Seminar on Fixed Point Theory Cluj-Napoca, **Volume 1, 2000, 19-30**
http://www.math.ubbcluj.ro/~nodeacj/journal.htm

DEFFICIENT SPLINE FUNCTIONS FOR THE NUMERICAL SOLUTION OF THE NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. A new collocation method for the numerical solution of the second order neutral delay differential equations is given by using the defficient spline functions of the lower degree. The spline approximating solutions are effectively constructed by a new modified collocation method. For the cubic and quartic splines the estimation of the errors as well as the convergence properties are investigated.
Keywords: neutral differential equations, spline approximation, algorithm's convergence
AMS Subject Classification: 65L10, 65Q05.

1 Introduction

In recent years a great number of dynamical processes have been described and investigated by differential equations with deviating argument. Because of the versatility of such equations in the modelling of processes in various applications, especially in physics, engineering, biomathematics, medical sciences, economics, etc, neutral delay differential equations provide the best and sometimes the only realistic simulation of observed phenomena. Since the solutions of such equations in general are not found explicitely, then the methods for their approximate solutions assume a great importance.

The idea of using spline functions to approximate the solution of delay differential equations has been applied in a number of papers, for instance [3], [5], [8], [11]-[15].

In this paper a modified collocation method to construct defficient spline approximation solutions for the second order neutral delay differential equations is proposed. For the lower degree splines the estimation of the errors and the convergence of the algorithm and investigated.

2 The description of modified spline collocation method

One consider the following second order neutral delay differential equation problem:

$$(1)\qquad \begin{cases} y''(x) = f(x, y(x), y(g(x)), y'(g(x))),\ x \in [a, b] \\ y(x) = \varphi(x),\ y'(x) = \varphi'(x),\ x \in [\alpha, a],\ \alpha \le a\ (\alpha = \inf\limits_{x\in[a,b]} g(x)) \end{cases}$$

$$\alpha \le g(x) \le x,\ x \in [a, b]$$

$$\varphi \in C^{m-2}[a, b],\ m > 2,\ f : [a, b] \times C^1[a, b] \times C^1[\alpha, b] \times C[\alpha, b] \to \mathbb{R}.$$

Assumptions.

H_1. For any $y \in C^1[\alpha, b]$ the mapping $x \mapsto f(x, y(x), y(\cdot), y'(\cdot))$ is a continuous on $[a, b]$.

H_2. The following Lipschitz condition holds:

$$\|f(x, y_1(x), z_1(\cdot), u_1(\cdot)) - f(x, y_2(x), z_2(\cdot), u_2(\cdot)\| \le$$

$$\le L_1[\|y_1 - y_2\|_{[\alpha,x]} + \|z_1 - z_2\|_{[\alpha,x-\delta]} + \|u_1 - u_2\|_{[\alpha,x-\delta]}] + L_2\|z_1 - z_2\|_{[\alpha,x]}$$

with: $L_1 \ge 0,\ 0 \le L_2 < 1,\ \delta > 0$, for any $x \in [a, b]$, $y_1, y_2 \in C^1[a, b]$, $z_1, z_2, u_1, u_2 \in C[\alpha, b]$.

Under the conditions H_1 and H_2 the problem (1) has a unique solution $y \in C^2[a, b] \cap C^1[\alpha, b]$ (see [7]).

As it is known (see [7], [10]) jump discontinuities can occur in various higher order derivatives of the solution y, even if f, g, φ are analytic in their arguments. Such jump discontinuities are caused by the delay function g and propagate from the point a as the order of derivatives increases.

If denote the jump discontinuities by $\{\xi_j\}$, it is also known that ξ_i are the roots of equation

$$g(\xi_i) = \xi_{i-1}, \quad \xi_0 = a$$

$\xi_0 = a$ is a jump discontinuity of φ (or of its derivative). Since in (1) the delay function g does nor depend of y (no state depending argument), we can consider the jump discontinuities for sufficiently high order derivatives to be such that:

$$\xi_0 < \xi_1 < \cdots < \xi_{k-1} < \xi_k < \cdots < \xi_M$$

We shall construct for the problem (1) a deficient polynomial spline approximating function of degree $m \ge 3$ and defficiency 2, ($s \in C^{m-2}$) denoted by $s : [a, b] \to \mathbb{R}$, which will be defined on each interval $[\xi_{k-1}, \xi_k]$. For this construction we shall use successively the collocation method as in [14]:

Let consider the first interval $[\xi_0, \xi_1]$, which is $[a, \xi_1]$ divided by an uniform partition defined by the knots

$$\xi_0 = t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots < t_N = \xi_1.$$

$$t_j := t_0 + jh, \quad h = \frac{\xi_1 - \xi_0}{N}.$$

On the first interval $[t_0, t_1]$, the spline component is defined by

$$s_0(t) := \varphi(t_0) + \frac{\varphi'(t_0)}{1!}(t - t_0) + \cdots + \frac{\varphi^{(m-2)}(t_0)}{(m-2)!}(t - t_0)^{m-2} + \tag{2}$$

$$+\frac{a_0}{(m-1)!}(t - t_0)^{m-1} + \frac{b_0}{m!}(t - t_0)^m$$

$$\Rightarrow \left\{ \begin{array}{l} s_0''\left(t_0 + \frac{1}{2}\right) = f\left(t_0 + \frac{h}{2}, s_0\left(t_0 + \frac{h}{2}\right), \right. \\ \left. \qquad \varphi\left(g\left(t_0 + \frac{h}{2}\right)\right), \varphi'\left(g\left(t_0 + \frac{h}{2}\right)\right)\right) \\ s_0''(t_1) = f(t_1, s_0(t_1), \varphi(g(t_1)), \varphi(g(t_1)), \varphi'(g(t_1))) \end{array} \right.$$

Having determined the polynomial (2), on the next interval $[t_1, t_2]$, we define:

$$s_1(t) := \sum_{j=0}^{m-2} \frac{s_0^{(j)}(t_1)}{j!}(t - t_2) + \frac{a_1}{(m-1)!}(t - t_1)^{m-1} + \frac{b_1}{m!}(t - t_1)^m, \tag{3}$$

where $s_0^{(j)}(t_1)$, $0 \le j \le m-2$ are left-hand limits of derivatives as $t \to t_1$ of the segment of s defined on $[t_0, t_1]$ and a_1 and b_1 are determined from the following collocation conditions:

$$\left\{ \begin{array}{l} s_1''\left(t_1 + \frac{h}{2}\right) = f\left(t_1 + \frac{h}{2}, s_1\left(t_1 + \frac{h}{2}\right), \right. \\ \left. \qquad s_0\left(g\left(t_1 + \frac{h}{2}\right)\right), s_0'\left(g\left(t_1 + \frac{h}{2}\right)\right)\right) \\ s_1''(t_2) = f(t_2, s_1(t_2), s_0(g(t_2)), s_0'(g(t_2))). \end{array} \right.$$

Continuing in this manner:
$t \in [t_k, t_{k+1}]$:

$$s_k(t) = \sum_{j=0}^{m-2} \frac{s_k^{(j)}(t_k)}{j = 0}(t - t_k)^j + \frac{a_k}{(m+1)!}(t - t_k)^{m-1} + \frac{b_k}{m!}(t - t_k)^m,$$

$$s_{k-1}^{(j)}(t_k) = \lim_{t \nearrow t_k} s_{k-1}^{(j)}(t), \quad t \in [t_{k-1}, t_k].$$

$$\Rightarrow s_k''\left(t_k + \frac{h}{2}\right) = f\left(t_k + \frac{h}{2}, s_k\left(t_k + \frac{h}{2}\right), \right.$$

$$\left. s_{k-1}\left(g\left(t_k + \frac{h}{2}\right)\right), s_{k-1}'\left(g\left(t_k + \frac{h}{2}\right)\right)\right)$$

$$s_k''(t_{k-1}) = f(t_{k+1}, s_k(t_{k+1}), s_{k-1}(g(t_{k+1})), s_{k-1}'(g(t_{k+1}))).$$

In general, on the interval $[\xi_j, \xi_{k+1}]$ $(j = \overline{0, M-1})$ which is also divided by an uniform partition with the points:

$$t_k := t_0 + kh, \quad k = \overline{0, N}; \quad t_0 := \xi_j, \quad t_N = \xi_{j+1}, \quad h := \frac{\xi_{j+1} - \xi_j}{N}$$

if we denote by s, $s \in S_m$, $s \in C^{m-2}$, the spline function approximating the solution of (1), then on the interval $[t_k, t_{k+1}]$, s is defined by

$$s(t) := \sum_{j=0}^{m-q} \frac{s^{(j)}(t_k)}{i!}(t - t_k)^i + \frac{a_k}{(m-1)!}(t - t_k)^{m-1} + \frac{b_k}{m!}(t - t_k)^m, \tag{4}$$

where $s^{(i)}(t_k)$, $0 \leq i \leq m-2$ are the left-hand limits of the derivatives of s defined on $[t_{k-1}, t_k]$, and the parameters a_k, b_k are determined from the following collocation conditions:

$$\begin{cases} s_j''\left(t_k + \frac{h}{2}\right) = f\left(t_k + \frac{h}{2}, s_j\left(t_k + \frac{h}{2}\right),\right. \\ \qquad \left. s_{j-1}\left(g\left(t_k + \frac{h}{2}\right)\right), s_{j-1}'\left(g\left(t_k + \frac{h}{2}\right)\right)\right) \\ s_j''(t_{k+1}) = f(t_{k+1}, s_j(t_{k+1}), s_{j-1}(g(t_{k+1})), s_{j-1}'(g(t_{k+1}))). \end{cases}$$

$$j = \overline{0, M}, \quad k = \overline{0, N-1}, \quad s_j := s|_{I_j}, \quad I_j := [t_j, t_{j+1}].$$

This procedure yield a spline $s \in S_m$, $s \in C^{m-2}$ (with defficiency 2) over the entire interval $[\xi_j, \xi_{j+1}]$ with the knots $\{t_k\}_{k=0}^N$. It remains to prove that for h sufficiently small, the parameters a_k, b_k, $0 \leq k \leq N$ can be uniquely determined from (5).

Theorem 1. *Let consider the delay neutral differential equations problem (1). If f satisfies the assumption H_1, H_2, $\varphi \in C^{m-2}$, $\alpha \leq g(t) \leq t$, $t \in [\alpha, b]$, and if h is small enough, then there exists a unique spline approximating solution s of the problem (1) given by the above construction.*

Proof.

$$s(t) = \sum_{j=0}^{m-2} \frac{s^{(j)}(t_k)}{i!}(t - t_k)^i + \frac{a_k}{(m-1)!}(t - t_k)^{m-1} + \frac{b_k}{m!}(t - t_k)^m =$$

$$=: A_k(t) + \frac{a_k}{(m-1)!}(t - t_k)^{m-1} + \frac{b_k}{m!}(t - t_k)^m$$

$$s_k'(t) = A_k'(t) + \frac{a_k}{(m-2)!}(t - t_k)^{m-2} + \frac{b_k}{(m-1)!}(t - t_k)^{m-1}$$

$$s_k''(t) = A_k''(t) + \frac{a_k}{(m-3)!}(t - t_k)^{m-3} + \frac{b_k}{(m-2)!}(t - t_k)^{m-2}.$$

The system (5) is:

$$a_k = \left(\frac{2}{h}\right)^{m-3} (m-3)! \Big\{ f\left(t_k + \frac{h}{2}, A_k\left(t_k + \frac{h}{2}\right) + \frac{a_k}{(m-1)!}\left(\frac{h}{2}\right)^{m+1} + \right.$$

$$+\frac{b_k}{m!}\left(\frac{h}{2}\right)^m, s_{k-1}\left(g\left(t_k+\frac{h}{2}\right)\right), s'_{k-1}\left(g\left(t_k+\frac{h}{2}\right)\right)-$$

$$-\frac{n_k}{(m-2)!}\left(\frac{h}{2}\right)^{m-2}-A''_k\left(t_k+\frac{h}{2}\right)\Big\}$$

$$b_k=\frac{(m-2)!}{h^{m-2}}\Big\{f\Big(t_{k+1}, A_k(t_{k+1})+\frac{a_k}{(m-1)!}h^{m-1}+$$

$$+\frac{b_k}{m!}h^m, s_{k-1}(g(t_{k+1})), s'_{k-1}(g(t_{k+1}))\Big)-$$

$$-\frac{a_k}{(m-3)!}h^{m-3}-A''_k(t_{k+1})\Big\}$$

$$\Rightarrow \begin{cases} a_k=F_1(a_k,b_k)\\ b_k=F_2(a_k,b_k)\end{cases}$$

Defined: $F:\mathbb{R}^2\to\mathbb{R}^2$

$$(a_k,b_k)\mapsto F(a_k,b_k):=(F_1(a_k,b_k),F_2(a_k,b_k))$$

$$d(F(a'_k,b'_k),F(a''_k,b''_k)):=|F_1(a'_k,b'_k)-F_1(a''_k,a''_k)|+$$

$$+|F_2(a'_k,b'_k)-F_2(a''_k,b''_k)|$$

$$|F_1(a'_k,b'_k)-F_1(a''_k,b''_k)|\le$$

$$\le h_1\left(\frac{2}{h}\right)^{m-3}(m-3)!\frac{h^{m-1}}{(m-1)!}\left|\frac{1}{2^{m-1}}|a'_k-a''_k|+\frac{h}{m}|b'_k-b''_k|\right|\le$$

$$\le\frac{L_1h^2}{2^2}\left||a'_k-a''_k|+\frac{h}{2m}|b'_k-b''_k|\right|.$$

Analogously

$$|F_2(a'_k,b'_k)-F_2(a''_k,b''_k)|\le\frac{L_1h^2}{m-1}\left||a'_k-a''_k|+\frac{h}{m}|b'_k-b''_k|\right|$$

$$d(F(a'_k,b'_k),F(a''_k,b''_k))\le\frac{L_1h^2}{2^2}\left\{|a'_k-a''_k|+\frac{h}{2m}|b'_k-b''_k|\right\}+$$

$$+\frac{L_1h^2}{m-1}\left\{|a'_k-a''_k|+\frac{h}{m}|b'_k-b''_k|\right\}\le 2L_1h^2d((a'_k,b'_k),(a''_k,b''_k))$$

$$2L_1h^2<1\ \Rightarrow\ h<\sqrt{\frac{1}{2L_1}}.$$

For h small enough the application f is a contraction, (5) has a unique solution, which can be find by iteration.

Theorem 2. ([3]) *For any spline function* $s \in S_m$, $s \in C^{m-2}[a,b]$, $m \geq 3$ *there exists a unique linear consistency relation between the values* $s(t_k)$ *and* $s''(t_k)$, $k = 0, 1, \ldots, N$, *namely*

$$\sum_{j=0}^{m-1} a_j^{(m)} s(t_{j+\nu}) = h^2 \sum_{j=0}^{m-1} b_j^{(m)} s''(t_{j+\nu}), \quad 0 \leq \nu \leq N - m + 1$$

where

$$a_j^{(m)} := (m-1)![Q_{m-1}(j+1) - 2Q_{m-1}(j) + Q_{m-1}(j-1)]$$

$$b_j^{(m)} := (m-1)! Q_{m-1}(j+1)$$

and

$$Q_k(x) := \frac{1}{(k-1)!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (x-i)_+^{k-1}.$$

Theorem 3. *The values* $s(t_k)$, $k = 0, 1, \ldots, N$ *of the above constructed spline function are exactly the values furnished by the discrete multistep method defined by the following recurrence relation:*

$$\sum_{j=0}^{m-1} a_j^{(m)} y_{j+k} = h^2 \sum_{j=0}^{m-1} y''_{j+k}, \quad k = 0, 1, 2, \ldots \tag{5}$$

if the starting values

$$y_0 = s(t_0), \quad y_1 = s(t_0 + h), \ldots, y_{m-2} = s(t_0 + (m-2)h) \tag{6}$$

are used.

Proof. For h small enough, only on set fo values $\{y_k\}_k$ is satisfying (6) with the starting values (7). But obviously the spline values $\{s_k\}_k$ are satisfying (6) on the basis of consistency relation and also the starting values (7). That means the spline values must coincide with the values given by the discrete multistep method (6). □

Because the spline approximating solution $s \in C^{m-2}[a,b]$, we shall define the derivatives $s^{(m-1)}$ and $s^{(m)}$ at the knots $\{t_k\}_{k=1}^{N-1}$ by the usual arithmetic mean:

$$s^{(j)} := \frac{1}{2}\left[s^{(j)}\left(t_k - \frac{1}{2}h\right) + s^{(j)}\left(t_k + \frac{h}{2}\right)\right], \quad j = m-1, m. \tag{7}$$

We need the following two lemmas:

Lemma 1. *Let* $s : [a,b] \to \mathbb{R}$ *be the spline approximating solution and* y *be the unique exact solution of (1).*

If

$$|s(t_k) - y(t_k)| < Kh^p, \quad |s(g(t_k) - y(t_k))| < Kh^p$$

$$s'(g(t_k)) - y'(g(t_k))| < Kh^p$$

where K is a constant, then there exists a constant K_1 independent of h, such that:

$$|s(t_k) - y(t_k)| < K_1 h^p, \quad |s''(t_k) - y''(t_k)| < K_1 h^p.$$

Proof. Using the Lipschitz condition H_2, we have:

$$|s''(t_k) - y''(t_k)| = |f(t_k, s(t_k), s(g(t_k)), s'(g(t_k)) -$$
$$-f(t_k, y(t_k), y(g(t_k)), y'(g(t_k)))| \leq$$
$$\leq L_1\{|s(t_k) - y(t_k)| + |s(g(t_k)) - y(g(t_k))| +$$
$$+|s'(g(t_k)) - y'(g(t_k))|\} +$$
$$+L_2|s(g(t_k)) - y(g(t_k))| \leq$$
$$\leq L_1[Kh^p + Kh_K^p h^p] + L_2 K h^p =$$
$$= 3KL_1 h^p + L_2 K h^p = K[3L_1 + L_2]h^p$$
$$\Rightarrow \; K_1 := \max\{K, K[3L_1 + L_2]\}.$$

Lemma 2. [3,15] *Let $y \in C^{m+1}[a,b]$ and s be the spline function of degree m and class C^{m-2} with the knots $\{t_k\}_k$.*

$$\begin{aligned} &|s^{(r)}(t_k) - y^{(r)}(t_k)| = O(h^{p_r}) \\ &|s^{(r)}(g(t_k)) - y^{(r)}(g(t_k))| = O(h^{p_r}), \end{aligned} \tag{8}$$

$0 \leq r \leq m-2,\ 0 \leq k \leq N$ *and*

$$\begin{aligned} &|s^{(m-1)}(t) - y^{(m-1)}(t)| = O(h), \\ &|s^{(m)}(t) - y^{(m)}(t)| = O(h), \end{aligned} \tag{9}$$

$t_k < t < t_{k+1},\ 0 \leq k \leq N$.

Then it follows:

$$|s(t) - y(t)| = O(h^p),\ \forall\ t \in [a,b] \tag{10}$$

where

$$p = \min_{r=0,1,\ldots,m}\{r + p_r, p_{m-1} = p_m = 1\}$$

and

$$\begin{aligned} &|s^{(m-1)}(t) - y^{(m-1)}(t)| = O(h), \\ &|s^{(m)}(t) - y^{(m)}(t)| = O(h) \end{aligned} \tag{11}$$

$t \in [a,b]$.

The proof is just a slight modification of the corresponding Lemma in [15].

In the following sections we shall investigate the cubic ($m = 3$) and the quartic ($m = 4$) spline approximations of the solution of (1).

3 Cubic spline function approximating the solution

By Theorem 3 for $m = 3$ the cubic approximating spline function yields the same values in the knots as discrete two-step method based of the following recurrence formula

$$(12) \qquad y_{k+1} - 2y_k + y_{k-1} = \frac{h^2}{6}[y''_{k+1} + 4k''_k + y''_{k-1}] =$$

$$= \frac{h^2}{6}[f_{k+1} + 4f_k + f_{k-1}]$$

where

$$f_j := f(t_j, y(t_j), y(g(t_j)), y'(g(t_j)))$$

if the starting values $y_0 = s(t_0)$, $y_1 = s(t_0 + h)$,

$$y_0(g(t_0)) = \varphi(g(t_0)), \quad y_1(g(t_1)) = s(g(t_0 + h)).$$

The two step method (13) has the degree of exactness two provided the starting values have second degree of exactness.

As in [3] it is not difficult to prove that the starting value have the same degree of exactness as the recurrence formula (13).

Therefore we can conclude that

$$|s(t_k) - y(t_k)| = O(h^3),$$

and from the Lemma 1, directly that

$$|s''(t_k) - y''(t_k)| = O(h^3),$$

and similarly

$$|s(g(t_k)) - y(g(t_k))| = O(h^3)$$
$$|s''(g(t_k)) - y''(g(t_k))| = O(h^3).$$

In a similar manner as in [14] we can prove that

$$|s'(t_k) - y'(g(t_k))| = O(h^2)$$

$$|s'(g(t_k)) - y'(g(t_k))| = O(h^2).$$

Theorem 4. *If $f \in C^2([a,b] \times C^1[a,b] \times C^1[\alpha,b] \times C[\alpha,b])$ and s is the cubic spline function of defficiency 2 approximating the solution of (1), then there exists a constant K independent of h such that, for h sufficiently small and $t \in [a,b]$*

$$|y^{(i)}(t) - s^{(i)}(t)| \le Kh^{3-i}, \quad i = 0, 1$$

and

$$|y''(t) - s''(t)| < Kh, \quad |y'''(t) - s'''(t)| < Kh$$

hold, provided $s''(t_k)$ and $s'''(t_k)$ are given by (8) for $m = 3$.

Proof. Applying the Lemma 2 for $m = 3$, $p_0 = 3$, $p_1 = 2$, $p_2 = 2$, two times successively, first for s and then for s' in the role of s the first two inequalities follow. The last two inequalities follows from (12) and the theorem is proved.

4 Quartic spline functions approximating a solution

According to the Theorem 3 for $m = 4$, the quartic spline function s above by collocation method constructed furnishes values at the knots which coincide with the values of the discrete 3-steps method defined by

$$(13) \qquad y_{k+1} - y_k - y_{k-1} + y_{k-2} = \frac{h^2}{12}[y''_{k+1} + 11y''_k + 11y''_{k-1} + y''_{k-2}] =$$

$$= \frac{h^2}{12}[f_{k+1} + 11f_k + 11f_{k-1} + f_{k-2}]$$

where

$$f_j := f(t_j, y(t_j), t(g(t_j)), y'(g(t_j))$$

provided $y_0 = s(t_0)$, $y_1 = s(t_0 + h)$,

$$y_0(g(t_0)) = \varphi(g(t_1)), \quad y_1(g(t_1)) = s(g(t_0 + h)).$$

The 3-steps method (14) has degree of exactness 4 if the starting values have the same exactness. Also for this case, it is easy to conclude that the starting values have degree of exactness 4. From this fact and from the Lemma 2 for $p = 4$ it follows that:

$$|s(t_k) - y(t_k)| = O(h^4), \quad |s''(t_k) - y''(t_k)| = O(h^4)$$

$$|s(g(t_k)) - y(g(t_k))| = O(h^4), \quad s''(g(t_k)) - y''(g(t_k))| = O(h^4),$$

$h = 0, 1, \ldots, N$.

In a similar manner as in [15] it may be shown that the following estimations hold:

$$|s'(t_k) - y'(t_k)| = O(h^3), \quad |s'(g(t_k)) - y'(g(t_k))| = O(h^3), \; k = 0, 1, \ldots, N$$

and also that:

$$|s'''(t) - y'''(t)| = O(h), \quad |s^{IV}(t) - y^{IV}(t)| = O(h),$$

$t_k < t < t_{k+1}, \quad 0 \le k \le N$.

So, the conditions of Lemma 2 are satisfied for $m = 4$, $p_0 = 4$, $p_1 = 3$, $p_2 = 4$. Applying Lemma 2 for s, and successively for s' and s'' in the role of s, we have the following theorem.

Theorem 5. *If $f \in C^3([a, b] \times C^1[a, b] \times C^1[a, b] \times C[a, b])$ and s is the quartic spline function approximating the solution of (1), then for h small enough and $t \in [a, b]$ there exists a constant K independent of h, such that*

$$|y^{(j)}(t) - s^{(j)}(t)| < Kh^{4-j}, \quad j = 0, 1, 2$$

and

$$|y'''(t) - s'''(t)| < Kh, \quad |y^{IV}(t) - s^{IV}(t)| < Kh$$

hold, provided that $s'''(t_k)$ and $s^{IV}(t_k)$ are calculated by (8) with $m = 4$.

The collocation methods of approximating the solution of the second order neutral delay differential equations given here for $m = 3$ and $m = 4$ have many advantages over the other methods, because they give a global smooth approximation solution, are convergent, and also permit to know the behaviour of the approximate solutions.

5 Numerical example

Consider the following neutral delay differential equation problem:

$$y''(t) = -\frac{1}{2}y(t) + \frac{1}{2}y(t-\pi) - y'(t-\pi) + \cos t, \quad t \geq 0$$

$$y(t) = 1, \quad -\pi \leq t \leq 0.$$

The exact solution of this problem, on the interval $[0, \pi]$ is:

$$y(t) = 1 - 2\cos t + 2\cos\left(\frac{\sqrt{3}}{3}t\right), \quad t \in [0, \pi].$$

Let give in the following table the approximations of the solution by cubic spline function of continuity class C^1.

k	a_k	b_k	$s(t_k)$	$y(t_k)$	$e(t_k) = \|s(t_k) - y(t_k)\|$
0	3.20652	0.01397	1.00049	1.00000	3.95876E-04
1	2.79085	0.01218	1.00164	1.00000	1.53228E-03
2	2.42944	0.01161	1.00442	1.00001	3.31299E-03
3	2.11567	0.00927	1.00676	1.00001	5.65587E-03
4	1.84169	0.00897	1.00958	1.00002	8.49593E-03

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