ \frac{3}{2}, & \text{if} \quad (t_1, t_2) \in [1, 2) \times [2, 4] \cup [2, 4] \times [1, 2) \\ 2, & \text{if} \quad (t_1, t_2) \in [2, 4] \times [2, 4], (t_1 \neq t_2) \\ 0, & \text{if} \quad t_1 = t_2 \end{cases}$$

We consider now the multivalued operator $T : [0, 4] \to P([0, 4])$ given by:

$$T(x) = \begin{cases} [3, 4), & \text{if} \quad x \in [0, 1) \\ [2, 3) & \text{if} \quad x \in [1, 2) \\ \{\frac{5}{2}\}, & \text{if} \quad x \in [2, 4] \end{cases}$$

Then T satisfy the contractive condition from Theorem 2.1. (with a=$\frac{1}{2}$, $b = \frac{1}{4}$ and $c = \frac{1}{2}$) and the fixed points set $F_T = \{\frac{5}{2}\}$.

Next, we shall discuss data dependence problem for the set of all fixed points of such multivalued expansive-type operators.

The second main result is:

Theorem 2.2. *Let (X, d) be a complete metric space and $T_1, T_2 : X \to P_{b,cl}(X)$ two surjective multivalued operators. We suppose:*

i) there exist $a_i, b_i, c_i \in \mathbb{R}_+$, with $b_i < 1$ and $c_i > 1$ such that

$$d(y_1, y_2) \geq a_i d(x_1, x_2) + b_i d(x_1, y_1) + c_i d(x_2, y_2),$$

for each $x_1, x_2 \in X$ and $(y_1, y_2) \in T_i(x_1) \times T_i(x_2)$, $i \in \{1, 2\}$ with $y_1 \neq y_2$.

ii) there exists $\eta > 0$ such that $\delta(T_1^{-1}(y), T_2^{-1}(y)) \leq \eta$, for each $y \in X$.

Then

a) $F_{T_i} \in P_{cl}(X)$, for $i \in \{1, 2\}$

b) $H(F_{T_1}, F_{T_2}) \leq \dfrac{\eta}{1 - \max\{k_1, k_2\}}$, where $k_i = \dfrac{1 - b_i}{a_i + c_i}$ for $i \in \{1, 2\}$.

Proof. a) From Theorem 2.1 we get $F_{T_i} \in P(X)$, for $i \in \{1, 2\}$. We shall prove that F_T is closed, where T is T_1 or T_2. Let $(x_n)_{n \in \mathbb{N}} \subset F_T$ such that $\lim\limits_{n \to \infty} x_n = x^*$. We suppose, by contradiction, that $x^* \notin F_T$, i.e. $x^* \notin T(x^*)$. Then, for $x_n \in T(x_n)$ and every $y \in T(x^*)$, we have:

$$d(x_n, y) \geq ad(x_n, x^*) + bd(x_n, x_n) + cd(x^*, y).$$

Taking now $\inf\limits_{y\in T(x^*)}$ in the previous relation, we get:

$$D(x_n, T(x^*)) \geq ad(x_n, x^*) + cD(x^*, T(x^*)).$$

When n tends to infinite we obtain $(c-1)D(x^*, T(x^*)) \leq 0$ and hence $D(x^*, T(x^*)) = 0$. So $x^* \in T(x^*)$, that is a contradiction.

b) For the second part, let us consider any $x_0 \in F_{T_1}$, i.e. $x_0 \in T_1(x_0)$. Obviously $x_0 \in T_1^{-1}(x_0)$. Let us observe that for each $x \in T_2^{-1}(x_0)$ we have

$$d(x_0, x) \leq \delta(T_1^{-1}(x_0), T_2^{-1}(x_0)) \leq \eta.$$

On the other hand, from the surjectivity of T_2 we can deduce that there exists $x_1 \in X$ such that $x_0 \in T_2(x_1)$ or equivalently $x_1 \in T_2^{-1}(x_0)$. Obviously, $d(x_0, x_1) \leq \eta$.

Using the same construction as in the proof of Theorem 2.1 we obtain the sequence $(x_n)_{n\in\mathbb{N}}$ having the properties:

(α) $x_{n-1} \in T_2(x_n)$; $n \in \mathbb{N}$, $n \geq 1$

(β) $d(x_n, x_{n+m}) \leq \dfrac{k_2^n}{1-k_2} d(x_0, x_1)$; $n \in \mathbb{N}$, $m \in \mathbb{N}$, $m \geq 1$.

As before, the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in X and its limit x^* is a fixed point for T_2. For $m \to \infty$ the relation (β) becomes:

$$d(x_n, x^*) \leq \frac{k_2^n}{1-k_2} d(x_0, x_1) \leq \frac{k_2^n}{1-k_2}\eta, \text{ for } n \in \mathbb{N}.$$

If we consider $n = 0$ we get

$$d(x_0, x^*) \leq \frac{\eta}{1-k_2}. \tag{2}$$

Let us consider now a fixed point $y_0 \in F_{T_2}$. Following the same method we obtain that there exist $y^* \in F_{T_1}$ such that

$$d(y_0, y^*) \leq \frac{\eta}{1-k_1}. \tag{3}$$

From (2) and (3), by using Lemma 1.1, the conclusion follows. □

If we consider $b = c = 0$ then Theorem 2.1 becomes:

Corollary 2.3. (the dual form of the Avramescu-Markin-Nadller theorem) *Let (X, d) be a complete metric space and $T : X \to P_{b,cl}(X)$ be a surjective multivalued operator. If there exists $a \in \mathbb{R}$, $a > 1$ such that:*

$d(y_1, y_2) \geq ad(x_1, x_2)$, for each $x_i \in X$ and each $y_i \in T(x_i)$, for $i \in \{1, 2\}$ and $y_1 \neq y_2$,

then $F_T = \{x^*\}$.

Remark 2.4. If T is a singlevalued operator, then we get Theorem 3 in [1] (the dual form of the Banach contraction principle).

Another result of this type is:

Theorem 2.5. *Let (X, d) be a complete metric space and $T_1, T_2 : X \to P_{b,cl}(X)$ be two surjective multivalued operators.*

We suppose that:

i) there exist $k_1, k_2 \in \mathbb{R}$, $k_1 > 1$, $k_2 > 1$ *such that*

$$d^2(y_1, y_2) \geq k_i \min\{d^2(x_1, y_1), d^2(x_2, y_2), d(x_1, y_1)d(x_1, x_2), d(x_2, y_2)d(x_1, x_2)\}, \quad (4)$$

for each $x_1, x_2 \in X$ *and each* $(y_1, y_2) \in T_i(x_1) \times T_i(x_2)$, *for* $i \in \{1, 2\}$.

ii) there exists $\eta > 0$ *such that for each* $y \in X$

$$\delta(T_1^{-1}(y), T_2^{-1}(y)) \leq \eta.$$

Then:

a) $F_{T_i} \in P_{cl}(X)$

b) $H(F_{T_1}, F_{T_2}) \leq \dfrac{1}{1 - \max\left\{k_1^{-\frac{1}{2}}, k_2^{-\frac{1}{2}}\right\}}\eta.$

Proof. a) Using the surjectivity of $T_i = T$, we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \neq x_{n-1}$ and $x_{n-1} \in T(x_n)$, for $n \in \mathbb{N}$, $n \geq 1$.

From (4) we have:

$$d^2(x_{n-1}, x_n) \geq k \min\{d^2(x_n, x_{n-1}), d^2(x_{n+1}, x_n),$$

$$d(x_n, x_{n-1})d(x_n, x_{n+1}), d(x_{n+1}, x_n)d(x_n, x_{n+1})\} =$$

$$= k \min\{d^2(x_{n+1}, x_n), d(x_n, x_{n-1})d(x_n, x_{n+1})\} =$$

$$= kd(x_n, x_{n+1}) \min\{d(x_{n+1}, x_n), d(x_n, x_{n-1})\}.$$

The following alternative is now possible:

I. $d^2(x_{n-1}, x_n) \geq kd^2(x_{n+1}, x_n)$ and hence

$$d(x_n, x_{n+1}) \leq \frac{1}{\sqrt{k}} d(x_{n-1}, x_n),$$

for $n \in \mathbb{N}$, $n \geq 1$.

II. $d^2(x_{n-1}, x_n) \geq kd(x_n, x_{n+1})d(x_n, x_{n-1})$ that means

$$d(x_n, x_{n+1}) \leq \frac{1}{k} d(x_{n-1}, x_n) \leq \frac{1}{\sqrt{k}} d(x_{n-1}, x_n),$$

for $n \in \mathbb{N}$, $n \geq 1$.

From the both cases, it results:

$$d(x_n, x_{n+1}) \leq \left(\frac{1}{\sqrt{k}}\right)^n d(x_0, x_1).$$

Obviously, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and hence it is convergent.

Let us denote $x^* = \lim\limits_{n\to\infty} x_n$. For $x^* \in X$, we consider $y^* \in T^{-1}(x^*)$ and from (4) we have

$$d^2(x_n, x^*) \geq k \min\{d^2(x_{n+1}, x_n), d^2(y^*, x^*), d(x_{n+1}, x_n)d(x_{n+1}, y^*), d(y^*, x^*)d(x_{n+1}, y^*)\}.$$

For $n \to \infty$, we conclude $0 \geq kd(x^*, y^*)$ and so $x^* = y^* \in T^{-1}(x^*)$, proving that $x^* \in F_T$.

b) For the second part, the proof goes similar with part b) in Theorem 2.2. □

Remark 2.6. If T_i $(i \in \{1, 2\})$ are singlevalued operator, then from Theorem 2.5 we get also Theorem 1 in Popa [2].

Remark 2.7. For other fixed point and date dependence theorems see also Rus-Petruşel-Sântămărian [5].

Remark 2.8. It is an open question to prove some strict fixed point results for such expansive-type multifunctions.

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