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ON THE ESSENTIALITY OF THE MÖNCH TYPE MAPS

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Abstract. In this note, some recent results by R.P. Agarwal and D. O'Regan [1] concerning essential maps in the sense of Mönch are extended to maps satisfying some compactness conditions introduced by D. O'Regan and R. Precup [4].

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1 Essential maps of Mönch type

We begin by presenting the fixed point theorem of Mönch [3], which has been particularly useful in establishing existence of solutions to nonlinear boundary value problems in Banach spaces [2].

Theorem 1.1. *Let X be a Banach space, D be a closed, convex subset of X and $x_0 \in D$. Let $A : D \to D$ be a continuous map with the property:*

$$(1)\qquad \left(\left. \begin{array}{l} C \subset D,\ C \text{ - countable} \\ \\ \overline{C} = \overline{co}(\{x_0\} \cup A(C)) \end{array} \right\} \Rightarrow \ \overline{C} \text{ is compact} \right)$$

Then A has a fixed point in D.

The following result is a Leray-Schauder type theorem.

Theorem 1.2. *Let X be a Banach space, U an open subset of X and $x_0 \in U$. Let $A : \overline{U} \to X$ be a continuous map with the property:*

$$(2)\qquad \left(\left. \begin{array}{l} C \subset \overline{U},\ C \text{ - countable} \\ \\ \overline{C} = \overline{co}(\{x_0\} \cup A(C)) \end{array} \right\} \Rightarrow \ \overline{C} \text{ is compact} \right)$$

If $x \neq (1-\lambda)x_0 + \lambda A(x)$, for any $x \in \partial U$ and $\lambda \in (0,1)$, then A has a fixed point in $\overline{U}$.

Remark 1.1. Note that (1) and (2) can be replaced by the more general assumptions

$$(1')\qquad \left(\left. \begin{array}{l} M \subset D \\ M = co(\{x_0\} \cup A(M)) \\ \overline{M} = \overline{C},\ C \subset M,\ C \text{ - countable} \end{array} \right\} \Rightarrow \ \overline{M} \text{ is compact} \right)$$

and

$$\left(\left.\begin{array}{l} M\subset \overline{U}\\ M = co(\{x_0\}\cup A(M))\\ \overline{M}=\overline{C},\ C\subset M,\ C \text{ - countable}\end{array}\right\} \Rightarrow \overline{M} \text{ is compact}\right) \tag{2'}$$

respectively.

The conditions (1') and (2') are useful to establish generalizations of Mönch's fixed point theorems for multivalued maps (see [4]).

Definition 1.1. We let $M(\overline{U},X)$ denote the set of all continuous maps $A:\overline{U}\to X$, which satisfy (2') with $x_0=0$.

Definition 1.2. We let $A\in M_{\partial U}(\overline{U},X)$ if $A\in M(\overline{U},X)$ and $x\neq A(x)$ for $x\in\partial U$.

Definition 1.3. A map $A\in M_{\partial U}(\overline{U},X)$ is essential if for every $G\in M_{\partial U}(\overline{U},X)$ with $G/_{\partial U}=A/_{\partial U}$ there exists $x\in U$ with $x=G(x)$.

These definitions, corresponding to condition (2), have been given by R.P. Agarwal and D. O'Regan [1].

2 Main results

The first result is an example of the zero map being essential in $M_{\partial U}(\overline{U},X)$.

Theorem 2.1. *Let X be a Banach space and let U an open subset of X with $0\in U$. Then the zero map is essential in $M_{\partial U}(\overline{U},X)$.*

Proof.

Let $\theta\in M_{\partial U}(\overline{U},X)$ with $\theta/_{\partial U}=0$.

We must show that there exists $x\in U$ with $\theta(x)=x$.

Let $Q=\overline{co}(\theta(\overline{U}))$ and let $F:Q\to Q$ be given by

$$F(x)=\begin{cases}\theta(x), & x\in\overline{U}\\ 0, & x\in Q\setminus\overline{U}\end{cases}$$

Now $Q\neq\emptyset$ is convex and closed and F is continuous.

We first show F satisfies the Mönch condition (1') (with $x_0=0$).

To see this let $M\subset Q$ with $M=co(\{0\}\cup F(M))$ and $\overline{M}=\overline{C}$, where $C\subset M$ is countable. Then

$$M\subseteq co(\{0\}\cup\theta(\overline{U}\cap M))$$

and so

$$M\cap\overline{U}\subseteq M\subseteq co(\{0\}\cup\theta(\overline{U}\cap M)). \tag{3}$$

Notice that the following relations are true:

$$M\cap\overline{U}\subseteq Q; \tag{4}$$

$$\overline{M\cap\overline{U}}=\overline{M}\cap\overline{\overline{U}}=\overline{C}\cap\overline{\overline{U}}=\overline{C\cap\overline{U}}; \tag{5}$$

$$C \cap \overline{U} \subseteq M \cap \overline{U}; \tag{6}$$

$$C \cap \overline{U} \text{ is countable.} \tag{7}$$

Since $\theta \in M_{\partial U}(\overline{U}, X)$, from (3)-(7) and (2') we have that $\overline{M \cap \overline{U}}$ is compact. Then, because θ is continuous, $\theta(\overline{M \cap \overline{U}})$ is compact and by Mazur's lemma, $co(\{0\} \cup \theta(\overline{U \cap \overline{M}}))$ is relatively compact.

Thus, since $M \subseteq co(\{0\} \cup \theta(\overline{\overline{U} \cap M}))$, we have that $\overline{M}$ is compact.

Consequently $F : Q \to Q$ satisfies (1') (with $x_0 = 0$).

Consequently, Theorem 1.1 guarantees that there exists $x \in Q$ with $F(x) = x$.

Now if $x \notin U$ we have $0 = F(x) = x$, which is a contradiction since $0 \in U$.

Thus $x \in U$ so $x = F(x) = \theta(x)$.

The following result is a nonlinear alternative of Leray-Schauder type for Mönch type maps.

Theorem 2.2. *Let X be a Banach space and let U be an open subset of X with $0 \in U$. Suppose $A \in M(\overline{U}, X)$ satisfies $x \neq \lambda A(x)$ for any $x \in \partial U$ and $\lambda \in (0, 1]$. Then A is essential in $M_{\partial U}(\overline{U}, X)$. In particular A has a fixed point in U.*

Proof. Let $H \in M_{\partial U}(\overline{U}, X)$, with $H/_{\partial U} = A/_{\partial U}$.

We will show that H has a fixed point in U. Consider

$$B = \{x \in \overline{U} |\ x = tH(x) \text{ for some } t \in [0, 1]\}.$$

Now $B \neq \emptyset$ since $0 \in U$ and B is closed since H is continuous. In addition $B \cap \partial U = \emptyset$ since $x \neq \lambda A(x)$ for any $x \in \partial U$ and $\lambda \in (0, 1]$ and $H/_{\partial U} = A/_{\partial U}$. We now claim that there exists a continuous function $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(B) = 1$. The claim is immediate from Urysohn's theorem since B and ∂U are closed.

Define a map $R_\mu : \overline{U} \to X$ by $R_\mu(x) = \mu(x)H(x)$.

We first show that R_μ satisfies the Mönch condition (2').

To see this let $M \subset \overline{U}$ with $M \subset co(\{0\} \cup R_\mu(M))$ and $\overline{M} = \overline{C}$, where $C \subset M$ is countable.

Let $y \in R_\mu(M)$ be arbitrary. Then there exists $x \in M$ such that $y = \mu(x)H(x)$.

a) If $x \in \partial U$ then $\mu(x) = 0$ and so $y = 0$;

b) If $x \in B \cap M$ then $\mu(x) = 1$ $(y = H(x))$ and so $y \in H(B \cap M) \subset H(M)$;

c) If $x \in M \setminus \{\partial U \cup B\}$ then $\mu(x) \in (0, 1)$ and so

$$y = R_\mu(x) = \mu(x)H(x) \in co(H(M)).$$

From a), b), c) we have $R_\mu(M) \subset co(\{0\} \cup H(M))$.

In addition, since $co(\{0\} \cup H(M))$ is convex and

$$\{0\} \cup co(\{0\} \cup H(M)) = co(\{0\} \cup H(M))$$

we have that

$$M \subset co(\{0\} \cup R_\mu(M)) \subset co(\{0\} \cup co(\{0\} \cup H(M))) =$$

$$= co(co(\{0\} \cup H(M)) = co(\{0\} \cup H(M)).$$

Since $H \in M_{\partial U}(\overline{U}, X)$ we obtain that $\overline{M}$ is compact. Thus $R_\mu \in M(\overline{U}, X)$, with $R_\mu/_{\partial U} = 0$.

Now since the zero map is essential in $M_{\partial U}(\overline{U}, X)$, there exists $x \in U$ with $x = R_\mu(x)$. Consequently $x \in B$ and so $\mu(x) = 1$ i.e. $x = H(x)$.

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