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COEFFICIENT BOUNDS FOR CERTAIN BAZILEVIČ MAPS

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Abstract. Following Babalola [3], we obtain the best possible upper bound for the coefficients of functions in the class $B_n^\lambda(\gamma)$, using a technique due to Nehari and Netanyahu [9] and an application of certain integral iteration of Carathéodory-type functions. The sharp bound on the Fekete-Szego functional in $B_n^\lambda(\gamma)$ is also obtained.

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1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$.

In a recent paper, Babalola [4] provided a new approach to the study of the well-known Bazilevič functions, given as

$$f(z) = \left\{ \frac{\alpha}{1+\beta^2} \int_0^z [p(t) - \mathrm{i}\beta] t^{-\left(1+\frac{\mathrm{i}\alpha\beta}{1+\beta^2}\right)} g(t)^{\left(\frac{\alpha}{1+\beta^2}\right)} \mathrm{d}t \right\}^{\frac{1+\mathrm{i}\beta}{\alpha}},$$

where the parameter β is no longer assumed to be zero, as in many previous works (see e.g. [1, 3, 8, 10, 13, 14]). The new method involved a modification of the class of Carathéodory functions. The modified class is denoted here by P_ξ and consists of analytic functions

$$h(z) = \xi + p_1 z + \dots$$

on E, with positive real part, where $\mathrm{Re}\,\xi = 1$.

The class P_ξ is of Carathéodory-type. We see that $h \in P_\xi$ if and only if $h(0) = \xi$ and $\mathrm{Re}\, h(z) > 0$. The well-known class P of Carathéodory maps coincides with P_ξ for $\xi = 1$ and it is easy to see that $p \in P$ if and only if $h(z) = p(z)+\xi-1 \in P_\xi$. The function given by $H_0(z) = (\xi+(2-\xi)z)/(1-z) = \xi + 2z + 2z^2 + \dots$ plays a central role in the study of the class P_ξ, especially regarding extremal problems.

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Using the new definition, Babalola inspired new investigations of the class of Bazilevič functions [4, 5]. In particular, using the Sălăgean derivative operator, D^n, $n = 0, 1, 2, \ldots$, defined by $D^n f(z) = D(D^{n-1} f(z)) = z(D^{n-1} f(z))'$, with $D^0 f(z) = f(z)$ (see [12]), he gave the following definition.

Definition 1.1 ([5]). Let $\eta > 0$, $\lambda = \eta + \mathrm{i}\mu$ and $\xi = \lambda/\eta$ be some constants. A function $f \in A$ belongs to the class $B_n(\lambda)$ if and only if

$$\frac{D^n f(z)^\lambda}{\eta \lambda^{n-1} z^\lambda} \in P_\xi, \quad z \in E.$$

We note that we obtain the class of Bazilevič functions in the case $\lambda = \alpha/(1 + \mathrm{i}\beta)$, $\alpha > 0$.

Now, denote by $P_\xi(\gamma)$ the subclass of functions $h \in P_\xi$ with $\mathrm{Re}\, h(z) > \gamma$, where $0 \le \gamma < 1$ and $z \in E$.

Definition 1.2. With all parameters defined above, a function $f \in A$ belongs to the class $B_n^\lambda(\gamma)$ if and only if

$$\frac{D^n f(z)^\lambda}{\eta \lambda^{n-1} z^\lambda} \in P_\xi(\gamma), \quad z \in E.$$

If $\xi = 1$ (that is $\lambda = \eta$) in Definition 1.2, we get the class $T_n^\eta(\gamma)$ introduced in [10] (see also [2]).

Following Babalola [2], we define an integral iteration of $h \in P_\xi(\gamma)$ as follows.

Definition 1.3. Let $h \in P_\xi(\gamma)$. The nth complex-parameter integral iteration of $h(z)$, $z \in E$, is defined by

$$h_n(z) = \frac{\lambda}{z^\lambda} \int_0^z t^{\lambda-1} h_{n-1}(t)\, \mathrm{d}t, \quad n = 1, 2, \ldots,$$

with $h_0(z) = h(z) = \xi + (1-\gamma)p_1 z + \ldots$

In series form, the above iteration gives $h_n(z) = \xi + (1-\gamma)\sum_{k=1}^\infty p_{n,k} z^k$, where $p_{n,k} = (1-\gamma)\lambda^n p_k/(\lambda + k)^n$ is such that

$$|p_{n,k}| \le \frac{2(1-\gamma)|\lambda|^n}{|(\lambda+k)|^n}, \quad k = 1, 2, \ldots.$$

The function $H_n(z)$, defined by

$$H_n(\gamma, z) = \frac{\lambda}{z^\lambda} \int_0^z t^{\lambda-1} H_{n-1}(\gamma, t) \mathrm{d}t, \quad n = 1, 2, \ldots,$$

where $H_0(\gamma, z) = \gamma + (1-\gamma)(1+z)/(1-z) + \xi - 1 = [\xi + (2(1-\gamma) - \xi)z]/(1-z)$, also plays a central role for extremal problems with respect to the iteration $h_n(z)$.

In the present paper, we follow the work of Babalola [3], using a technique due to Nehari and Netanyahu [9] and an application of the integral iteration $h_n(z)$, to obtain the best possible bounds for the coefficients of the functions in the class $B_n^\lambda(\gamma)$ and their Fekete-Szego functional. The two coefficient problems dealt with in this paper are well-known in the theory of geometric functions (see [7, 11]). In the next section, we state (and prove, where necessary) the relevant lemmas which we then apply, in Section 3, to prove our results.

2. PRELIMINARY LEMMAS

In [4], Babalola noted that most of the inequalities for P remain unperturbed by the new normalization. The proofs of the first two lemmas are similar to those of given by Nehari and Netanyahu for [9, Lemmas I and II].

Lemma 2.1. *If* $p(z) = \xi + b_1 z + b_2 z^2 + \dots$ *and* $q(z) = \xi + c_1 z + c_2 z^2 + \dots$ *belongs to* P_ξ, *then* $r(z) = \xi + \frac{1}{2}\sum_{k=1}^{\infty} b_k c_k z^k$ *also belongs to* P_ξ.

Lemma 2.2. *Let* $h(z) = \xi + \sum_{k=1}^{\infty} d_k z^k$ *and* $\xi + G(z) = \xi + \sum_{k=1}^{\infty} b'_k z^k$ *be functions in* P_ξ. *Set*

$$\beta_m = \frac{1}{2^m}\left[\xi + \frac{1}{2}\sum_{\epsilon=1}^{m} \binom{m}{\epsilon} d_\epsilon\right], \quad \beta_0 = \xi.$$

If B_ν *is defined by*

$$\sum_{m=1}^{\infty} (-1)^{m+1}\beta_{m-1} G^m(z) = \sum_{\nu=1}^{\infty} B_\nu z^\nu,$$

then $|B_\nu| \le 2$, $\nu = 1, 2, \dots$

Corollary 2.3. *Let* $h_n(z)$ *be the nth integral iteration of* $h_0(z) = \xi + \sum_{k=0}^{\infty} p_k z^k$ *with* $\operatorname{Re} h_n(z) > \gamma$ *and let* $\xi + G(z) = \xi + \sum_{k=0}^{\infty} b'_k z^k$ *be a function in* P_ξ. *Define* β_m *as in the previous lemma and* ϕ_m *as*

$$\phi_m = \frac{(1-\gamma)\lambda^n}{(\lambda+m)^n}\beta_m, \quad \phi_0 = (1-\gamma)\xi. \tag{2}$$

If A_ν *is defined by*

$$\sum_{m=1}^{\infty} (-1)^{m+1}\phi_{m-1} G^m(z) = \sum_{\nu=1}^{\infty} A_\nu z^\nu, \tag{3}$$

then

$$|A_\nu| \le \frac{2(1-\gamma)|\lambda|^n}{|\lambda+\nu|^n}, \quad \nu = 1, 2, \dots \tag{4}$$

Proof. The proof follows as in [3], in view of (2). □

Lemma 2.4 ([3]). *Let* $J(z) = \sum_{k=0}^{\infty} c_k z^k$ *be a power series. Then the* m^{th} *integer product of* $J(z)$ *is*

$$J^m(z) = \left(\sum_{k=0}^{\infty} c_k z^k\right)^m = \sum_{k=0}^{\infty} c_k^{(m)} z^k,$$

where $c_k^{(1)} = c_k$ *and*

$$c_k^{(m)} = \sum_{j=0}^{k} c_j c_{k-j}^{(m-1)}, \quad m \geq 2.$$

LEMMA 2.5 ([3, p. 145]). *Let* $m = 1, 2, \ldots,$ $n = 0, 1, 2, \ldots$ *and* ρ_l, $l = 1, 2, \ldots,$ *take values in the set* $M = \{0, 1, 2, \ldots, m\}$ *such that* $\rho_1 + \rho_2 + \ldots + \rho_m = m$. *If* $\alpha > 0$ *is a real number, then we have the inequality*

$$\prod_{l=1}^{m} \frac{\alpha^{\rho_l}}{(\alpha + l)^{\rho_l}} \leq \frac{\alpha}{\alpha + m - 1}.$$

LEMMA 2.6 ([5]). *Let* $h = \xi + p_1 z + p_2 z^2 + \ldots \in P_\xi$. *Then, for any real number* τ, *we have the sharp inequality*

$$\left| p_2 - \tau \frac{p_1^2}{2} \right| \leq 2 \max \{1, |1 - \tau|\}.$$

Before we state and prove our main result, we compute the leading coefficients A_ν, in the expression (3), as follows: From (3) we have

$$\sum_{m=1}^{\infty} (-1)^{m+1} \phi_{m-1} G^m(z) = \phi_0 G(z) - \phi_1 G^2(z) + \cdots = \sum_{\nu=1}^{\infty} A_\nu z^\nu, \tag{5}$$

with $G(z) = \sum_{\nu=1}^{\infty} b'_\nu z^\nu$, and, applying Lemma 2.4, we have

$$G^m(z) = \left(\sum_{\nu=1}^{\infty} b'_\nu z^\nu \right)^m = \sum_{\nu=m}^{\infty} G_\nu^{(m)} z^\nu, \quad m = 1, 2, \ldots. \tag{6}$$

$G_\nu^{(m)}$ has the general form

$$G_\nu^{(m)} = \sum_{\rho \in J_{\nu,m}} G_{\nu,\rho} \prod_{l=1}^{m} (b'_l)^{\rho_l}, \text{ where } G_{\nu,\rho} = \frac{m!}{\rho_1! \rho_2! \ldots \rho_l!}, \tag{7}$$

for some multi-index $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$ and the set $J_{\nu,m} = \{\rho | \sum_{l=1}^{m} \rho_l = m,\ \sum_{l=1}^{m} l\rho_l = \nu\}$. Using (6) and (7) in (5), we obtain

$$\sum_{m=1}^{\infty} (-1)^{m+1} \phi_{m-1} G^m(z) = \sum_{\nu=1}^{\infty} \left(\sum_{m=1}^{\nu} (-1)^{m+1} \phi_{m-1} G_\nu^{(m)} \right) z^\nu,$$

which implies that

$$\sum_{\nu=1}^{\infty} \left[\sum_{m=1}^{\nu} (-1)^{m+1} \phi_{m-1} G_\nu^{(m)} \right] z^\nu = \sum_{\nu=1}^{\infty} A_\nu z^\nu,$$

with

$$A_\nu = \sum_{m=1}^{\nu} (-1)^{m+1} \phi_{m-1} G_\nu^{(m)}.$$

By Corollary 2.3, the coefficients A_ν satisfy inequality (4), if $\xi + G(z) = \xi + b'_1 z + b'_2 z^2 + \ldots$ is a function in the class P_ξ, and, by Lemma 2.1, we may set $b'_l = \frac{1}{2} b_l c_l$, where $\xi + b_1 z + b_2 z^2 + \ldots$ is in P_ξ and $H(z) = \xi + c_1 z + c_2 z^2 + \ldots$ is an arbitrary function in P_ξ. Then, taking into account also (7), we have

$$|A_\nu| = \left| \sum_{m=1}^{\nu} (-1)^{m+1} \frac{\phi_{m-1}}{2^m} \left(\sum_{j=1}^{\nu} G_j \prod_{l=1}^{m} b_l^{\rho_l} c_l^{\rho_l} \right) \right| \leq \frac{2(1-\gamma)|\lambda|^n}{|\lambda+\nu|^n}. \tag{8}$$

Using (2) in (8), yields

$$|A_\nu| = \left| \sum_{m=1}^{\nu} \frac{(-1)^{m+1}(1-\gamma)\lambda^n}{2^m(\lambda+m-1)^n\lambda} \beta_{m-1} \left(\sum_{j=1}^{\nu} G_j \prod_{l=1}^{m} b_l^{\rho_l} c_l^{\rho_l} \right) \right| \leq \frac{2(1-\gamma)|\lambda|^{n-1}}{|\lambda+\nu|^n}.$$

Using Lemma 2.5, we get, for $\nu = 1, 2, \ldots$,

$$\begin{aligned} &\sum_{m=1}^{\nu} (-1)^{m+1} \frac{\beta_{m-1}}{2^m|\lambda|} \left(\sum_{j=1}^{\nu} G_j \prod_{l=1}^{m} \frac{(1-\gamma)^{\rho_l}\eta^{\rho_l+1}|\lambda|^{n\rho_l-\rho_l-1}}{|\lambda+l|^{n\rho_l}} b_l^{\rho_l} c_l^{\rho_l} \right) \\ &\leq \sum_{m=1}^{\nu} \frac{(-1)^{m+1}(1-\gamma)|\lambda|^n}{|\lambda+m-1|^n\,|\lambda|} \frac{\beta_{m-1}}{2^m} \left(\sum_{j=1}^{\nu} G_j \prod_{l=1}^{m} b_l^{\rho_l} c_l^{\rho_l} \right). \end{aligned} \tag{9}$$

Using (8) in (9), we get

$$\begin{aligned} &\left| \sum_{m=1}^{\nu} \frac{(-1)^{m+1}\beta_{m-1}}{2^m\lambda} \left(\sum_{j=1}^{\nu} G_j \prod_{l=1}^{m} \frac{(1-\gamma)^{\rho_l}\eta^{\rho_l+1}\lambda^{n\rho_l-\rho_l-1}}{(\lambda+l)^{n\rho_l}} b_l^{\rho_l} c_l^{\rho_l} \right) \right| \\ &\leq \frac{2(1-\gamma)|\lambda|^{n-1}}{|\xi||\lambda+\nu|^n}, \end{aligned} \tag{10}$$

which implies that

$$\left| \sum_{m=1}^{\nu} \frac{(-1)^{m+1}(1-\gamma)^m\eta^{m+1}\lambda^{mn-m-2}\beta_{m-1}}{2^m} \Phi_\nu \right| \leq \frac{2(1-\gamma)|\lambda|^{n-1}}{|\xi||\lambda+\nu|^n}, \tag{11}$$

where

$$\Phi_\nu = \sum_{j=1}^{\nu} G_j \prod_{l=1}^{m} \frac{b_l^{\rho_l} c_l^{\rho_l}}{(\lambda+l)^{n\rho_l}}.$$

3. MAIN RESULT

THEOREM 3.1. *Let $\eta > 0$, μ be a real number, $\lambda = \eta + i\mu$, $\xi = \lambda/\eta$ and $0 \leq \gamma < 1$. If $f \in B_n^\lambda(\gamma)$, then*

$$|a_k| \leq \frac{2(1-\gamma)|\lambda|^{n-1}}{|\xi||\lambda+k-1|^n},\ k = 2, 3, \ldots$$

The inequalities are sharp. The equalities are obtained for $f(z)$ satisfying

$$\frac{D^n f(z)^\lambda}{\eta\lambda^{n-1}z^\lambda} = \frac{\xi + [2(1-\gamma)-\xi]z^{k-1}}{1-z^{k-1}},\ k = 2, 3, \dots$$

Proof. Let $f \in B_n^\lambda(\gamma)$. Then there exists an analytic function $h \in P_\xi(\gamma)$ such that

$$\frac{D^n f(z)^\lambda}{\eta\lambda^{n-1}z^\lambda} = h(z) = \gamma + (1-\gamma)p(z) + \xi - 1,$$

for some $p(z) = 1 + p_1 z + p_2 z^2 + \ldots \in P$. Hence

$$\frac{f(z)}{z} = \left(1 + (1-\gamma)\eta\lambda^{n-1}\sum_{k=1}^{\infty}\frac{p_k z^k}{(\lambda+k)^n}\right)^{\frac{1}{\lambda}}.$$

Expanding binomially and employing Lemma 2.4, we have

$$f(z) = z + \sum_{k=2}^{\infty}\tilde{B}_1 C_{k-1}^{(1)} z^2 + \sum_{k=2}^{\infty}\tilde{B}_2 C_{k-1}^{(2)} z^3 + \ldots + \sum_{k=2}^{\infty}\tilde{B}_m C_{k-1}^{(m)} z^k + \ldots, \tag{12}$$

where

$$\tilde{B}_m = \frac{(1-\gamma)^m \eta^m \lambda^{m(n-2)} \prod_{j=0}^{m-1}(1-j\lambda)}{m!}$$

and $C_k^{(m)}$, $m = 1, 2, \ldots;\ k = m, m+1, \ldots$ is defined by

$$\sum_{k=1}^{\infty} C_k^{(m)} z^k = \left(\sum_{k=1}^{\infty} q_k z^k\right)^m, \tag{13}$$

where $q_k = \frac{p_k}{(\lambda+k)^n}$ and $C_k^{(m)}$ has a similar description as $G_\nu^{(m)}$ in (7)

$$C_k^{(m)} = \sum_{j=1}^{k} C_{k,\rho}\prod_{l=1}^{m} q_l^{\rho_l}.$$

Comparing the coefficients in (1) and (12), we have

$$a_k = \sum_{m=1}^{k-1}\frac{(1-\gamma)^m \eta^m \lambda^{m(n-2)} \prod_{j=0}^{m-1}(1-j\lambda)}{m!}\left(\sum_{j=1}^{k-1} C_j \prod_{l=1}^{m} q_l^{\rho_l}\right). \tag{14}$$

Now, comparing (14) and the term in absolute value in (11) with $\nu = k-1$ and noting that C_j in (14) and G_j in (11) have similar descriptions as C_j mentioned earlier, we conclude that the inequalities in (4) hold if we are able to find two members $h(z) = \xi + d_1 z + d_2 z^2 + \ldots$ and $H(z) = \xi + c_1 z + c_2 z^2 + \ldots$ of P_ξ which give rise to the constants β_m (as required by Lemma 2.2) and c_l. For $H \in P_\xi$, a natural choice is the function $H(z) = \frac{\xi+(2-\xi)z}{1-z} = \xi + 2z + 2z^2 + \ldots$, which turns out to be suitable. Thus, we have $c_l = 2,\ l = 1, 2, \dots$ Then (11) yields

$$\left|\sum_{m=1}^{k-1}(-1)^{m+1}(1-\gamma)^m\eta^{m+1}\lambda^{mn-m-2}\beta_{m-1}\Phi_{k-1}\right| \le \frac{2(1-\gamma)|\lambda|^{n-1}}{|\xi||(\lambda+k-1)|^n}. \tag{15}$$

Also, comparing (14) and the terms in the absolute value in (15), we have

$$(-1)^{m+1}\frac{\beta_{m-1}}{\lambda} = \frac{\prod_{j=0}^{m-1}(1-j\lambda)}{m!\eta\lambda^{m-1}},$$

that is

$$\beta_{m-1} = \frac{\prod_{j=1}^{m-1}(j\lambda-1)}{m!\eta\lambda^{m-2}}, \quad \beta_0 = \xi. \tag{16}$$

Now, we define

$$\frac{1}{2^{m-1}}\left[\xi + \frac{1}{2}\sum_{\epsilon=1}^{m-1}\binom{m-1}{\epsilon}d_\epsilon\right] = \frac{\prod_{j=1}^{m-1}(j\lambda-1)}{m!\,\eta\,\lambda^{m-2}}, \tag{17}$$

for some d_ϵ, $\epsilon = 1, 2, \dots, m-1$, and we need to find $h(z)_k$ corresponding to each a_k, $k = 2, 3, 4, \dots$, such that the coefficients d_ϵ of each $h(z)_k$ satisfy (17). In view of (15), we consider the following cases for $m = 1, 2, \dots, k-1$, $k = 2, 3, 4, \dots$

(i) For $k = 2$, $m = 1$, using (16), we have $\beta_0 = \xi$ and, by (17), we have $d_\epsilon = 0$, for all ϵ. Hence we obtain $h(z)_2 = \xi$.

(ii) For $k = 3$, $m = 1, 2$, using (17), we have $d_1 = -2/\eta$. Hence, we obtain

$$h(z)_3 = \xi - \frac{-1}{\eta} + \frac{1}{\eta}\left(\frac{1-z}{1+z}\right) = \xi - \frac{-2}{\eta}z + \dots$$

(iii) For $k = 4$, $m = 1, 2, 3$, using (17), we have

$$\frac{1}{4}\left[\xi + \frac{1}{2}(2d_1 + d_2)\right] = \frac{(\lambda-1)(2\lambda-1)}{6\,\eta\,\lambda}$$

and, taking $d_1 = 0$, we obtain $\frac{d_2}{2} = \frac{\lambda^2-6\lambda+2}{3\eta\lambda}$, where

$$|d_2| = \frac{2}{3}\left|\frac{\lambda^2-6\lambda+2}{\eta\,\lambda}\right| \le 2.$$

We define

$$h(z)_4 = \frac{2(\lambda-1)(2\lambda-1)}{3\,\eta\,\lambda} - \left(\frac{\lambda^2-6\lambda+2}{3\,\eta\,\lambda}\right)\left(\frac{1-z^2}{1+z^2}\right),$$

where

$$|\lambda^2+2| \le (3\eta+6)|\lambda|.$$

Then

$$h(z)_4 = \xi + \frac{2(\lambda^2-6\lambda+2)}{3\eta\lambda}z^2 + \dots.$$

(iv) For $k = 5$, we have $m = 1, 2, 3, 4$. So, using (17), we get

$$\frac{1}{8}\left[\xi + \frac{1}{2}(3d_1 + 3d_2 + d_3)\right] = \frac{(\lambda-1)(2\lambda-1)(3\lambda-1)}{24\,\eta\,\lambda^2},$$

where, taking $d_1 = d_2 = 0$, we obtain

$$\frac{d_3}{2} = \frac{6\lambda^3 - 11\lambda^2 + 6\lambda - 1}{3\,\eta\,\lambda^2},$$

with

$$|d_3| = \frac{2}{3}\left|\frac{3\lambda^3 - 11\lambda^2 + -6\lambda - 1}{\eta\,\lambda^2}\right| \leq 2,$$

and we define

$$h(z)_5 = \frac{6\lambda^3 - 11\lambda^2 + 6\lambda - 1}{3\,\eta\,\lambda^2} - \left(\frac{3\lambda^3 - 11\lambda^2 + 6\lambda - 1}{3\,\eta\,\lambda^2}\right)\left(\frac{1-z^3}{1+z^3}\right),$$

where

$$|3\lambda^3 - 11\lambda^2 - 1| \leq (3\eta|\lambda| - 6)|\lambda|.$$

Hence

$$h(z)_5 = \xi + \frac{2(3\lambda^3 - 11\lambda^2 + 6\lambda - 1)}{3\,\eta\,\lambda^2}z^3 + \ldots$$

(v) For $k \geq 6$, we have $m = 1, 2, 3, \ldots, k-1$, and we set $d_1 = \frac{-2\xi}{(m-1)}$, $d_2 = d_4 = \cdots = d_\tau = \sigma$, where τ equals $m-1$, if $m-1$ is even, and $m-2$, otherwise. Also, $d_3 = d_5 = \cdots = d_\omega = 0$, where ω equals $m-1$, if $m-1$ is odd, and $m-2$, otherwise. Thus, we have

$$\frac{\sigma_m}{2} = \frac{2^{m-1}\xi\prod_{j=1}^{m-1}\left(\frac{j\lambda-1}{j\lambda}\right)}{m\left(\begin{pmatrix} m-1 \\ 2 \end{pmatrix} + \begin{pmatrix} m-1 \\ 4 \end{pmatrix} + \cdots + \begin{pmatrix} m-1 \\ \tau \end{pmatrix}\right)},$$

for all $\epsilon = 1, 2, \ldots, m-1$ such that $|d_\epsilon| \leq 2$. Thus, setting $m = k-1$, we find that $h(z)_k$, $k \geq 6$, is given by

$$h(z)_k = \xi - \frac{2\xi}{k-2}z + \frac{\sigma_{k-1}}{2}z^2 + \frac{\sigma_{k-1}}{2}z^4 + \ldots$$

That $h(z)_k$ belongs to P_ξ follows from the fact that P_ξ is, like P, a convex family. The proof is complete.

□

Theorem 3.2. *Let $\eta > 0$, μ be a real number, $\lambda = \eta + i\mu$, $\xi = \lambda/\eta$ and $0 \leq \gamma < 1$. If $f \in B_n^\lambda(\gamma)$, then*

$$\left|a_3 - \rho a_2^2\right| \leq \frac{2(1-\gamma)|\lambda|^{n-1}}{|\xi||\lambda+2|^n}\max\{1, |1-M|\},$$

where

$$M = \frac{(2\rho + \lambda - 1)(1-\gamma)(\lambda+2)^2\eta\lambda^{n-2}}{(\lambda+1)^{2n}}.$$

The inequalities are sharp. For each ρ, equalities are obtained by the same extremal function defined in Theorem 3.1.

Proof. Careful computations for (6) yield

$$a_2 = \frac{(1-\gamma)\eta\lambda^{n-2}p_1}{(\lambda+1)^n}$$

$$a_3 = \frac{(1-\gamma)\eta\lambda^{n-2}p_2}{(\lambda+2)^n} + \frac{(1-\lambda)(1-\gamma)^2\eta^2\lambda^{2n-4}p_1^2}{2(\lambda+1)^{2n}}.$$

Hence,

$$\left|a_3 - \rho a_2^2\right| = \frac{(1-\gamma)\eta\lambda^{n-2}}{(\lambda+2)^n}\left|p_2 - \frac{(2\rho+\lambda-1)(1-\gamma)(\lambda+2)^n\eta\lambda^{n-2}}{(\lambda+1)^{2n}}\frac{p_1^2}{2}\right|.$$

By choosing

$$\tau = \frac{(2\rho+\lambda-1)(1-\gamma)(\lambda+2)^n\eta\lambda^{n-2}}{(\lambda+1)^{2n}}$$

and using Lemma 2.6, the result then follows. □

For $\lambda = \alpha/(1+\mathrm{i}\beta)$, we have the following corollaries for generalized Bazilevič maps (with $g(z) = z$), whose family is denoted here by $B_n^{\alpha,\beta}(\gamma)$.

Corollary 3.3. *Let $f \in B_n^{\alpha,\beta}(\gamma)$. Then*

$$|a_k| \le \frac{2(1-\gamma)\alpha^n}{\sqrt{(1+\beta^2)(\alpha^2+\beta^2)[\alpha^2+2\alpha(k-1)+(1+\beta^2)(k-1)^2]^n}},\ k = 2, 3, \ldots$$

The inequalities are sharp.

Corollary 3.4. *Let $f \in B_n^{\alpha,\beta}(\gamma)$. Then*

$$\left|a_3 - \rho a_2^2\right| \le \frac{2(1-\gamma)\alpha^n\sqrt{(1+\beta^2)^n}}{\sqrt{(\alpha^2+\beta^2)[(\alpha+2)^2+4\beta^2]^n}}\max\{1, |1-T|\},$$

where

$$T = \frac{(1-\gamma)(\alpha+2+\mathrm{i}\beta)^2\alpha^{n-1}[2\rho+\alpha-1+\mathrm{i}\beta(2\rho-1)]}{(1+\beta^2)(1+\mathrm{i}\beta)^{n+1}}.$$

The inequalities are sharp.

Finally, we remark that, with appropriate choices of the defining parameters, our results agree with the existing results.

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