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MULTIPLE POSITIVE SOLUTIONS FOR A CLASS OF A NONLINEAR BOUNDARY VALUE PROBLEM ON $\mathbb{R}^N_+$

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Abstract. In this article, we study the multiplicity of positive solutions for the following elliptic problem:

$$\begin{cases} -\Delta w + B(y)w = A(y)w^s, & y \in \mathbb{R}^N_+, \\ \dfrac{\partial w}{\partial \nu} = \mu H(y)|w|^{r-2}w + G(y)|w|^{p-2}w, & y \in \partial\mathbb{R}^N_+, \end{cases}$$

where $\mu > 0$, $1 < r < 2 < s < p < 2^*$,

$$2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } N \geq 3, \\ \infty & \text{if } N = 2, \end{cases}$$

and the upper half plane of $\mathbb{R}^N$ is denoted by $\mathbb{R}^N_+ = \{(y^{N-1}, y_N) \in \mathbb{R}^{N-1} \times \mathbb{R} |\ y_N > 0\}$ and the maps A, B, G, and H satisfy some appropriate conditions. We show that, assuming $\mu > 0$ is sufficiently small, the equation has at least two positive solutions using the decomposition of the Nehari manifold.

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Key words. Nehari manifold, Palais-Smale sequence, fibering method, Ekeland's variational principle, nonlinear boundary condition.

1. INTRODUCTION

In this article, we are interested in the multiplicity results of non-trivial positive solutions for the following elliptic problem:

$$\begin{cases} -\Delta w + B(y)w = A(y)w^s, & y \in \mathbb{R}^N_+, \\ \dfrac{\partial w}{\partial \nu} = \mu H(y)|w|^{r-2}w + G(y)|w|^{p-2}w, & y \in \partial\mathbb{R}^N_+, \end{cases} \tag{1}$$

where $\mu > 0$, $1 < r < 2 < s < p < 2^*$,

$$2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } N \geq 3, \\ \infty & \text{if } N = 2, \end{cases}$$

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and the upper half plane of $\mathbb{R}^N$ is denoted by $\mathbb{R}^N_+ = \{(y^{N-1}, y_N) \in \mathbb{R}^{N-1} \times \mathbb{R}|\ y_N > 0\}$.

We suppose that the maps A, B, G, and H satisfy the following assumptions:

(a1) $A(y) \in C(\mathbb{R}^N_+) \cap L^r(\mathbb{R}^N_+)$;
(b1) $B(y) \in L^\infty(\mathbb{R}^N_+)$ and $B(y) \geq B_0 > 0$ in $\mathbb{R}^N_+$;
(g1) $G \in C(\partial\mathbb{R}^N_+)$ and here is a positive number $R_h < r$ such that $G(y) \geq 1+C_0 \exp(-R_h|y|)$ for some $C_0 < 1$ and for all $y \in \partial\mathbb{R}^N_+$ and $G(y) \to 1$ as $|y| \to \infty$;
(h1) $H \in L^{\frac{p}{p-r}}(\partial\mathbb{R}^N_+) \setminus \{0\}$ with $H_\pm(y) = \pm\max\{\pm H(y), 0\} \neq 0$.

We admit that the variational functional I_μ in $H^1(\mathbb{R}^N_+)$ of problem (1) is

$$I_\mu(w) = \frac{1}{2}\|w\|^2_{H^1} - \frac{1}{s+1}\int_{\mathbb{R}^N_+} A|w|^{s+1}\mathrm{d}y - \frac{\mu}{r}\int_{\partial\mathbb{R}^N_+} H|w|^r\mathrm{d}\sigma - \frac{1}{p}\int_{\partial\mathbb{R}^N_+} G|w|^p\mathrm{d}\sigma, \tag{2}$$

where $\mathrm{d}\sigma$ is used to represent the measure on the boundary and $\|w\|_{H^1} = (\int_{\mathbb{R}^N_+}(|\nabla w|^2 + B(y)|w|^2)\mathrm{d}y)^{\frac{1}{2}}$ is the norm in $H^1(\mathbb{R}^N_+)$. We know that $I_\mu \in C^1(H^1(\mathbb{R}^N_+))$ and the solutions of problem (1) are the critical points of the variational functional I_μ.

When nonlinear components are added to an equation, or when a problem with Dirichlet boundary conditions of the form -$\Delta w = g(y, w)$ is taken into consideration, there has been a lot of interest in the study of existence and multiplicity; see for example [1, 2, 5, 9, 11, 14, 20, 22], etc.

On the other hand, nonlinear boundary conditions have just recently been taken into account. For the semi-linear (or linear) elliptic equations with a nonlinear boundary condition, see for example [7, 8, 12, 13, 15, 16, 21], etc. Recently, in [23], for the nonlinear boundary value problem Wu studied the existence of multiple solutions by using the Nehari manifold and the fibering maps method.

Our work is motivated by Wu [23]. In this paper, we discuss the Nehari manifold and studied carefully the relation between the Nehari manifold and the fibering map, we will show that the existence of the two positive solutions for the problem (1).

This paper is organized as follows. In Section 2, we give some notations and preliminaries. In Section 3, we prove the existence of a local minimum for I_μ. In Section 4, we prove Theorem 1.1.

THEOREM 1.1. *Eq.* (1) *has at least two solutions if there exists* $\mu^0 > 0$ *such that* $\mu \in (0, \mu^0)$.

2. PRELIMINARIES

In this paper, the Sobolev trace constant is denoted by C_p for the embedding of $H^1(\mathbb{R}^N_+)$ into $L^p(\partial\mathbb{R}^N_+)$:

$$C_p = \inf_{w\in H^1(\mathbb{R}^N_+)\setminus\{0\}} \frac{(\int_{\mathbb{R}^N_+}(|\nabla w|^2 + B(y)|w|^2)\mathrm{d}y)^{\frac{1}{2}}}{(\int_{\partial\mathbb{R}^N_+} G|w|^p\mathrm{d}\sigma)^{\frac{2}{p}}} > 0. \tag{3}$$

In particular, $(\int_{\partial\mathbb{R}^N_+}|w|^p\mathrm{d}\sigma)^{\frac{1}{p}} \leq C_p^{-\frac{1}{2}}\|w\|_{H^1}$ for all $w \in H^1(\mathbb{R}^N_+)$.

We define the Palais-Smale (denoted by (PS)) sequences, (PS)-values, and (PS)-conditions in $H^1(\mathbb{R}^N_+)$ for I_μ as given.

Definition 2.1. (i) For $\gamma \in \mathbb{R}$, a sequence $\{w_n\}$ is a $(PS)_\gamma$- sequence in $H^1(\mathbb{R}^N_+)$ for I_μ if $I_\mu(w_n) = \gamma + O(1)$ and $I'_\mu(w_n) = O(1)$ strongly in $H^*(\mathbb{R}^N_+)$ as $n \to \infty$;

(ii) I_μ satisfies the $(PS)_\gamma$-condition in $H^1(\mathbb{R}^N_+)$ if all $(PS)_\gamma$-sequences in $H^1(\mathbb{R}^N_+)$ for I_μ contains a convergent subsequence.

The variational functional I_μ, is not bounded below on the whole space $H^1(\mathbb{R}^N_+)$, but it is bounded below and has a minimizer on an appropriate subset of $H^1(\mathbb{R}^N_+)$, then this minimizer is a critical point of I_μ. So it is a solution of the given problem (1). It is of help us to consider the variational functional I_μ on the Nehari manifold $N_\mu = \{w \in H^1(\mathbb{R}^N_+) \setminus \{0\} \mid \langle I'_\mu(w), w\rangle = 0\}$. Clearly, $w \in N_\mu$ iff

$$\|w\|^2_{H^1} - \int_{\mathbb{R}^N_+} A|w|^{s+1}\mathrm{d}y - \mu\int_{\partial\mathbb{R}^N_+} H|w|^r\mathrm{d}\sigma - \int_{\partial\mathbb{R}^N_+} G|w|^p\mathrm{d}\sigma = 0. \tag{4}$$

Furthermore, we prove the following lemmas.

Lemma 2.2. *The variational functional I_μ is bounded below and coercive on N_μ.*

Proof. If $w \in N_\mu$, then by the Sobolev trace, Hölder inequalities, (2) and (4), we have

$$\begin{aligned} I_\mu(w) &= \left(\frac{1}{2}-\frac{1}{p}\right)\|w\|^2_{H^1} - \left(\frac{1}{s+1}-\frac{1}{p}\right)\int_{\mathbb{R}^N_+} A|w|^{s+1}\mathrm{d}y \\ &\quad - \left(\frac{\mu}{r}-\frac{\mu}{p}\right)\int_{\partial\mathbb{R}^N_+} H|w|^r\mathrm{d}\sigma \\ &\geq \left(\frac{1}{2}-\frac{1}{p}\right)\|w\|^2_{H^1} - \left(\frac{1}{s+1}-\frac{1}{p}\right)\int_{\mathbb{R}^N_+} A|w|^{s+1}\mathrm{d}y \\ &\quad - \mu\left(\frac{p-r}{pr}\right)\|H\|_{L^{\frac{p}{p-r}}} C_p^{\frac{-r}{2}}\|w\|^r_{H^1}. \end{aligned} \tag{5}$$

Thus, I_μ is coercive and bounded below on N_μ. $\square$

The behaviour of the function $\Phi_w : t \to I_\mu(tw)$ for $t > 0$ is strongly related to the Nehari manifold N_μ. These maps, referred to as fibering maps, were first introduced by Drabek and Pohozaev in [17–19] and are examined by Brown and Zhang [5]. If $w \in H^1(\mathbb{R}^N_+)$, then we have

$$\Phi_w(t)$$
$$= \frac{t^2}{2}\|w\|^2_{H^1} - \frac{t^{s+1}}{s+1}\int_{\mathbb{R}^N_+} A|w|^{s+1}\mathrm{d}y - \frac{\mu t^r}{r}\int_{\partial\mathbb{R}^N_+} H|w|^r\mathrm{d}\sigma - \frac{t^p}{p}\int_{\partial\mathbb{R}^N_+} G|w|^p\mathrm{d}\sigma;$$

$$\Phi'_w(t) = t\|w\|^2_{H^1} - t^s\int_{\mathbb{R}^N_+} A|w|^{s+1}\mathrm{d}y - \mu t^{r-1}\int_{\partial\mathbb{R}^N_+} H|w|^r\mathrm{d}\sigma - t^{p-1}\int_{\partial\mathbb{R}^N_+} G|w|^p\mathrm{d}\sigma;$$

$$\Phi''_w(t) = \|w\|^2_{H^1} - st^{s-1}\int_{\mathbb{R}^N_+} A|w|^{s+1}\mathrm{d}y - \mu(r-1)t^{r-2}\int_{\partial\mathbb{R}^N_+} H|w|^r\mathrm{d}\sigma$$
$$- (p-1)t^{p-2}\int_{\partial\mathbb{R}^N_+} G|w|^p\mathrm{d}\sigma.$$

Here we see that

$$t\Phi'_w(t) = \|tu\|^2_{H^1} - \int_{\mathbb{R}^N_+} A|tu|^{s+1}\mathrm{d}y - \mu\int_{\partial\mathbb{R}^N_+} H|tu|^r\mathrm{d}\sigma - \int_{\partial\mathbb{R}^N_+} G|tu|^p\mathrm{d}\sigma$$

and so, for $w \in H^1(\mathbb{R}^N_+) \setminus \{0\}$ and $t > 0$, $tw \in N_\mu$ iff $\Phi'_w(t) = 0$, i.e., points on the Nehari manifold correspond to positive critical points of Φ_w. In particular, $\Phi'_w(1) = 0$ iff $w \in N_\mu$. Thus, we split N_μ into three parts corresponding to local minima, points of inflection and local maxima and so we define $N^+_\mu = \{w \in N_\mu \mid \Phi''_w(1) > 0\}$; $N^0_\mu = \{w \in N_\mu \mid \Phi''_w(1) = 0\}$; $N^-_\mu = \{w \in N_\mu \mid \Phi''_w(1) < 0\}$. Next, we acquire some basic properties of N^+_μ, N^-_μ, and N^0_μ.

LEMMA 2.3. *Assume that w_* is a local minimizer for I_μ on N_μ and that $w_* \notin N^0_\mu$. Then $I'_\mu(w_*) = 0$ in $H^*(\mathbb{R}^N_+)$.*

Proof. Our proof corresponds to that of Theorem 2.3 in [6], (or [3]) . □

For every $w \in N_\mu$ and by (4), we have

$$\Phi''_w(1)$$
$$= \|w\|^2_{H^1} - s\int_{\mathbb{R}^N_+} A|w|^{s+1}\mathrm{d}y - \mu(r-1)\int_{\partial\mathbb{R}^N_+} H|w|^r\mathrm{d}\sigma$$
$$- (p-1)\int_{\partial\mathbb{R}^N_+} G|w|^p\mathrm{d}\sigma \tag{6}$$
$$= (2-p)\|w\|^2_{H^1} - (s+p-1)\int_{\mathbb{R}^N_+} A|w|^{s+1}\mathrm{d}y - \mu(r-p)\int_{\partial\mathbb{R}^N_+} H|w|^r\mathrm{d}\sigma$$
$$= (2-r)\|w\|^2_{H^1} - (s+r-1)\int_{\mathbb{R}^N_+} A|w|^{s+1}\mathrm{d}y - (p-r)\int_{\partial\mathbb{R}^N_+} G|w|^p\mathrm{d}\sigma.$$

LEMMA 2.4. (i) *For every* $w \in N_\mu^+ \cup N_\mu^0$, *we have* $\int_{\partial\mathbb{R}_+^N} H|w|^r \mathrm{d}\sigma > 0$;

(ii) *For every* $w \in N_\mu^-$, *we have* $\int_{\partial\mathbb{R}_+^N} G|w|^p \mathrm{d}\sigma > 0$.

Proof. The outcomes now instantly follow from (6). □

Let

$$\Pi_* = \left[\frac{C_p}{p-r}\right]^{\frac{p-r}{p-2}} \left[\frac{p-2}{\|H\|_{L^{\frac{p}{p-r}}}}\right] \left[\frac{2-r}{\|G\|_\infty}\right]^{\frac{2-r}{p-2}}.$$

Then, we prove the following results.

LEMMA 2.5. *For every* $\mu \in (0, \Pi_*)$, *we have* $N_\mu^0 = \emptyset$.

Proof. We argue, by contradiction. Then there exists $\mu \in (0, \Pi_*)$ such that $N_\mu^0 \neq \emptyset$. Then for $w \in N_\mu^0$, by (6), the Sobolev trace and the Hölder inequalities we have

$$\begin{aligned}
&(2-p)\|w\|_{H^1}^2 = (s+p-1)\int_{\mathbb{R}_+^N} A|w|^{s+1}\mathrm{d}y + \mu(r-p)\int_{\partial\mathbb{R}_+^N} H|w|^r \mathrm{d}\sigma \\
&(2-p)\|w\|_{H^1}^2 \geq \mu(r-p)\int_{\partial\mathbb{R}_+^N} H|w|^r \mathrm{d}\sigma \\
&\|w\|_{H^1}^2 \leq \mu\left(\frac{p-r}{p-2}\right)\int_{\partial\mathbb{R}_+^N} H|w|^r \mathrm{d}\sigma \\
&\|w\|_{H^1}^2 \leq C_p^{\frac{r}{(r-2)}} \left[\mu\left(\frac{p-r}{p-2}\right)\|H\|_{L^{\frac{p}{p-r}}}\right]^{\frac{2}{2-r}}.
\end{aligned} \tag{7}$$

Similarly by the Sobolev trace inequality and by (6), we have

$$\begin{aligned}
&(2-r)\|w\|_{H^1}^2 = (s+r-1)\int_{\mathbb{R}_+^N} A|w|^{s+1}\mathrm{d}y + (p-r)\int_{\partial\mathbb{R}_+^N} G|w|^p \mathrm{d}\sigma \\
&(2-r)\|w\|_{H^1}^2 > (p-r)\int_{\partial\mathbb{R}_+^N} G|w|^p \mathrm{d}\sigma \\
&\|w\|_{H^1}^2 > C_p^{\frac{p}{(p-2)}} \left[\frac{2-r}{(p-r)\|G\|_\infty}\right]^{\frac{2}{p-2}}.
\end{aligned} \tag{8}$$

From (7) and (8), we have

$$C_p^{\frac{p}{(p-2)}} \left[\frac{2-r}{(p-r)\|G\|_\infty}\right]^{\frac{2}{p-2}} \leq C_p^{\frac{r}{(r-2)}} \left[\mu\left(\frac{p-r}{p-2}\right)\|H\|_{L^{\frac{p}{p-r}}}\right]^{\frac{2}{2-r}}$$

$$\mu \geq \left[\frac{C_p}{p-r}\right]^{\frac{p-r}{p-2}} \left[\frac{p-2}{\|H\|_{L^{\frac{p}{p-r}}}}\right] \left[\frac{2-r}{\|G\|_\infty}\right]^{\frac{2-r}{p-2}} = \Pi_*,$$

which is a contradiction. Thus, here completes the proof. □

Denote

$$\Psi(\mathbb{R}^N_+) = \left\{ w \in H^1(\mathbb{R}^N_+) \setminus \{0\} \mid \int_{\partial\mathbb{R}^N_+} G|w|^p \mathrm{d}\sigma > 0 \right\} \subset H^1(\mathbb{R}^N_+).$$

To better understand fibering maps and the Nehari manifold, we observe the function $\chi_w : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\chi_w(t) = t^{2-r}\|w\|^2_{H_1} - t^{s-r+1}\int_{\mathbb{R}^N_+} A|w|^{s+1}\mathrm{d}y - t^{p-r}\int_{\partial\mathbb{R}^N_+} G|w|^p \mathrm{d}\sigma,$$

for $t > 0$. Consequently, $tw \in N_\mu$ iff $\chi_w(t) = \int_{\partial\mathbb{R}^N_+} G|w|^p \mathrm{d}\sigma$. Moreover,

$$\begin{aligned} \chi_w'(t) = (2-r)t^{1-r}\|w\|^2_{H_1} - (s-r+1)t^{s-r}\int_{\mathbb{R}^N_+} A|w|^{s+1}\mathrm{d}y \\ - (p-r)t^{p-r-1}\int_{\partial\mathbb{R}^N_+} G|w|^p \mathrm{d}\sigma. \end{aligned} \tag{9}$$

Hence, it is simple to observe that, if $tw \in N_\mu$, then $t^{r-1}\chi_w'(t) = \Phi_w''(t)$. Hence $\chi_w(t) < 0$ (or > 0) iff $tw \in N^-_\mu$ (or N^+_μ).

Let $w \in \Psi(\mathbb{R}^N_+)$. Then, by (9) we have $t = t^{\max}(w)$ is a critical point of χ_w and obviously χ_w is $\uparrow$ on $(0, t^{\max}(w))$ and $\downarrow$ on $(t^{\max}(w), \infty)$. Furthermore, if $\mu \in (0, \Pi_*)$, then $\chi_w(t^{\max}(w)) > \int_{\partial\mathbb{R}^N_+} \mu H|w|^r \mathrm{d}\sigma$.

Then, we prove the following lemma.

Lemma 2.6. *For every $w \in \Psi(\mathbb{R}^N_+)$, we obtain:*

(i) *If $\int_{\partial\mathbb{R}^N_+} H|w|^r \mathrm{d}\sigma \le 0$, then there is a unique $t_- = t_-(w) > t^{\max}(w)$ such that $t_-(w) \in N^-_\mu$ and χ_w is increasing on $(0, t_-)$ and decreasing on (t_-, ∞). Moreover,*

$$I_\mu(t_- w) = \sup_{t \ge 0} I_\mu(tw); \tag{10}$$

(ii) *If $\int_{\partial\mathbb{R}^N_+} H|w|^r \mathrm{d}\sigma > 0$, then there are unique $0 < t_+ = t_+ w < t^{\max}(w) < t_-$ such that $t_+ w \in N^+_\mu$, $t_- w \in N^-_\mu$, χ_w is $\downarrow$ on $(0, t_+)$, $\uparrow$ on (t_+, t_-) and $\downarrow$ on (t_-, ∞). Moreover,*

$$I_\mu(t_+ w) = \inf_{0 \le t \le t^{\max}(w)} I_\mu(tw);\; I_\mu(t_- w) = \sup_{t \ge t_+} I_\mu(tw); \tag{11}$$

(iii) *$t_-(w)$ is a continuous function on $\Psi(\mathbb{R}^N_+)$;*

(iv)

$$N^-_\mu = \left\{ w \in \Psi(\mathbb{R}^N_+) \mid \frac{1}{\|w\|_{H^1}} t_-\left(\frac{w}{\|w\|_{H^1}}\right) = 1 \right\}.$$

Proof. For a fixed $w \in \Psi(\mathbb{R}^N_+)$, we have the following.

(i) Assume $\int_{\partial\mathbb{R}^N_+} H|w|^r \mathrm{d}\sigma \le 0$. Then $\chi_w(t) = \int_{\partial\mathbb{R}^N_+} \mu H|w|^r \mathrm{d}\sigma$ has a unique solution $t_- > t^{\max}(w)$ and $\chi'_w(t_-) < 0$. Hence χ_w has a unique turning point at $t = t_-$ and $0 > \Phi''_w(t)$. Thus $t_- w \in N^-_\mu$ and (10) are true.

(ii) Assume $\int_{\partial\mathbb{R}^N_+} H|w|^r \mathrm{d}\sigma > 0$. Since

$$\chi_w(t^{\max}(w)) > \int_{\partial\mathbb{R}^N_+} \mu H|w|^r \mathrm{d}\sigma,$$

the equation

$$\chi_w(t) = \int_{\partial\mathbb{R}^N_+} \mu H|w|^r \mathrm{d}\sigma$$

has exactly two solutions $t_+ < t^{\max}(w) < t_-$ such that $0 < \chi'_w(t_+)$ and $0 > \chi'_w(t_-)$. Hence there are exactly two multiple of w lying in N_μ, i.e. $t_+ w \in N^+_\mu$ and $t_- w \in N^-_\mu$.

Thus, χ_w has turning points at $t = t_+$ and $t = t_-$ with $\Phi''_w(t_+) > 0$ and $\Phi''_w(t_-) < 0$. Thus, χ_w is $\downarrow$ on $(0, t_+)$, $\uparrow$ on (t_+, t_-) and $\downarrow$ on (t_-, ∞). Hence, (11) must be true.

(iii) By the external property of $t_-(w)$ and the uniqueness of $t_-(w)$, $t_-(w) \in \Psi(\mathbb{R}^N_+)$ is a continuous function.

(iv) For $w \in N^-_\mu$, we have $W = \frac{w}{\|w\|_{H^1}}$. By parts (i) and (ii), here is a unique $t_-(W) > 0$ such that $t_-(W)W \in N^-_\mu$ or $t_-\left(\dfrac{w}{\|w\|_{H^1}}\right)\dfrac{1}{\|w\|_{H^1}} w \in N^-_\mu$. Since $w \in N^-_\mu$, we have $t_-\left(\dfrac{w}{\|w\|_{H^1}}\right)\dfrac{1}{\|w\|_{H^1}} = 1$, this implies that

$$N_\mu \subset \left\{ w \in H^1(\mathbb{R}^N_+) \mid \frac{1}{\|w\|_{H^1}} t_-\left(\frac{w}{\|w\|_{H^1}}\right) = 1 \right\}.$$

On the other hand, let $w \in \Psi(\mathbb{R}^N_+)$ such that $\dfrac{1}{\|w\|_{H^1}} t_-\left(\dfrac{w}{\|w\|_{H^1}}\right) = 1$. Then $t_-\left(\dfrac{w}{\|w\|_{H^1}}\right)\dfrac{w}{\|w\|_{H^1}} \in N^-_\mu$. Hence,

$$N^-_\mu = \left\{ w \in \Psi(\mathbb{R}^N_+) \mid \frac{1}{\|w\|_{H^1}} t_-\left(\frac{w}{\|w\|_{H^1}}\right) = 1 \right\}.$$

Thus, the proof is complete. □

3. MAIN RESULT

Here, by Lemma 2.5, we write $N_\mu = N^+_\mu \cup N^-_\mu$ for all $\mu \in (0, \Pi_*)$. Furthermore, by Lemma 2.6 $N^+_\mu \neq \emptyset$ and $N^-_\mu \neq \emptyset$, and by Lemma 2.2 we define $\beta^+_\mu = \inf_{w \in N^+_\mu} I_\mu(w)$ and $\beta^-_\mu = \inf_{w \in N^-_\mu} I_\mu(w)$. Therefore, we have the following results.

THEOREM 3.1. *We have:*

(i) *for all* $\mu \in (0, \Pi_*)$, $\beta_\mu^+ < 0$.

(ii) *if* $\mu \in (0, \frac{r\Pi_*}{2})$, *then* $\beta_\mu^- > c_0$ *for some* $c_0 > 0$.

In particular, for all $\mu \in (0, \frac{r\Pi_*}{2})$, *we have* $\beta_\mu^+ = \inf_{w \in N_\mu^+} I_\mu(w)$.

Proof. (i) For $w \in N_\mu^+$, by (6), we have

$$(2-p)\|w\|_{H^1}^2 > (s+p-1)\int_{\mathbb{R}_+^N} A|w|^{s+1}\mathrm{d}y + \mu(r-p)\int_{\partial\mathbb{R}_+^N} H|w|^r\mathrm{d}\sigma$$

$$(2-p)\|w\|_{H^1}^2 > \mu(r-p)\int_{\partial\mathbb{R}_+^N} H|w|^r\mathrm{d}\sigma$$

$$\|w\|_{H^1}^2 < \mu\left(\frac{p-r}{p-2}\right)\int_{\partial\mathbb{R}_+^N} H|w|^r\mathrm{d}\sigma.$$

Then by Lemma 2.4 and (5), we have

$$\begin{aligned} I_\mu(w) &= \left(\frac{1}{2}-\frac{1}{p}\right)\|w\|_{H^1}^2 - \left(\frac{1}{s+1}-\frac{1}{p}\right)\int_{\mathbb{R}_+^N} A|w|^{s+1}\mathrm{d}y \\ &\quad - \left(\frac{\mu}{r}-\frac{\mu}{p}\right)\int_{\partial\mathbb{R}_+^N} H|w|^r\mathrm{d}\sigma \\ &< \left(\frac{1}{2}-\frac{1}{p}\right)\|w\|_{H^1}^2 - \left(\frac{\mu}{r}-\frac{\mu}{p}\right)\int_{\partial\mathbb{R}_+^N} H|w|^r\mathrm{d}\sigma \\ &< -\mu\frac{(p-r)(2-r)}{2pr}\int_{\partial\mathbb{R}_+^N} H|w|^r\mathrm{d}\sigma < 0, \end{aligned}$$

and thus $\beta_\mu^+ < 0$.

(ii) For $w \in N_\mu^-$, we have

$$(2-r)\|w\|_{H^1}^2 < (s+r-1)\int_{\mathbb{R}_+^N} A|w|^{s+1}\mathrm{d}y + (p-r)\int_{\partial\mathbb{R}_+^N} G|w|^p\mathrm{d}\sigma,$$

which gives

$$\left(\frac{2-r}{p-r}\right)\|w\|_{H^1}^2 < \int_{\partial\mathbb{R}_+^N} G|w|^p\mathrm{d}\sigma,\ \forall\ w \in N_\mu^-.$$

Moreover, by (3), we have

$$\left(\frac{2-r}{p-r}\right)\|w\|_{H^1}^2 < C_p^{\frac{-p}{2}}\|G\|_\infty\|w\|_{H^1}^p.$$

Consequently, we have

$$\left(\frac{2-r}{p-r}\right)\frac{C_p^{\frac{p}{2}}}{\|G\|_\infty} < \|w\|_{H^1}^{p-2}.$$

Then, we get

$$\left(\frac{2-r}{(p-r)\|G\|_\infty}\right)^{\frac{1}{p-2}} C_p^{\frac{p}{2(p-2)}} < \|w\|_{H^1},\ \forall\ w \in N_\mu^-.$$

By (5), we get

$$\begin{aligned}
I_\mu(w) &\geq \left(\frac{1}{2}-\frac{1}{p}\right)\|w\|_{H^1}^2 - \left(\frac{1}{s+1}-\frac{1}{p}\right)\int_{\mathbb{R}_+^N} A|w|^{s+1}\mathrm{d}y \\
&\quad - \mu\left(\frac{p-r}{pr}\right)\|H\|_{L^{\frac{p}{p-r}}} C_r^{\frac{-r}{2}}\|w\|_{H^1}^r \\
&\geq \left(\frac{1}{2}-\frac{1}{p}\right)\|w\|_{H^1}^2 - \mu\left(\frac{p-r}{pr}\right)\|H\|_{L^{\frac{p}{p-r}}} C_p^{\frac{-r}{2}}\|w\|_{H^1}^r \\
&\geq \|w\|_{H^1}^\delta\left[\left(\frac{p-2}{2p}\right)\|w\|_{H^1}^{2-r} - \mu\left(\frac{p-r}{pr}\right)\|H\|_{L^{\frac{p}{p-r}}} C_p^{\frac{-r}{2}}\right].
\end{aligned} \tag{12}$$

From (12), if $\mu \in (0, \frac{r\Pi_*}{2})$, then $\beta_\mu^- > c_0$ for some $c_0 > 0$. Hence, the proof is now complete. □

Next, we consider the below elliptic PDE:

$$\begin{cases} -\Delta w + B(y)w = A(y)w^s, & y \in \mathbb{R}_+^N; \\ \dfrac{\partial w}{\partial \nu} = |w|^{p-2}w, & y \in \partial\mathbb{R}_+^N. \end{cases} \tag{13}$$

We admit the variational functional I_∞ in $H^1(\mathbb{R}_+^N)$ of problem (13)

$$I^\infty(w) = \frac{1}{2}\|w\|_{H^1}^2 - \frac{1}{s+1}\int_{\mathbb{R}_+^N} A|w|^{s+1}\mathrm{d}y - \frac{1}{p}\int_{\partial\mathbb{R}_+^N} G|w|^p\mathrm{d}\sigma.$$

Consider the minimizing problem $\beta^\infty = \inf_{w\in N^\infty} I^\infty(w)$, where

$$N^\infty = \{w \in H^1(\mathbb{R}_+^N)\setminus\{0\} \mid \langle (I^\infty)'(w), w\rangle = 0\}.$$

In a similar way as in del Pino and Flores [16], eq. (13) has a positive ground state solution $\bar{W}$ such that $I^\infty(\bar{W}) = \beta^\infty > 0$. Moreover, if $\mu \in (0, \frac{r\Pi_*}{2})$, then we have the following proposition for the (PS)-sequence of I_μ.

Proposition 3.2. *If $\{w_n\}$ is a $(PS)_\gamma$-sequence in $H^1(\mathbb{R}_+^N)$ for I_μ with $\gamma < \beta_\mu^+ + \beta^\infty$, then there exists a subsequence $\{w_n\}$ and a nonzero $\bar{w}$ in $H^1(\mathbb{R}_+^N)$ such that $\{w_n\}$ converges $\bar{w}$ strongly in $H^1(\mathbb{R}_+^N)$ and $I_\mu(\bar{w}) = \gamma$. Furthermore, $\bar{w}$ is a solution of problem* (1).

Proof. By Lemma 2.1 and the Rellich-Kondrachov theorem there exists a subsequence $\{w_n\}$, and $\bar{w} \in H^1(\mathbb{R}_+^N)$ is a solution of problem (1) such that

- $\{w_n\}$ converges to $\bar{w}$ weakly in $H^1(\mathbb{R}_+^N)$;
- $\{w_n\}$ converges to $\bar{w}$ strongly in $L_{loc}^p(\partial\mathbb{R}_+^N)$;
- $\{w_n\}$ converges to $\bar{w}$ almost everywhere in $\overline{\mathbb{R}_+^N}$.

Here we claim that $\bar{w} \neq 0$. If not, then by $H \in L^{\frac{p}{p-r}}(\partial\mathbb{R}^N_+)$ and (g1), we get

$$\int_{\partial\mathbb{R}^N_+} H|w_n|^r \mathrm{d}\sigma \to 0 \text{ as } n \to \infty$$

and

$$\int_{\partial\mathbb{R}^N_+} G|w_n|^p \mathrm{d}\sigma \to \int_{\partial\mathbb{R}^N_+} |w_n|^p \mathrm{d}\sigma + O(1).$$

Thus,

$$\|w_n\|^2_{H^1} = \int_{\mathbb{R}^N_+} A|w_n|^{s+1} \mathrm{d}y + \int_{\partial\mathbb{R}^N_+} |w_n|^p \mathrm{d}\sigma + O(1)$$

and

$$\begin{aligned}
\left(\frac{1}{2} - \frac{1}{p}\right) \int_{\partial\mathbb{R}^N_+} G|w_n|^p \mathrm{d}\sigma &= \frac{1}{2}\|w_n\|^2_{H^1} - \frac{1}{2}\int_{\mathbb{R}^N_+} A|w_n|^{s+1} \mathrm{d}y \\
&\quad - \frac{1}{r}\int_{\partial\mathbb{R}^N_+} H|w_n|^r \mathrm{d}\sigma - \frac{1}{p}\int_{\partial\mathbb{R}^N_+} G|w_n|^p \mathrm{d}\sigma + O(1) \\
&\leq \frac{1}{2}\|w_n\|^2_{H^1} - \frac{1}{s+1}\int_{\mathbb{R}^N_+} A|w_n|^{s+1} \mathrm{d}y \\
&\quad - \frac{1}{r}\int_{\partial\mathbb{R}^N_+} H|w_n|^r \mathrm{d}\sigma - \frac{1}{p}\int_{\partial\mathbb{R}^N_+} G|w_n|^p \mathrm{d}\sigma + O(1) \\
&= \gamma + O(1).
\end{aligned}$$

From the Sobolev trace embedding and $\{w_n\} \subset N^-_\mu$ we get $\|w_n\|_{H^1} > c$ for some $c > 0$ and $\beta^\infty \leq \gamma$. This contradicts $\gamma < \beta^+_\mu + \beta^\infty < \beta^\infty$. Thus, $\bar{w}$ is a nontrivial solution of the problem (1) and $\beta^+_\mu \leq I_\mu(\bar{w})$.

Take $w_n = v_n + \bar{w}$ with v_n converges to 0 weakly $H^1(\mathbb{R}^N_+)$. Then, by the fact that v_n converges 0 weakly in $H^1(\mathbb{R}^N_+)$ and the Brezis-Lieb lemma [4], we obtain

$$\begin{aligned}
\int_{\partial\mathbb{R}^N_+} G|w_n|^p \mathrm{d}\sigma &= \int_{\partial\mathbb{R}^N_+} G|v_n + \bar{w}|^p \mathrm{d}\sigma \\
&= \int_{\partial\mathbb{R}^N_+} |v_n|^p \mathrm{d}\sigma + \int_{\partial\mathbb{R}^N_+} G|\bar{w}|^p \mathrm{d}\sigma + O(1).
\end{aligned}$$

Since, we have that $\{w_n\}$ is a bounded sequence in $H^1(\mathbb{R}^N_+)$, then $\{v_n\}$ is also bounded sequence in $H^1(\mathbb{R}^N_+)$. Furthermore, by $H \in L^{\frac{p}{p-r}}(\partial\mathbb{R}^N_+)$, the Hölder inequality and the Egorov theorem, we obtain

$$\begin{aligned}
\int_{\partial\mathbb{R}^N_+} H|v_n|^r \mathrm{d}\sigma &= \int_{\partial\mathbb{R}^N_+} H|w_n|^r \mathrm{d}\sigma - \int_{\partial\mathbb{R}^N_+} H|\bar{w}|^r \mathrm{d}\sigma + O(1) \\
&= O(1).
\end{aligned}$$

Thus for large n, we are able to draw that

$$\begin{aligned}
&\beta_\mu^+ + \beta^\infty \\
&> I_\mu(v_n + \bar{w}) \\
&\geq I_\mu(\bar{w}) + \frac{1}{2}\|v_n\|_{H^1}^2 - \frac{1}{s+1}\int_{\mathbb{R}_+^N} A|v_n|^{s+1}\mathrm{d}y - \frac{1}{p}\int_{\partial\mathbb{R}_+^N} |v_n|^p \mathrm{d}\sigma + O(1) \\
&\geq \beta_\mu^+ + \frac{1}{2}\|v_n\|_{H^1}^2 - \frac{1}{s+1}\int_{\mathbb{R}_+^N} A|v_n|^{s+1}\mathrm{d}y - \frac{1}{p}\int_{\partial\mathbb{R}_+^N} |v_n|^p \mathrm{d}\sigma + O(1)
\end{aligned}$$

or

$$\frac{1}{2}\|v_n\|_{H^1}^2 - \frac{1}{s+1}\int_{\mathbb{R}_+^N} A|v_n|^{s+1}\mathrm{d}y - \frac{1}{p}\int_{\partial\mathbb{R}_+^N} |v_n|^p \mathrm{d}\sigma < \beta^\infty + O(1). \tag{14}$$

Also from the facts that $I'_\mu(w_n) = O(1)$ in $H^*(\mathbb{R}_+^N)$, $\bar{w}$ is a solution of problem (1) and $\{w_n\}$ is uniformly bounded, we have

$$O(1) = \langle I'_\mu(w_n), w_n\rangle = \|v_n\|_{H^1}^2 - \int_{\mathbb{R}_+^N} A|v_n|^{s+1}\mathrm{d}y - \int_{\partial\mathbb{R}_+^N} G|v_n|^p \mathrm{d}\sigma + O(1). \tag{15}$$

We claim that (14) and (15) can be hold simultaneously only if $\{v_n\}$ has a subsequence $\{v_{n_i}\}$ such that $\{v_{n_i}\} \to 0$ strongly. If not, then $\|v_n\|_{H^1}$ is bounded away from zero, that is $\|v_n\| > c$ for some $c > 0$.

Since,

$$\frac{1}{2}\|v_n\|_{H^1}^2 - \frac{1}{2}\int_{\mathbb{R}_+^N} A|v_n|^{s+1}\mathrm{d}y < \frac{1}{2}\|v_n\|_{H^1}^2 - \frac{1}{s+1}\int_{\mathbb{R}_+^N} A|v_n|^{s+1}\mathrm{d}y,$$

then by (14), we have

$$\frac{1}{2}\int_{\partial\mathbb{R}_+^N} |v_n|^p \mathrm{d}\sigma - \frac{1}{p}\int_{\partial\mathbb{R}_+^N} |v_n|^p \mathrm{d}\sigma < \beta^\infty + O(1)$$

or

$$\int_{\partial\mathbb{R}_+^N} |v_n|^p \mathrm{d}\sigma < \left(\frac{2p}{p-2}\right)\beta^\infty + O(1).$$

For large enough n, by (14) and (15), we have

$$\begin{aligned}
\beta^\infty &> \left(\frac{1}{2} - \frac{1}{p}\right)\int_{\partial\mathbb{R}_+^N} |v_n|^p \mathrm{d}\sigma + O(1) \\
&\geq \frac{1}{2}\|v_n\|_{H^1}^2 - \frac{1}{s+1}\int_{\mathbb{R}_+^N} A|v_n|^{s+1}\mathrm{d}y - \frac{1}{p}\int_{\partial\mathbb{R}_+^N} |v_n|^p \mathrm{d}\sigma + O(1) \\
&= \beta^\infty.
\end{aligned}$$

This is a contradiction. Thus, w_n converges $\bar{w}$ strongly in $H^1(\mathbb{R}_+^N)$ and $I_\mu(\bar{w}) = \gamma$. □

THEOREM 3.3. *For every* $\mu \in (0, \frac{r\Pi_*}{2})$, *the functional* I_μ *has a minimizer* $w^{\min}$ *in* N_μ^+ *and satisfies the following.*

(i) $I_\mu(w^{\min}) = \beta_\mu^+$;

(ii) $w^{\min}$ *is a positive solution of problem* (1);

(iii) *As* $\mu \to 0 \implies \|w^{\min}\|_{H^1} \to 0$.

Proof. By the Ekeland variational principle [10] (or by [22]), there exists $\{w_n\} \subset N_\mu^+$ such that $I_\mu(w_n) = \beta_\mu^+ + O(1)$ and $I'_\mu(w_n) = O(1)$ in $H^*(\mathbb{R}_+^N)$. Then by Proposition 3.2, there exists a subsequence $\{w_n\}$ and $w^{\min} \in N_\mu^+$, a solution of the problem (1) such that w_n converges to $w^{\min}$ strongly in $H^1(\mathbb{R}_+^N)$ and $I_\mu(w^{\min}) = \beta_\mu^+$. Since $I_\mu(w^{\min}) = I_\mu(|w^{\min}|)$ and $|w^{\min}| \in N_\mu^+$, by Lemma 2.3, we may suppose that $w^{\min} \geq 0$. Moreover, the Hopf lemma and the maximum principle, we get $w^{\min} > 0$ in $\mathbb{R}_+^N$. Hence, by (6), we have

$$\|w^{\min}\|_{H^1}^{2-r} \leq \mu C_p^{\frac{-r}{2}} \left(\frac{p-r}{p-2}\right) \|H\|_{L^{\frac{p}{p-r}}}.$$

Thus, as $\mu \to 0$ we have $\|w^{\min}\|_{H^1} \to 0$. □

4. PROOF OF THE MAIN THEOREM

Here, suppose $\bar{W}(y)$ is a positive solution of problem (13) such that $\beta^\infty = I^\infty(w)$. Then by Lemma 3.3 in [11], there exists a positive number C_* such that

$$|\bar{W}(y)| \leq C_* \exp(-|y|) \text{ for all } y \in \mathbb{R}_+^N. \tag{16}$$

Let $W_l(y) = \bar{W}(y + le)$, for $l \in \mathbb{R}$ and $e \in \mathbb{S}$, where $\mathbb{S} = \{y \in \partial\mathbb{R}_+^N \mid |y| = 1\}$. Clearly $\int_{\partial\mathbb{R}_+^N} H|W_l|^r \mathrm{d}\sigma = 0$ as $l \to \infty$. Then the results are as follows.

LEMMA 4.1. *Let* $w^{\min}$ *be a positive solution of problem* (1) *as in Theorem* 3.3. *Then for all* $\mu \in (0, \frac{r\Pi_*}{2})$ *there exists* $0 < l^0$ *such that for* $l^0 < l$, *we have*

$$\sup_{t \geq 0} I_\mu(w^{\min} + tW_l) < \beta_\mu^+ + \beta^\infty.$$

Proof. Since

$$\begin{aligned}
&I_\mu(w^{\min} + tW_l) \\
&= \frac{1}{2}\|w^{\min} + tW_l\|_{H^1}^2 - \frac{1}{s+1}\int_{\mathbb{R}_+^N} A(w^{\min} + tW_l)^{s+1}\mathrm{d}y \\
&\quad - \frac{\mu}{r}\int_{\partial\mathbb{R}_+^N} H(w^{\min} + tW_l)^r \mathrm{d}\sigma - \frac{1}{p}\int_{\partial\mathbb{R}_+^N} G(w^{\min} + tW_l)^p \mathrm{d}\sigma \\
&= \frac{1}{2}\|w^{\min}\|_{H^1}^2 - \frac{1}{s+1}\int_{\mathbb{R}_+^N} A(w^{\min})^{s+1}\mathrm{d}y - \frac{\mu}{r}\int_{\partial\mathbb{R}_+^N} H(w^{\min})^r \mathrm{d}\sigma
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{p}\int_{\partial\mathbb{R}^N_+} G(w^{\min})^p \mathrm{d}\sigma + \frac{1}{2}\|tW_l\|^2_{H^1} - \frac{1}{s+1}\int_{\mathbb{R}^N_+} A(tW_l)^{s+1}\mathrm{d}y \\
&- \frac{1}{p}\int_{\partial\mathbb{R}^N_+} G(tW_l)^p \mathrm{d}\sigma - \frac{\mu}{r}\left[\int_{\partial\mathbb{R}^N_+} H(w^{\min}+tW_l)^r \mathrm{d}\sigma - \int_{\partial\mathbb{R}^N_+} H(w^{\min})^r \mathrm{d}\sigma\right] \\
&- \frac{1}{p}\left[\int_{\partial\mathbb{R}^N_+} G(w^{\min}+tW_l)^p \mathrm{d}\sigma - \int_{\partial\mathbb{R}^N_+} G(w^{\min})^p \mathrm{d}\sigma - \int_{\partial\mathbb{R}^N_+} G(tW_l)^p \mathrm{d}\sigma\right] \\
&+ \frac{1}{s+1}\left[\int_{\mathbb{R}^N_+} A(w^{\min})^{s+1}\mathrm{d}y + \int_{\mathbb{R}^N_+} A(tW_l)^{s+1}\mathrm{d}y\right] \\
&+ t\left(\int_{\partial\mathbb{R}^N_+} G(w^{\min})^{p-1}W_l \mathrm{d}\sigma + \int_{\partial\mathbb{R}^N_+} H(w^{\min})^{r-1}W_l \mathrm{d}\sigma\right). \\
\le\ & \beta^+_\mu + I^\infty(tW_l) \\
&- \int_{\partial\mathbb{R}^N_+} G\left[(w^{\min}+tW_l)^p - (w^{\min})^p - t^r W_l^p - p(w^{\min})^{p-1}tW_l\right]\mathrm{d}\sigma \\
&+ \frac{\Pi_* t^r}{2}\int_{\partial\mathbb{R}^N_+} HW_l^r \mathrm{d}\sigma - t^p\int_{\partial\mathbb{R}^N_+}(G-1)W_l^p \mathrm{d}\sigma \\
&+ \frac{1}{s+1}\left[\int_{\mathbb{R}^N_+} A[(w^{\min})^{s+1} + (tW_l)^{s+1}]\mathrm{d}y\right].
\end{aligned}
$$

We know that, by Brown and Zhang [6], we have $I^\infty(tW_l) \le \beta^\infty\ \forall\ l \in \mathbb{R}$. We remark that $(w+v)^p - w^p - v^p - pw^{p-1}v \ge 0$ for all $(w,v) \in [0,\infty)\times[0,\infty)$. Hence,

$$
\int_{\partial\mathbb{R}^N_+} G\left[(w^{\min}+tW_l)^p - (w^{\min})^p - t^p W_l^p - p(w^{\min})^{p-1}tW_l\right]\mathrm{d}\sigma \ge 0.
$$

Thus, we have

$$
\begin{aligned}
&I_\mu(w^{\min}+tW_l) \\
&\le \beta^+_\mu + \beta^\infty + \frac{\Pi_* t^r}{2}\int_{\partial\mathbb{R}^N_+} HW_l^r \mathrm{d}\sigma - t^p\int_{\partial\mathbb{R}^N_+}(G-1)W_l^p \mathrm{d}\sigma \\
&\quad + \frac{1}{s+1}\left[\int_{\mathbb{R}^N_+} A[(w^{\min})^{s+1} + (tW_l)^{s+1}]\right]\mathrm{d}y.
\end{aligned}
\tag{17}
$$

Since, $I_\mu(w^{\min}+tW_l) \to I_\mu(w^{\min}) = \beta^+_\mu < 0$ as $t \to 0$ and $I_\mu(w^{\min}+tW_l) \to -\infty$ as $t \to 0$. Then easily we can find $0 < t_1 < t_2$ such that

$$
I_\mu(w^{\min}+tW_l) < \beta^+_\mu + \beta^\infty\ \forall\ t \in [0,t_1]\cup[t_2,\infty). \tag{18}
$$

Hence, we need to prove that there exists $l^0 > 0$ such that for $l^0 < l$,

$$
\sup_{t_1 \le t \le t_2} I_\mu(w^{\min}+tW_l) < \beta^+_\mu + \beta^\infty.
$$

By the condition (g1) and by del Pino and Flores [16, Lemma 2.2], we have

$$
\begin{aligned}
\int_{\partial\mathbb{R}^N_+}(G-1)W_l^p\mathrm{d}\sigma &= \int_{\partial\mathbb{R}^N_+}(G(y-le)-1)W_0^p(y)\mathrm{d}\sigma \\
&\geq D_0\int_{B^N_+(1)}(G(y-le)-1)\mathrm{d}\sigma \\
&\geq \bar{C}_0\exp(-R_h l)
\end{aligned}
\tag{19}
$$

where $D_0 = \min_{y\in B^N_+(1)} W_0^p(y)$ and $B^N_+(1) = \{y \in \partial\mathbb{R}^N_+ \mid |y| = 1\}$.

By (16) and the condition (h1), we also have

$$
\begin{aligned}
\int_{\partial\mathbb{R}^N_+} HW_l^r\mathrm{d}\sigma &\leq \int_{\partial\mathbb{R}^N_+} C_0^r|H|\exp(-r|y+le)\mathrm{d}\sigma \\
&\leq \bar{C}_1\exp(-rl).
\end{aligned}
$$

Since $R_h = r$ and $t_1 < t < t_2$, by (17) – (19) we can find $l^0 > 0$ such that

$$\sup_{t\geq 0} I_\mu(w^{\min} + tW_l) < \beta^+_\mu + \beta^\infty \text{ for all } l^0 \leq l.$$

Hence, the proof is complete. □

Further, we prove the existence of a local minimum for I_μ on N^-_μ.

THEOREM 4.2. *For every $\mu \in (0, \frac{r\Pi_*}{2})$, the functional I_μ has a minimizer w_0^* in N^-_μ and satisfies the following.*

(i) $I_\mu(w_0^*) = \beta^-_\mu$;

(ii) *w_0^* is a positive solution of the problem* (1).

Proof. First, we claim that $\beta^-_\mu < \beta^+_\mu + \beta^\infty$. Let

$$w^1_* = \left\{w \in \Psi(\mathbb{R}^N_+) \mid \frac{1}{\|w\|_{H^1}}t_-\left(\frac{w}{\|w\|_{H^1}}\right) > 1\right\}$$

and

$$w^2_* = \left\{w \in \Psi(\mathbb{R}^N_+) \mid \frac{1}{\|w\|_{H^1}}t_-\left(\frac{w}{\|w\|_{H^1}}\right) < 1\right\}.$$

By Lemma 2.6, N^-_μ disconnects $\Psi(\mathbb{R}^N_+)$ in two parts w^1_* and w^2_* and $\Psi(\mathbb{R}^N_+)\setminus N^-_\mu = w^1_* \cup w^2_*$. Moreover, for all $w \in N^+_\mu$, we have $1 < t^{\max}(w) < t_-(w)$ and $\int_{\partial\mathbb{R}^N_+} G|w|^p\mathrm{d}\sigma > 0$. Since

$$t_-(w) = \frac{1}{\|w\|_{H^1}}t_-\left(\frac{w}{\|w\|_{H^1}}\right),$$

then $N_\mu \subset w^1_*$. In particular, $w^{\min} \in w^1_*$. We claim that for each $l^0 \leq l$ there exists $s_0 > 0$ such that $w^{\min} + s_0W_l \in w^2_*$. Indeed, $\int_{\partial\mathbb{R}^N_+} G|w^{\min} + sW_l|^p\mathrm{d}\sigma > 0$ for all $s \geq 0$. By Lemma 2.6, for all $s \geq 0$ there is a unique

$$0 < t_-\left(\frac{w^{\min} + sW_l}{\|w^{\min} + sW_l\|_{H^1}}\right)$$

such that

$$t_-\left(\frac{w^{\min}+sW_l}{\|w^{\min}+sW_l\|_{H^1}}\right)\frac{w^{\min}+sW_l}{\|w^{\min}+sW_l\|_{H^1}}\in N_\mu^-.$$

First, we have to find a $D>0$ such that $0<t_-\left(\frac{w^{\min}+sW_l}{\|w^{\min}+sW_l\|_{H^1}}\right)<D$ for all $l\geq 0$. Instead, there exists a sequence $\{s_n\}$ such that

$$s_n\to\infty \text{ and } t_-\left(\frac{w^{\min}+s_nW_l}{\|w^{\min}+s_nW_l\|_{H^1}}\right)\to\infty \quad \text{as} \quad n\to\infty.$$

Let $v_n=\frac{w^{\min}+s_nW_l}{\|w^{\min}+s_nW_l\|_{H^1}}$. Since $t_-(v_n)v_n\in N_\mu^-\subset N_\mu$ and by the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}\int_{\partial\mathbb{R}_+^N}G|v_n|^p\mathrm{d}\sigma&=\frac{1}{\|w^{\min}+s_nW_l\|_{H^1}}\int_{\partial\mathbb{R}_+^N}G(w^{\min}+s_nW_l)^p\mathrm{d}\sigma\\&=\frac{1}{\left\|\frac{w^{\min}}{s_n}+W_l\right\|_{H^1}}\int_{\partial\mathbb{R}_+^N}G\left(\frac{w^{\min}}{s_n}+W_l\right)^p\mathrm{d}\sigma\\&\to\frac{\int_{\partial\mathbb{R}_+^N}G|W_l|^p\mathrm{d}\sigma}{\|W_l\|_{H^1}^p}\text{ as }n\to\infty.\end{aligned}$$

Then, we get

$$\begin{aligned}&I_\mu(t_-(v_n)v_n)\\&=\frac{1}{2}[t_-(v_n)]^2-\frac{[t_-(v_n)]^{s+1}}{s+1}\int_{\mathbb{R}_+^N}Av_n^{s+1}\mathrm{d}y-\frac{\mu[t_-(v_n)]^r}{r}\int_{\partial\mathbb{R}_+^N}Hv_n^r\mathrm{d}\sigma\\&-\frac{[t_-(v_n)]^p}{p}\int_{\partial\mathbb{R}_+^N}G(v_n)^p\mathrm{d}\sigma.\end{aligned}$$

$I_\mu(t_-(v_n)v_n)$ tends to $-\infty$ as $n\to\infty$. Thus, we have a contradiction because I_μ is bounded below on N_μ.

Let $s_0=\frac{|D^2-\|w^{\min}\|_{H^1}^2|^{\frac{1}{2}}}{\|W_0\|_{H^1}}+1$. Then, we have

$$\begin{aligned}\|w^{\min}+s_0W_l\|_{H^1}^2&=\|w^{\min}\|_{H^1}^2+s_0^2\|W_l\|_{H^1}^2+2s_0\langle w^{\min},W_l\rangle\\&>\|w^{\min}\|_{H^1}^2+|D^2-\|w^{\min}\|_{H^1}^2|\\&+2l_0\left(\int_{\partial\mathbb{R}_+^N}\mu H(w^{\min})^{r-1}W_l\mathrm{d}\sigma+\int_{\partial\mathbb{R}_+^N}G(w^{\min})^{p-1}W_l\mathrm{d}\sigma\right)\\&>D^2>\left[t_-\left(\frac{w^{\min}+s_0W_l}{\|w^{\min}+s_0W_l\|_{H^1}}\right)\right]^2,\end{aligned}$$

that is $w^{\min}+s_0W_l\in w_*^2$.

Let

$$f(s)=\frac{1}{\|w^{\min}+sW_l\|_{H^1}}t_-\left(\frac{w^{\min}+sW_l}{\|w^{\min}+sW_l\|_{H^1}}\right)>0.$$

By Lemma 2.6 (iii), $f(s)$ is a continuous function on $[0,\infty)$, since $1<f(0)$ and $f(s_0)<1$. By the intermediate value theorem, there exists $s_1\in(0,s_0)$ such that

$$f(s_1)=\frac{1}{\|w^{\min}+s_1W_l\|_{H^1}}t_-\left(\frac{w^{\min}+s_1W_l}{\|w^{\min}+s_1W_l\|_{H^1}}\right)=1.$$

Thus, $w^{\min}+s_1W_l\in N_\mu^-$ and $I_\mu(w^{\min}+s_1W_l)\geq\beta_\mu^-$. Moreover, by Lemma 4.1, we have $\beta_\mu^-\leq I_\mu(w^{\min}+s_1W_l)<\beta_\mu^++\beta^\infty$.

By the Ekeland variational principle [10] (or [22]), there exists $\{w_n\}\subset N_\mu^-$ such that $I_\mu(w_n)=\beta_\mu^-+O(1)$ and $I'_\mu(w_n)=O(1)$ in $H^*(\mathbb{R}_+^N)$.

Since, $\beta_\mu^-<\beta_\mu^++\beta^\infty$ by Proposition 3.2, there exists a subsequence $\{w_n\}$ and $w_0^*\in H^1(\mathbb{R}_+^N)$ such that $w_n\to w_0^*$ strongly in $H^1(\mathbb{R}_+^N)$ which implies that $w_0^*\in N_\mu^-$ and as $n\to\infty$ then $I_\mu(w_n)$ converges to $I_\mu(w_0^*)=\beta_\mu^-$.

Since $I_\mu(w_0^*)=I_\mu(|w_0^*|)$ and $|w_0^*|\in N_\mu^-$, by Lemma 2.3 we may suppose that $w_0^*\geq 0$. Furthermore, by the Hopf Lemma and the maximum principle, we get that w_0^* is a positive solution of problem (1). □

Now, we are ready for the proof of Theorem 1.1.

Proof of Theorem 1.1. From Theorems 3.3 and 4.2, for $\mu\in(0,\frac{r\Pi_*}{2})$, eq. (1) has two positive solutions $w^{\min}$ and w_0^* such that $w^{\min}\in N_\mu^+$ and $w_0^*\in N_\mu^-$. Since $N_\mu^+\cap N_\mu^-=\emptyset$, then $w^{\min}$ and w_0^* are different. □

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