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AN APPLICATION OF THE WEAKLY PICARD OPERATORS TECHNIQUE TO A DIRICHLET PROBLEM

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Abstract. The purpose of this paper is to study the following elliptic equation:

$$-\Delta u = f(x,u) \text{ on } \Omega \tag{1}$$

where Ω is a smooth bounded open set of R^n and $f \in C(\Omega \times R^n)$.

We use the weakly Picard operators' theory to establish an integral equation equivalent with equation (1) and some differential inequalities for the Dirichlet problem:

$$\begin{cases} -\Delta u = f(x,u) & \text{on} \quad \Omega \\ u = \varphi & \text{on} \quad \partial\Omega \end{cases} \tag{2}$$

where $\varphi \in C(\partial\Omega)$.

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1 Introduction

1.1 Weakly Picard operators

We begin with some notions and results from the weakly Picard operators theory ([3], [4]).

Let (X,d) be a metric space and $A : X \to X$.

A is a weakly Picard operator if for all $x \in X$ the sequence $A^n(x)$ converge to $x^*(x)$ for $n \to \infty$, where x^* is a fixed point of A.

If A has a unique fixed point then A is a Picard operator.

1.2 The A^∞ operator

Let (X,d) be a metric space and $A : X \to X$ a weakly Picard operator.

Then, we define $A^\infty : X \to X$ with $A^\infty(x) = \lim\limits_{n\to\infty} A^n(x)$.

Theorem 1. ([4]) *Let $(X,d,\leq)$ be a metric space orderly and $A : X \to X$. Assume that:*

a) A is a weakly Picard operator.

b) A is monotone increasing.

Then A^∞ is monotone increasing.

Theorem 2. ([4]) *Let* $(X, d, \leq)$ *be a metric space orderly and* $A, B : X \to X$. *Suppose that:*

a) $A \leq B$.

b) A, B *are weakly Picard operators.*

c) A *or* B *is monotone increasing.*

Then $x \leq y$ *for* $x, y \in X$ *implies that* $A^\infty(x) \leq B^\infty(y)$.

2 An integral equation equivalent with an elliptic equation

From the theory of partial differential equations we see that the equation (1) is equivalent with the integral equation:

$$u(x) = \int_\Omega f(s, u(s))G(x, s)ds + \int_{\partial\Omega} u(s)\frac{\partial G(x, s)}{\partial n_s} d\sigma_s. \tag{3}$$

We define

$$A : C(\overline{\Omega}) \to C(\overline{\Omega})$$

where

$$A(u)(x) = \int_\Omega f(s, u(s))G(x, s)ds + \int_{\partial\Omega} u(s)\frac{\partial G(x, s)}{\partial n_s} d\sigma_s.$$

We have get the fixed point equation

$$u = A(u). \tag{4}$$

We have

Theorem 3. *If* $f(x, \cdot)$ *if* L_f *Lipschitz with* $L_f > 0$ *small enough for all* $x \in \overline{\Omega}$, *then* A *is a weakly Picard operator.*

Proof. We decompose $C(\overline{\Omega})$ in the following partition:

$$C(\overline{\Omega}) = \bigcup_{\varphi \in C(\partial\Omega)} X_\varphi$$

where $X_\varphi = \{u \in C(\overline{\Omega}) | u|_{\partial\Omega} = \varphi\}$. It is clear that

i) X_φ is an invariant set for $A : A(X_\varphi) \subset X_\varphi$.

ii) $A|_{X_\varphi}$ is a contraction, i.e. $A|_{X_\varphi}$ is a Picard operator.

From (i) and (ii), we easily get that A is a weakly Picard operator.

3 Differential inequalities

Next, we consider the problems:

$$\begin{cases} -\Delta u = f(x, u) & \text{on} \quad \Omega \\ u = \varphi & \text{on} \quad \partial\Omega \end{cases} \tag{5}$$

and

$$\begin{cases} -\Delta u = f(x,u) & \text{on} \quad \Omega \\ u = \psi & \text{on} \quad \partial\Omega \end{cases} \tag{6}$$

where $\varphi, \psi \in C(\partial\Omega)$.

Theorem 5. ([1], [2], [5]) *Suppose that:*

a) $\varphi \le \psi$.

b) $f(x,\cdot)$ *is monotone increasing for all* $x \in \Omega$.

c) $f(x,\cdot)$ *is* L_f *Lipschitz, with* $L_f > 0$ *small enough, for all* $x \in \Omega$.

Then $u^* \le v^*$, *where* u^* *is the solution of problem (5) and* v^* *is the solution of problem (6).*

Proof. As we have see:

$$(5) \;\Leftrightarrow\; u(x) = \int_\Omega f(s,u(s))G(x,s)ds + \int_{\partial\Omega} \varphi(s)\frac{\partial G(x,s)}{\partial n_s}d\sigma_s \tag{7}$$

$$(6) \;\Leftrightarrow\; u(x) = \int_\Omega f(s,u(s))G(x,s)ds + \int_{\partial\Omega} \psi(s)\frac{\partial G(x,s)}{\partial n_s}d\sigma_s \tag{8}$$

c) implies that equations (7) and (8) have the unique solutions u^* and v^*.

Suppose that $\overline{\varphi}$ and $\overline{\psi}$ are the continuous extensions of φ and ψ on $\overline{\Omega}$, such that $\overline{\varphi} \le \overline{\psi}$.

From b) and c), by Theorem 1, we get that A^* is monotone increasing and so

$$A^\infty(\overline{\varphi}) \le A^\infty(\overline{\psi}).$$

But $A^\infty(C(\overline{\Omega}) = F_A$ (the set of fixed point of A), and so, since u^* and v^* are the unique solutions of (7) and (8), we have $u^* \le v^*$.

Next, we consider the problems:

$$\begin{cases} -\Delta u = f(x,u) & \text{on} \quad \Omega \\ u = \varphi & \text{on} \quad \partial\Omega \end{cases} \tag{9}$$

and

$$\begin{cases} -\Delta v = g(x,v) & \text{on} \quad \Omega \\ v = \psi & \text{on} \quad \partial\Omega \end{cases} \tag{10}$$

where $f, g \in C(\Omega \times R^n)$ and $\varphi, \psi \in C(\partial\Omega)$.

Theorem 6. ([1], [2], [5]) *Suppose that:*

a) $f(x,\cdot) \le g(x,\cdot)$ *for all* $x \in \Omega$.

b) $\varphi \le \psi$.

c) $f(x,\cdot)$ *is* L_f *Lipschitz and* $g(x,\cdot)$ *is* L_g *Lipschitz with* $L_f > 0$, $L_g > 0$ *small enough, for all* $x \in \Omega$.

d) $f(x,\cdot)$ *or* $g(x,\cdot)$ *is monotone increasing for all* $x \in \Omega$.

Then $u^* \le v^*$, *where* u^* *and* v^* *are the solutions of problem (9) and (10).*

Proof. Suppose that $g(x,\cdot)$ is monotone increasing. Then

$$(9) \;\Leftrightarrow\; u(x) = \int_\Omega f(s,u(s))G(x,s)ds + \int_{\partial\Omega} \varphi(s)\frac{\partial G(x,s)}{\partial n_s}d\sigma_s = A(u)(s) \tag{11}$$

$$(10) \;\Leftrightarrow\; v(x) = \int_{\Omega} g(s, v(s))G(x, s)ds + \int_{\partial\Omega} \psi(s)\frac{\partial G(x, s)}{\partial n_s}d\sigma_s = B(v)(s) \qquad (12)$$

c) implies that equation (11) and (12) have the unique solutions u^* and v^*.

Suppose that $\overline{\varphi}$ and $\overline{\psi}$ are the continuous extensions of φ and ψ on $\overline{\Omega}$, such that $\overline{\varphi} \leq \overline{\psi}$. Since $g(x, \cdot)$ is monotone increasing and $f(x, \cdot) \leq g(x, \cdot)$ for all $x \in \Omega$, we have

$$f(x, \overline{\varphi}) \leq g(x, \overline{\varphi}) \leq g(x, \overline{\psi}),$$

and so $A(\overline{\varphi}) \leq B(\overline{\psi})$. By Theorem 2 we get $A^{\infty}(\overline{\varphi}) \leq B^{\infty}(\overline{\psi})$.

But $A^{\infty}(C(\overline{\Omega}) = F_A$ and $B^{\infty}(C(\overline{\Omega})) = F_B$, and so $u^* \leq v^*$.

References

[1] J. Peetre, I.A. Rus, *Sur la positivité de la fonction de Green*, Math. Scand. 21(1967), 80-89.

[2] R. Precup, *Ecuaţii cu derivate parţiale*, Transilvania Press, Cluj-Napoca, 1997.

[3] I.A. Rus, *Generalized contractions and applications*, Cluj University Press, 2001.

[4] I.A. Rus, *Weakly Picard operators and applications* (to appear).

[5] M.E. Taylor, *Partial Differential Equations*, Springer, 1996.