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DATA DEPENDENCE FOR MULTIVALUED CONTRACTIONS ON GENERALIZED COMPLETE METRIC SPACES

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Abstract. A data dependence result for multivalued contractions in generalized metric spaces.

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1 Introduction

In 1985 T.C. Lim proved (see [2]) that if T_1 and T_2 are multivalued a-contractions on a complete metric space X such that $H(T_1(x), T_2(x)) \leq \eta$, for every $x \in X$ then as regard to the Hausdorff-Pompeiu distance between their fixed points set we have:

$$H(F_{T_1}, F_{T_2}) \leq \frac{\eta}{1-a}.$$

The purpose of this note is to prove that the data dependence problem for multivalued contractions on a generalized complete metric space has also a positive answer.

Let us consider now some notations, notions and preliminary results.

Definition 1. The pair (X, d) will be called a generalized metric space if X is an arbitrary nonempty set and the functions $d : X \times X \to [0, \infty]$ fulfils all the standard conditions for a metric.

In a generalized metric space (where the generalized metric d is allowed to take on $+\infty$ as well), just as in a metric space, we can define open and closed balls, convergence of sequences, completeness of the space, etc. (see for example [5]).

If (X, d) is a generalized metric space, we will define the following classes of subsets of X:

$P_{cl}(X) = \{Y|\ Y \text{ is a nonempty and closed subset of } X\}$

$P_{b,cl}(X) = \{Y \in P_{cl}(X)|\ Y \text{ is bounded}\}$.

Also, we can define the generalized Hausdorff-Pompeiu metric H is the set $P_{cl}(X)$ by

$$H(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B} d(a, b), \sup_{b\in B}\inf_{a\in A} d(a, b)\right\} \text{ for } A, B \in P_{cl}(X).$$

If $T : X \to P_{cl}(X)$ is a multivalued operator then a fixed point for T is an element $x^* \in X$ such that $x^* \in T(x^*)$. The fixed points set for T will be denoted by F_T.

Now, we shall consider the notion of successive approximations sequence for a multivalued operator T.

Definition 2. The sequence $(x_n)_{n\in\mathbb{N}}$, $x_n \in X$ for $n = 0, 1, \ldots$ is called a successive approximations sequence starting from $x_0 \in X$ for the multivalued operator T from a generalized metric space X into itself if $x_{n+1} \in T(x_n)$ for each $n = 0, 1, 2, \ldots$.

Definition 3. Let (X, d) be a generalized metric space and $T : X \to P_{cl}(X)$ be a multivalued operator. Then T is said to be an a-contraction if there exists $a \in [0, 1[$ such that

$$H(T(x), T(y)) \le ad(x, y), \text{ for } x, y \in X \text{ with } d(x, y) < +\infty.$$

2 Main result

First, we shall prove the following auxiliary result:

Lemma 1. *Let (X, d) be a generalized metric space and $A, B \in P_{cl}(X)$. Then for each $q > 1$ and each $a \in A$ there exists $b \in B$ such that $d(a, b) \le qH(A, B)$.*

Proof. Suppose, by contradiction, that there exists $q > 1$ and $a \in A$ such that for each $b \in B$ we have $d(a, b) > qH(A, B)$.

If there exists $b \in B$ such that $d(a, b) < +\infty$ then $D(a, B) > qH(A, B)$ and hence $H(A, B) \ge D(a, B) > qH(A, B)$ is a contradiction.

On the other hand, if for each $b \in B$ we have $d(a, b) = +\infty$ then $D(a, B) = +\infty$ and hence $H(A, B) = +\infty$. Contradiction again with the fact that $qH(A, B) < d(a, b) = +\infty$. The proof is complete. □

The main result of this note is the following:

Theorem 1. *Let (X, d) be a generalized metric space and $T_i : X \to P_{cl}(X)$ be a_i-contraction for $i \in \{1, 2\}$. Let us suppose that:*

i) for each $x_0 \in X$, $y_0 \in X$ there exists the successive approximations sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ for T_1 and respectively T_2 such that $d(x_{k-1}, x_k) < +\infty$ and $d(y_{k-1}, y_k) < +\infty$ for every $k \in \mathbb{N}^*$;

ii) there exists $\eta > 0$ such that $H(T_1(x), T_2(x)) \le \eta$, for every $x \in X$.

Then

$$H(F_{T_1}, F_{T_2}) \le \frac{\eta}{1 - \max\{a_1, a_2\}}.$$

Proof. Using Covitz-Nadler [1] Theorem, we deduce that F_{T_i} is nonempty and closed in X. Let $x_0 \in F_{T_1}$ be arbitrary. Then $x_0 \in T_1(x_0)$ and from Lemma 1, there exists $x_1 \in T_2(x_0)$ such that $d(x_0, x_1) \le qH(T_1(x_0), T_2(x_0)) \le q\eta$. Let $1 < q < \min\{1/a_1, 1/a_2\}$. For $x_1 \in T_2(x_0)$ it follows from Lemma 1, that there is $x_2 \in T_2(x_1)$ such that

$$d(x_1, x_2) \le qH(T_2(x_0), T_2(x_1)) \le qa_2(x_0, x_1).$$

It is easy to see that, by this procedure, we get a successive approximations sequence for T_2 such that

$$d(x_n, x_{n+1}) \le (qa_2)^n d(x_0, x_1).$$

Hence

$$d(x_n, x_{n+p}) \leq (qa_2)^n d(x_0, x_1) + (qa_2)^{n+1} d(x_0, x_1) + \cdots + (qa_2)^{n+p-1} d(x_0, x_1) \leq$$

$$\leq (qa_2)^n d(x_0, x_1)[1 + qa_2 + \cdots + (qa_2)^{p-1}].$$

Obviously, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X and its limit $x_2^* \in F_{T_2}$.

Let us observe that $d(x_n, x_2^*) \leq \dfrac{(qa_2)^n}{1 - qa_2} d(x_0, x_1)$, for $n \geq 0$. For $n = 0$ we get

$$d(x_0, x_2^*) \leq \frac{d(x_0, x_1)}{1 - qa_2} \leq \frac{q\eta}{1 - qa_2}.$$

Interchanging the roles of T_1 and T_2 we obtain that for each $y_0 \in F_{T_2}$ there exists $x_1^* \in F_{T_1}$ such that

$$d(y_0, x_1^*) \leq \frac{q\eta}{1 - qa_1}.$$

Hence

$$H(F_{T_1}, F_{T_2}) \leq \frac{q\eta}{1 - q \max\{a_1, a_2\}}.$$

Letting $q \searrow 1$ we obtain the desired conclusion. $\square$

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