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ON FUNCTION SPACES WITH CAUCHY CONVERGENCE TOPOLOGY AND $\gamma_{\overline{c}_f}$-COVERS

RITU SEN

Abstract. In this paper, we investigate the countable tightness, the countable strong fan tightness, the countable fan tightness, the strictly Fréchet-Urysohn property and the selectively strictly A-property of the function space of all continuous functions from a metric space X endowed with the Cauchy convergence topology to the real line $\mathbb{R}$. Two new types of covers, namely, $\gamma_{\overline{c}_f}$-covers and $\gamma_{\overline{c}_f}$-shrinkable covers have been introduced. Using $\gamma_{\overline{c}_f}$-shrinkable covers investigations are made to study the selection principles of X and that of $C(X)$ with the Cauchy convergence topology for different collections of subsets $\mathcal{A}$ and $\mathcal{B}$ of X.

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1. INTRODUCTION

Cauchy convergence topology was first introduced and studied by M. H. Clapp and R. C. Shiflett in [5]. The motivation for defining such topology is to compare it with the compact convergence topology and the uniform convergent topology. In [7], Z. Li investigated some cardinal invariants and variants on tightness in the space of all real-valued continuous functions defined on a metric space X endowed with the Cauchy convergence topology. After that in [10], L. X. Peng and Y. Sun pointed out that the Cauchy convergence topology which was discussed by M. H. Clapp, R. C. Shiflett and Z. Li cannot be a topology. In the same paper, the authors then redefine the Cauchy convergence topology and reconsider some conclusions which appear in [5] and [7].

In this paper, we continue the investigations of different types of properties of $C(X)$ ($\equiv$ the family of all real-valued continuous functions on a metric space X) endowed with the Cauchy convergence topology as defined in [10].

Throughout this paper, X is a metric space, d is the metric on X and the topology of X is the induced topology by d.

In Section 2, we give some necessary definitions and results required for a smooth continuation of the paper.

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In Section 3, we investigate the countable tightness, the countable strong fan tightness, the countable fan tightness, the strictly Fréchet-Urysohn property and the selectively strictly A-property of the function space $C(X)$ endowed with the Cauchy convergence topology. Though countable strong fan tightness and countable fan tightness of $C(X)$ with the Cauchy convergence topology have been studied in [7], we here give some equivalent conditions for such properties. Also in this section we introduce a new type of covers, namely $\gamma_{\overline{c}_f}$-covers of a space X.

Section 4 is fully devoted to the study of $\gamma_{\overline{c}_f}$-shrinkable covers of a space X. Investigations are made for studying the selection principles of X and that of $C(X)$ with the Cauchy convergence topology for different collections of subsets $\mathcal{A}$ and $\mathcal{B}$ of X.

Throughout this paper, $\mathbb{R}$ denotes the set of all real numbers with the natural topology τ_d. We will use the symbol $\underline{0}$ to represent the zero-function on a space X. The set of all positive integers is denoted by $\mathbb{N}$. For a subset A of X, $\overline{A}$ denotes the closure of A with respect to the topology of X. Let Y^X denote the set of all functions from X to Y. In notation and terminology we will follow [6]. On some conclusions on function spaces of all real-valued continuous functions on a Tychonoff space X, one can follow [2] and [8]. As the space $C(X)$ is homogeneous, instead of working with any arbitrary function $f \in C(X)$, we can work with the function $\underline{0}$ (i.e. the constant function 0).

2. DEFINITIONS AND PRELIMINARIES

First of all we recall some notations that will be carried out throughout the paper.

Let (X, d) be a metric space. A sequence $S = \{x_n : n \in \mathbb{N}\}$ of points of X is said to be a Cauchy sequence in (X, d) if for any $\epsilon > 0$, there exists a natural number k such that $d(x_n, x_m) < \epsilon$, whenever $n, m \geq k$.

Notation 2.1. Let (X, d) and (Y, ρ) be metric spaces. We denote

$\mathcal{S}(X) = \{S \subset X : S$ is a Cauchy sequence in $(X, d)\}$.

$Y^X = \{f : f$ is a mapping from X to $Y\}$.

$C(X, Y) = \{f \in Y^X : f$ is continuous$\}$. In particular, if Y is the real line $\mathbb{R}$, then we write $C(X)$ instead of $C(X, \mathbb{R})$.

$V(f, A, \epsilon) = \{g \in Y^X : \rho(f(x), g(x)) < \epsilon$, for every $x \in A\}$, where A is a non-empty subset of X and $\epsilon > 0$.

$V^*(f, A, \epsilon) = \{g \in Y^X : \sup\{\rho(f(x), g(x)) : x \in A\} < \epsilon\}$, where A is a non-empty subset of X and $\epsilon > 0$. Similarly, we use the same notations $V^*(f, A, \epsilon)$ in $C(X, Y)$ or $C(X)$.

$\mathcal{D}$ is the family of all dense subsets of $C(X)$ with Cauchy convergence topology.

$\mathcal{S}$ is the family of all sequentially dense subsets of $C(X)$ with Cauchy convergence topology.

Also, for a topological space (X,τ) and a point $x \in X$, we denote

$$\Omega_x = \{A \subseteq X : x \in \overline{A} \setminus A\}.$$

Γ_x is the set of all sequences converging to x.

The following definition appears in [5]. Let (X,d) and (Y,ρ) be metric spaces. Given an element f of Y^X, we choose $\varphi_f = \{V(f,S,\epsilon) : S \in \mathcal{S}(X), \epsilon > 0\}$ as a neighbourhood base at f for the topology of Cauchy convergence. The topology of Cauchy convergence on Y^X is denoted by τ_{ch} in [5], whereas the same has been denoted by $\tau_{\mathrm{ch}(d)}$ in [10]. In this paper, we will use the notation $\tau_{\mathrm{ch}(d)}$ to denote the Cauchy convergence topology.

For a Cauchy sequence $S = \{x_n : n \in \mathbb{N}\}$ of X, we denote

$$\overline{S} = \begin{cases} S, \text{ if } \{x_n : n \in \mathbb{N}\} \text{ is not convergent in } X \\ \{x_0\} \bigcup S, \text{ if } \{x_n : n \in \mathbb{N}\} \text{ converges to } x_0. \end{cases}$$

$\overline{S}$ is said to be a Cauchy closed sequence [7] of X. Each Cauchy closed sequence is a Cauchy sequence. Let $\overline{\mathcal{S}}(X)$ be the family of all the Cauchy closed sequences of X.

Definition 2.2 ([10]). Let (X,d) and (Y,ρ) be metric spaces. Let $\tau_{\mathrm{ch}(d)}$ be a topology on Y^X such that a subset U of Y^X belongs to $\tau_{\mathrm{ch}(d)}$ if and only if for every $f \in U$ there exist $k \in \mathbb{N}$, $S_i \in \mathcal{S}(X)$ for every $i \leq k$ such that $f \in \bigcap_{i=1}^{k} V(f,S_i,\epsilon_i) \subset U$. The topology $\tau_{\mathrm{ch}(d)}$ on Y^X is called the Cauchy convergence topology or the topology of Cauchy convergence.

Proposition 2.3 ([10]). *Let (X,d) and (Y,ρ) be metric spaces. For any $f \in Y^X$, for any $S \in \mathcal{S}(X)$ and for any $\epsilon > 0$, we have*

$$V^*(f,S,\frac{\epsilon}{2}) \subset V(f,S,\frac{\epsilon}{2}) \subset V^*(f,S,\frac{3}{4}\epsilon) \subset V(f,S,\epsilon).$$

Proposition 2.4 ([10]). *Let (X,d) and (Y,ρ) be metric spaces. Then*

$$\varphi^* = \{V^*(f,S,\epsilon) : f \in Y^X, S \in \mathcal{S}(X), \epsilon > 0\}$$

forms a subbase of the Cauchy convergence topology $\tau_{\mathrm{ch}(d)}$ on Y^X.

Proposition 2.5 ([10]). *If X is a metric space, then for any Cauchy sequence S of X and for any $\epsilon > 0$, the following inclusions hold in $C(X)$:*

$$V^*(f,\overline{S},\tfrac{\epsilon}{2}) \subset V^*(f,S,\tfrac{\epsilon}{2}) \subset V^*(f,\overline{S},\epsilon) \subset V^*(f,S,\epsilon).$$

Proposition 2.6 ([10]). *Let (X,d) be a metric space. Then*

$$\overline{\varphi}^* = \{V^*(f,\overline{S},\epsilon) : f \in C(X), \overline{S} \in \overline{\mathcal{S}}(X), \epsilon > 0\}$$

forms a subbase of the Cauchy convergence topology $\tau_{\mathrm{ch}(d)}$ on $C(X)$.

Next, we recall two well known concepts, both defined in 1996 by M. Scheepers [12]. Given an infinite set, let $\mathcal{A}$ and $\mathcal{B}$ be collections of families of subsets of X.

- $S_1(\mathcal{A},\mathcal{B})$ denotes the principle: For any sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$, there is a sequence $\{b_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $b_n \in A_n$ and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.
- $S_{\text{fin}}(\mathcal{A},\mathcal{B})$ denotes the principle: For any sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$, there is a sequence $\{B_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, B_n is a finite subset of A_n and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

3. MAIN RESULTS I

In this section we discuss about countable tightness, countable strong fan tightness, countable fan tightness and selectively strictly A-property of $C(X)$ with the Cauchy convergence topology. We then introduce $\gamma_{\overline{c}_f}$-covers and use them to get some equivalent conditions for $(C(X), \tau_{\text{ch}(d)})$ to be a strictly Fréchet-Urysohn space.

Definition 3.1 ([10]). A family $\mathcal{U}$ of subsets of a space X is called a $\overline{c}_f$-cover of X, if for every finite subcollection $\mathcal{S}'$ of $\overline{\mathcal{S}}(X)$, there exists $U \in \mathcal{U}$ such that $\bigcup \mathcal{S}' \subset U$. If each element of $\mathcal{U}$ is open, then $\mathcal{U}$ is called an open $\overline{c}_f$-cover of X. Also, if $X \notin \mathcal{U}$, then $\mathcal{U}$ is called a proper $\overline{c}_f$-cover of X.

The collection of all $\overline{c}_f$-covers of X will be denoted by $\overline{\mathbb{C}}_f(X)$.

Lemma 3.2. *For a space X, the following hold.*

(i) *Let $\mathcal{U}$ be a proper $\overline{c}_f$-cover of X. Set $A = \{f \in C(X) :$ there exists $U \in \mathcal{U}$ such that $f(x) = 1$, for all $x \in X \setminus U\}$. Then $\underline{0} \in \overline{A} \setminus A$ in $(C(X), \tau_{\text{ch}(d)})$.*

(ii) *Let $A \subset (C(X), \tau_{\text{ch}(d)})$ and let $\mathcal{U} = \{f^{-1}(\frac{-1}{n}, \frac{1}{n}) : f \in A\}$, where $n \in \mathbb{N}$. If $\underline{0} \in \overline{A}$ and $X \notin \mathcal{U}$, then $\mathcal{U}$ is a proper $\overline{c}_f$-cover of X.*

Proof. (i) Since $\mathcal{U}$ is proper, $\underline{0} \notin A$. Let $V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_n\}, \epsilon)$ be any arbitrary basic $\tau_{\text{ch}(d)}$-open neighbourhood of $\underline{0}$, where $\{\overline{S}_1, ..., \overline{S}_n\}$ is a finite subcollection of $\overline{\mathcal{S}}(X)$ and $\epsilon > 0$. As $\mathcal{U}$ is a $\overline{c}_f$-cover of X, there exists $U \in \mathcal{U}$ such that $\bigcup_{i=1}^n \overline{S}_i \subset U$. By the normality of (X, d), there exists $h \in C(X)$ such that $h(x) = 0$, for all $x \in \bigcup_{i=1}^n \overline{S}_i$ and $h(X \setminus U) = \{1\}$. Then $h \in V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_n\}, \epsilon) \cap A$, so that $\underline{0} \in \overline{A} \setminus A$ in $(C(X), \tau_{\text{ch}(d)})$.

(ii) Let $\{\overline{S}_1, ..., \overline{S}_p\}$ be a finite subcollection of $\overline{\mathcal{S}}(X)$. Then

$$V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_p\}, \frac{1}{n})$$

is a basic $\tau_{\text{ch}(d)}$-open neighbourhood of $\underline{0}$, where $n \in \mathbb{N}$. So there exists $g \in V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_p\}, \frac{1}{n}) \cap A$. Thus $g(x) < \frac{1}{n}$, for all $x \in \bigcup_{i=1}^p \overline{S}_i$. Hence $\bigcup_{i=1}^p \overline{S}_i \subset g^{-1}(\frac{-1}{n}, \frac{1}{n})$, where $g \in A$ and $n \in \mathbb{N}$. Thus $\mathcal{U}$ is a proper $\overline{c}_f$-cover of X. ☐

Recall that a space X is said to have countable tightness [3] if for every $x \in X$ and $C \subset X$ such that $x \in \text{cl}(C)$, there is a countable subset C_0 with $C_0 \subset C$ and $x \in \text{cl}(C_0)$.

THEOREM 3.3. *For a space X, the following are equivalent:*

(i) $(C(X), \tau_{\mathrm{ch}(d)})$ *has countable tightness.*

(ii) *Every open $\overline{c}_f$-cover of X has a countable $\overline{c}_f$-subcover.*

Proof. (i) $\Rightarrow$ (ii) Let $\mathcal{U}$ be an open $\overline{c}_f$-cover of X. Then for every finite subcollection $\mathcal{S}' = \{\overline{S}_1, ..., \overline{S}_n\} \subset \overline{\mathcal{S}}(X)$, there exists $U_{\mathcal{S}'} \in \mathcal{U}$ such that $\bigcup_{i=1}^n \overline{S}_i \subset U_{\mathcal{S}'}$. By the normality of X, there exists $f_{\mathcal{S}'} \in C(X)$ such that $f_{\mathcal{S}'}(\bigcup_{i=1}^n \overline{S}_i) = \{0\}$ and $f_{\mathcal{S}'}(X \setminus U_{\mathcal{S}}) = \{1\}$.

Set

$$F = \{f_{\mathcal{S}'} : \mathcal{S}' \text{ is a finite subcollection of } \overline{\mathcal{S}}(X)\}.$$

Clearly $\underline{0} \in \mathrm{cl}(F)$. Now by (i), there exists a countable subset F' of F such that $\underline{0} \in \mathrm{cl}(F')$. Define $\mathcal{V} = \{U_{\mathcal{S}'} : f_{\mathcal{S}'} \in F'\}$. We claim that $\mathcal{V}$ is a $\overline{c}_f$-cover of X. Let $\mathcal{S}_1 = \{\overline{S}_1, ..., \overline{S}_k\}$ be a finite subcollection of $\overline{\mathcal{S}}(X)$ and $V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_k\}, 1)$ be a $\tau_{\mathrm{ch}(d)}$-neighbourhood of $\underline{0}$. Then $F' \cap V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_k\}, 1) \neq \emptyset$. Choose $f_{\mathcal{S}_1} \in F' \cap V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_k\}, 1)$. Then $|f_{\mathcal{S}_1}(x)| < 1$, for all $x \in \bigcup_{i=1}^k \overline{S}_i$. Hence $\bigcup_{i=1}^k \overline{S}_i \subset U_{\mathcal{S}_1} \in \mathcal{V}$.

(ii) $\Rightarrow$ (i) Let $\underline{0} \in C(X)$ and $\mathcal{G}$ be a subset of $C(X)$ such that $\underline{0} \in \mathrm{cl}(\mathcal{G})$. Then for any finite subcollection $\mathcal{S}' = \{\overline{S}_1, ..., \overline{S}_k\} \subset \overline{\mathcal{S}}(X)$, there exists $g_{\mathcal{S}',n} \in V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_k\}, \frac{1}{n}) \cap \mathcal{G}$, for $n \in \mathbb{N}$. Let $W_{\mathcal{S}',n} = \{x \in X : |g_{\mathcal{S}',n}(x)| < \frac{1}{n}\}$, for any $n \in \mathbb{N}$. Then $\bigcup_{i=1}^k \overline{S}_i \subset W_{\mathcal{S}',n}$.

For each $n \in \mathbb{N}$, define

$$\mathcal{W}_n = \{W_{\mathcal{S}',n} : \mathcal{S}' \text{ is a finite subcollection of } \overline{\mathcal{S}}(X)\}.$$

Then $\mathcal{W}_n$ is a $\overline{c}_f$-cover of X, for each $n \in \mathbb{N}$. By (ii), there exists a countable $\mathcal{V}_n \subseteq \mathcal{W}_n$ which is a $\overline{c}_f$-subcover of X, for each $n \in \mathbb{N}$. Let $\mathcal{G}' = \{g_{\mathcal{S}',n} : W_{\mathcal{S}',n} \in \mathcal{V}_n,\ n \in \mathbb{N}\}$. Then $\mathcal{G}' \subseteq \mathcal{G}$ and $\mathcal{G}'$ is countable. We claim that $\underline{0} \in \mathrm{cl}(\mathcal{G}')$. Let $V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_k\}, \frac{1}{n})$ be a $\tau_{\mathrm{ch}(d)}$-neighbourhood of $\underline{0}$, where $\mathcal{S}' = \{\overline{S}_1, ..., \overline{S}_k\}$ is a finite subcollection of $\overline{\mathcal{S}}(X)$ and $n \in \mathbb{N}$. Then $\bigcup_{i=1}^k \overline{S}_i \subset W_{\mathcal{S},n} \in \mathcal{V}_n$. Hence $g_{\mathcal{S}',n} \in V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_k\}, \frac{1}{n}) \cap \mathcal{G}'$. □

We next give some equivalent descriptions of countable strong fan tightness of $(C(X), \tau_{\mathrm{ch}(d)})$. Recall that the i-weight $\mathrm{iw}(X)$ of a space X is the smallest infinite cardinal number κ such that X can be mapped by a one-to-one continuous mapping onto a Tychonoff space of the weight not greater than κ.

Also a space X is said to have countable fan tightness [1, 2], if for each collection $\{A_n : n \in \mathbb{N}\}$ of subsets of X with $x \in \bigcap_{n\in\mathbb{N}} \mathrm{cl}(A_n)$, there exists a finite set $B_n \subset A_n$ such that $x \in \mathrm{cl}(\bigcup_{n\in\mathbb{N}} B_n)$. Also X is said to have countable strong fan tightness [11] if for each $x \in X$ and each sequence $\{A_n : n \in \mathbb{N}\}$ with $x \in \bigcap_{n\in\mathbb{N}} \mathrm{cl}(A_n)$, there exists $x_n \in A_n$ such that $x \in \mathrm{cl}(\{x_n : n \in \mathbb{N}\})$.

Theorem 3.4. *For a Tychonoff space X with* $\mathrm{iw}(X) = \aleph_0$*, the following statements are equivalent:*

(i) $(C(X), \tau_{\mathrm{ch}(d)})$ *has countable strong fan tightness.*

(ii) $(C(X), \tau_{\mathrm{ch}(d)})$ *satisfies* $S_1(\mathcal{D}, \Omega_{\underline{0}})$.

(iii) $(C(X), \tau_{\mathrm{ch}(d)})$ *satisfies* $S_1(\mathcal{D}, \mathcal{D})$.

(iv) $(C(X), \tau_{\mathrm{ch}(d)})$ *satisfies* $S_1(\overline{\mathbb{C}}_f(X), \overline{\mathbb{C}}_f(X))$.

Proof. (i) $\Rightarrow$ (ii) Trivial.

(ii) $\Rightarrow$ (iii) Given a sequence of dense subsets of $C(X)$, we first separate that sequence as a countable collection of sequences of dense subsets of $C(X)$. Let $\{D_{i,j} : i \in \mathbb{N}\}$ be a sequence of dense subsets of $(C(X), \tau_{\mathrm{ch}(d)})$ for each $j \in \mathbb{N}$ and let $D = \{d_j : j \in \mathbb{N}\}$ be a countable dense subset of $(C(X), \tau_{\mathrm{ch}(d)})$. By (ii), for every $j \in \mathbb{N}$ there is a family $\{d_j^i : i \in \mathbb{N}\}$ such that $d_j^i \in D_{i,j}$ and $\{d_j^i : i \in \mathbb{N}\} \in \Omega_{d_j}$. Note that $\{d_j^i : i, j \in \mathbb{N}\} \in \mathcal{D}$.

(iii) $\Rightarrow$ (iv) Let $\mathcal{U}_n \in \overline{\mathbb{C}}_f(X)$ for every $n \in \mathbb{N}$ and let D be a countable dense subset of $(C(X), \tau_{\mathrm{ch}(d)})$. Consider $D_i = \{f_{\mathcal{S}',U,d} \in C(X) : f(X \setminus U) = \{1\}$ and $f(\bigcup \mathcal{S}') = \{d\}$, for a finite subcollection $\mathcal{S}'$ of $\overline{\mathcal{S}}(X)$, $U \in \mathcal{U}_i$ such that $\bigcup \mathcal{S}' \subset U$ and $d \in D\}$. Since D is a dense subset of $(C(X), \tau_{\mathrm{ch}(d)})$, we have that D_i is a dense subset of $(C(X), \tau_{\mathrm{ch}(d)})$ for every $i \in \mathbb{N}$. By (iii), there exists a sequence $\{f_{\mathcal{S}'_i,U_i,d_i} : i \in \mathbb{N}\}$ such that for each i, $f_{\mathcal{S}'_i,U_i,d_i} \in D_i$ and $\{f_{\mathcal{S}'_i,U_i,d_i} : i \in \mathbb{N}\}$ is a dense subset of $(C(X), \tau_{\mathrm{ch}(d)})$. Note that $U_i \in \mathcal{U}_i$ for each $i \in \mathbb{N}$ and $\{U_i : i \in \mathbb{N}\} \in \overline{\mathbb{C}}_f(X)$.

(iv) $\Rightarrow$ (i) By Theorem 3.5 of [7]. □

Similarly we can have the following.

Theorem 3.5. *For a Tychonoff space X with* $\mathrm{iw}(X) = \aleph_0$*, the following statements are equivalent:*

(i) $(C(X), \tau_{\mathrm{ch}(d)})$ *has countable fan tightness.*

(ii) $(C(X), \tau_{\mathrm{ch}(d)})$ *satisfies* $S_{\mathrm{fin}}(\mathcal{D}, \Omega_{\underline{0}})$.

(iii) $(C(X), \tau_{\mathrm{ch}(d)})$ *satisfies* $S_{\mathrm{fin}}(\mathcal{D}, \mathcal{D})$.

(iv) $(C(X), \tau_{\mathrm{ch}(d)})$ *satisfies* $S_{\mathrm{fin}}(\overline{\mathbb{C}}_f(X), \overline{\mathbb{C}}_f(X))$.

Proof. Can be done as before. □

We next introduce the concept of $\gamma_{\overline{c}_f}$-covers of a space X.

Definition 3.6. A countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X is said to be a $\gamma_{\overline{c}_f}$-cover if it is infinite and for each finite subcollection $\mathcal{S}' \subset \overline{\mathcal{S}}(X)$, there exists $n_0 \in \mathbb{N}$ such that $\bigcup \mathcal{S}' \subset U_n$, for all $n \geq n_0$.

The collection of all $\gamma_{\overline{c}_f}$-covers of X will be denoted by $\Gamma_{\overline{\mathbb{C}}_f}$.

Recall here that a space X is said to be strictly Fréchet–Urysohn [4] if it fulfills the selection property $S_1(\Omega_x, \Gamma_x)$, for every $x \in X$.

THEOREM 3.7. *For a space X, the following are equivalent:*

(i) $(C(X), \tau_{\mathrm{ch}(d)})$ *is a strictly Fréchet-Urysohn space.*

(ii) X *satisfies* $S_1(\overline{\mathbb{C}}_f(X), \Gamma_{\overline{\mathbb{C}}_f})$.

(iii) $(C(X), \tau_{\mathrm{ch}(d)})$ *satisfies* $S_1(\mathcal{D}, \Gamma_{\underline{0}})$.

Proof. (i) $\Rightarrow$ (ii) Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $\overline{c}_f$-covers of X. For each finite subcollection $\mathcal{S}'$ of $\overline{\mathcal{S}}(X)$, there exists $U_{\mathcal{S}',n} \in \mathcal{U}_n$ such that $\bigcup \mathcal{S}' \subset U_{\mathcal{S}',n}$, for any $n \in \mathbb{N}$. Set

$$\mathcal{U}_{\mathcal{S}',n} = \{U \in \mathcal{U}_n : \bigcup \mathcal{S}' \subset U, \text{ for any finite subcollection } \mathcal{S}' \text{ of } \overline{\mathcal{S}}(X)\}.$$

For each $n \in \mathbb{N}$, let $A_n = \{f_{\mathcal{S}',U} \in C(X) : \mathcal{S}'$ is a finite subcollection of $\overline{\mathcal{S}}(X), U \in \mathcal{U}_{\mathcal{S}',n}$ and $f_{\mathcal{S}',n}(\bigcup \mathcal{S}') = \{0\}, f_{\mathcal{S}',U}(X \setminus U) = \{1\}\}$. Clearly $\underline{0} \in \mathrm{cl}(A_n)$, for each $n \in \mathbb{N}$. By (i), there exists $f_{\mathcal{S}'_n,U_n} \in A_n$ such that the sequence $\{f_{\mathcal{S}'_n,U_n} : n \in \mathbb{N}\}$ converges to $\underline{0}$ under $\tau_{\mathrm{ch}(d)}$. We claim that $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\overline{c}_f}$-cover of X. Let $V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_k\}, 1)$ be a $\tau_{\mathrm{ch}(d)}$-neighbourhood of $\underline{0}$, where $\{\overline{S}_1, ..., \overline{S}_k\}$ is a finite subcollection of $\overline{\mathcal{S}}(X)$. Then there exists $n_0 \in \mathbb{N}$ such that $f_{\mathcal{S}'_n,U_n} \in V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_k\}, 1)$, for all $n \geq n_0$. Hence for all $n \geq n_0$, $\bigcup_{i=1}^k \overline{S}_i \subset f^{-1}_{\mathcal{S}'_n,U_n}(-1, 1)$. Hence for all $n \geq n_0$, $\bigcup_{i=1}^k \overline{S}_i \subset U_n$.

(ii) $\Rightarrow$ (i) Let $\{A_n : n \in \mathbb{N}\}$ be a sequence of subsets of $C(X)$ such that $\underline{0} \in \bigcap_{n \in \mathbb{N}}(\overline{A}_n \setminus A_n)$. For each $n \in \mathbb{N}$, let $\mathcal{U}_n = \{f^{-1}(\frac{-1}{n}, \frac{1}{n}) : f \in A_n\}$. Let $M = \{n \in \mathbb{N} : X \in \mathcal{U}_n\}$.

If M is infinite, then for any basic open neighbourhood $V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_n\}, \epsilon)$ of $\underline{0}$ with $\epsilon > 0$, there exists $m \in M$ such that $\frac{1}{m} < \epsilon$. Then there exists $g_m \in A_m$ such that $X = g_m^{-1}(\frac{-1}{m}, \frac{1}{m})$, i.e. $g_m(X) \subset (\frac{-1}{m}, \frac{1}{m})$, i.e. $g_m \in V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_n\}, \epsilon)$. Hence $\{g_m : m \in M\}$ converges to $\underline{0}$.

If M is finite, there exists $n_0 \in \mathbb{N}$ such that $\{\mathcal{U}_m : m \geq n_0\}$ is a sequence of open $\overline{c}_f$-covers of X, $g^{-1}((\frac{-1}{m}, \frac{1}{m})) \neq X$, for $g \in A_m$. Thus there exists $U_m \in \mathcal{U}_m$ such that $\{U_m : m \geq n_0\}$ is a $\gamma_{\overline{c}_f}$-cover of X. Then there exists $f_m \in A_m$ such that $U_m = f_m^{-1}((\frac{-1}{m}, \frac{1}{m}))$.

It now suffices to prove that $\{f_m : m \geq n_0\}$ converges to $\underline{0}$.

Let $V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_k\}, \epsilon)$ be a neighbourhood of $\underline{0}$, where $\{\overline{S}_1, ..., \overline{S}_k\}$ is a finite subcollection of $\overline{\mathcal{S}}(X)$ and $\epsilon > 0$. Let $\mathcal{U}_{\mathcal{S}'} = \{U_m : \bigcup \mathcal{S}' \subset U_m, m \geq n_0,$ for any finite subcollection $\mathcal{S}'$ of $\overline{\mathcal{S}}(X)\}$. Then $\mathcal{U}_{\mathcal{S}'} \neq \emptyset$. If $\mathcal{U}_{\mathcal{S}'}$ is finite, let $\mathcal{U}_{\mathcal{S}'} = \{U_{m_j} : j \leq k\}$. For each $j \leq k$, by $U_{m_j} \neq X$, take $x_{m_j} \in X \setminus U_{m_j}$. Then $\{x_{m_j} : j \leq k\} \cup (\bigcup \mathcal{S}') \in \overline{\mathcal{S}}(X)$, so $U_m \cap (\{x_{m_j} : j \leq k\} \cup (\bigcup \mathcal{S}')) = \emptyset$, for all $m \geq n_0$, a contradiction. Hence $\mathcal{U}_{\mathcal{S}'}$ is infinite. Hence there exists $m \geq n_0$ such that $\bigcup \mathcal{S}' \subset U_m = f_m^{-1}((\frac{-1}{m}, \frac{1}{m}))$, $\frac{1}{m} < \epsilon$. Thus $f_m \in V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_k\}, \epsilon)$, for all $m \geq n_0$.

(i) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (ii) Let $\{\mathcal{U}_n : n \in \mathbb{N}\} \subset \overline{\mathbb{C}}_f(X)$ and let $D = \{d_j : j \in \mathbb{N}\}$ be a countable dense subset of $(C(X), \tau_{\mathrm{ch}(d)})$. Consider $D_i = \{f_{\mathcal{S}',U,i,j} \in C(X) : f_{\mathcal{S}',U,i,j}(\bigcup \mathcal{S}') = \{d_j\}, f_{\mathcal{S}',U,i,j}(X \setminus U) = \{1\}$, where $\mathcal{S}'$ is a finite subcollection of $\overline{\mathcal{S}}(X)$ with $\bigcup \mathcal{S}' \subset U \in \mathcal{U}_i\}$, for each $i \in \mathbb{N}$. Since D is a dense subset of $(C(X), \tau_{\mathrm{ch}(d)})$, D_i is a dense subset of $(C(X), \tau_{\mathrm{ch}(d)})$ for every $i \in \mathbb{N}$. By (iii), there exists a set $\{f_{\mathcal{S}'(i),U(i),i,j(i)} : i \in \mathbb{N}\}$ such that $f_{\mathcal{S}'(i),U(i),i,j(i)} \in D_i$ and $\{f_{\mathcal{S}'(i),U(i),i,j(i)} : i \in \mathbb{N}\} \in \Gamma_{\underline{0}}$.

We now prove that $\{U(i) : i \in \mathbb{N}\} \in \Gamma_{\overline{\mathbb{C}}_f}$. Let $V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_k\}, \epsilon)$ be a neighbourhood of $\underline{0}$, where $\{\overline{S}_1, ..., \overline{S}_k\}$ is a finite subcollection of $\overline{\mathcal{S}}(X)$ and $\epsilon > 0$. Since $\{f_{\mathcal{S}'(i),U(i),i,j(i)} : i \in \mathbb{N}\} \in \Gamma_{\underline{0}}$, there exists $i_0 \in \mathbb{N}$ such that $f_{\mathcal{S}'(i),U(i),i,j(i)} \in V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_k\}, \epsilon)$, for every $i > i_0$. It then follows that $\bigcup_{p=1}^k \overline{S}_p \subset U(i)$, for every $i > i_0$ and hence $\{U(i) : i \in \mathbb{N}\} \in \Gamma_{\overline{\mathbb{C}}_f}$. □

Definition 3.8 ([9]). A space X is said to be a selectively strictly A-space (in short, SSA) if for each sequence $\{A_n : n \in \mathbb{N}\}$ of subsets of X and each point $x \in X$ such that $x \in \overline{A_n} \setminus A_n$, for each $n \in \mathbb{N}$, there exists a sequence $\{T_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $T_n \subset A_n$ and $x \in \overline{\bigcup_{n\in\mathbb{N}} T_n} \setminus \bigcup_{n\in\mathbb{N}} \overline{T}_n$.

Theorem 3.9. *For a space X, the following are equivalent:*

(i) $(C(X), \tau_{\mathrm{ch}(d)})$ *is SSA.*

(ii) *For each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of $\overline{c}_f$-covers of X, there is a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ such that $\mathcal{V}_n \subset \mathcal{U}_n$, for each $n \in \mathbb{N}$, no $\mathcal{V}_n$ is a $\overline{c}_f$-cover of X but $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a $\overline{c}_f$-cover of X.*

Proof. (i) $\Rightarrow$ (ii) Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $\overline{c}_f$-covers of X. Then for each finite subcollection $\mathcal{S}' \subset \overline{\mathcal{S}}(X)$, there exists $U \in \mathcal{U}_n$, $n \in \mathbb{N}$ such that $\bigcup \mathcal{S}' \subset U$.

For $n \in \mathbb{N}$, and a finite subcollection $\mathcal{S}' \subset \overline{\mathcal{S}}(X)$, put $\mathcal{U}_{\mathcal{S}',n} = \{U \in \mathcal{U}_n : \bigcup \mathcal{S}' \subset U\}$. For each $U \in \mathcal{U}_{\mathcal{S}',n}$, pick a continuous function $f_{\mathcal{S}',n,U} : X \to [0,1]$ such that $f_{\mathcal{S}',n,U}(\bigcup \mathcal{S}') = \{0\}$ and $f_{\mathcal{S}',n,U}(X \setminus U) = \{1\}$ and let for each $n \in \mathbb{N}$, $A_n = \{f_{\mathcal{S}',n,U} : \mathcal{S}'$ is a finite subcollection of $\overline{\mathcal{S}}(X), U \in \mathcal{U}_{\mathcal{S}',n}\}$. Then $\underline{0} \in \overline{A}_n$, for each $n \in \mathbb{N}$. But $\underline{0} \notin A_n$, for if $\underline{0} \in A_m = f_{\mathcal{S}',m,U}$, for some m, then $f_{\mathcal{S}',m,U}(X \setminus U) = \{0\}$, a contradiction.

Now by (i), there exists a sequence $\{T_n : n \in \mathbb{N}\}$ such that $T_n \subset A_n$, for all n and $\underline{0} \in \overline{\bigcup_{n\in\mathbb{N}} T_n} \setminus \bigcup_{n\in\mathbb{N}} \overline{T}_n$.

For $n \in \mathbb{N}$, denote by $\mathcal{V}_n$, the set of corresponding sets U for each $f_{\mathcal{S}',n,U} \in T_n$. We claim that no $\mathcal{V}_n$ is a $\overline{c}_f$-cover. If not, let $\mathcal{S}' \subset \overline{\mathcal{S}}(X)$ be a finite subcollection of $\overline{\mathcal{S}}(X)$ and consider the neighbourhood $V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_k\}, \epsilon)$ of $\underline{0}$, where $\{\overline{S}_1, ..., \overline{S}_k\}$ is a finite subcollection of $\overline{\mathcal{S}}(X)$ and $\epsilon > 0$. Then there exists $U \in \mathcal{V}_n$ such that $\bigcup_{i=1}^k \overline{S_i} \subset U$. Then for $n \in \mathbb{N}$, $f_{\mathcal{S}',n,U} \in T_n \cap V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_k\}, \epsilon)$, i.e. $\underline{0} \in \overline{T}_n$, a contradiction.

We now prove that $\bigcup \mathcal{V}_n$ is a $\overline{c}_f$-cover of X. Let $\mathcal{S}'$ be a finite subcollection of $\overline{\mathcal{S}}(X)$. As $\underline{0} \in \overline{\bigcup_{n\in\mathbb{N}} T_n}$, there exists $m \in \mathbb{N}$ and $f_{\mathcal{S}',m,U} \in T_m$ such that $f_{\mathcal{S}',m,U}(x) = 0$, for all $x \in \bigcup \mathcal{S}'$, which implies that $\bigcup \mathcal{S}' \subset U \in \mathcal{V}_m \subset \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$.

(ii) $\Rightarrow$ (i) Let $\{A_n : n \in \mathbb{N}\}$ be a sequence of subsets of $(C(X), \tau_{\mathrm{ch}(d)})$ such that for each $n \in \mathbb{N}$, $\underline{0} \in \overline{A}_n \setminus A_n$.

For each $n \in \mathbb{N}$, let $\mathcal{U}_n = \{f^{-1}(\frac{-1}{n}, \frac{1}{n}) : f \in A_n\}$. Then if $X \in \mathcal{U}_n$ for infinitely many n, the conclusion follows trivially. In fact, there exists an increasing sequence $n_1 < n_2 < ... < n_k < ...$ in $\mathbb{N}$ and $f_{n_k} \in A_{n_k}$, $k \in \mathbb{N}$, such that $f_{n_k}^{-1}(\frac{-1}{n_k}, \frac{1}{n_k}) = X$. Put $T_{n_k} = \{f_{n_k}\}, k \in \mathbb{N}$ and $T_n = \emptyset$, for $n \neq n_k$, $k \in \mathbb{N}$. Then $\underline{0} \in \overline{\bigcup_{n\in\mathbb{N}} T_n} \setminus \bigcup_{n\in\mathbb{N}} \overline{T}_n$ (because the sequence $\{f_{n_k} : k \in \mathbb{N}\}$ actually converges to $\underline{0}$)]. So let $X \notin \mathcal{U}_n$, for each n. Clearly $\mathcal{U}_n$ is a $\overline{c}_f$-cover of X, for each n.

By (ii), there exists a sequence $\mathcal{V}_n$ such that $\mathcal{V}_n \subset \mathcal{U}_n$ for each n, no $\mathcal{V}_n$ is a $\overline{c}_f$-cover of X, but $\bigcup \mathcal{V}_n$ is a $\overline{c}_f$-cover of X. Let $T_n \subset A_n$ be such that $\mathcal{V}_n = \{f^{-1}(\frac{-1}{n}, \frac{1}{n}) : f \in T_n\}$, $n \in \mathbb{N}$. Now $\underline{0} \notin \overline{T}_n$, for $n \in \mathbb{N}$, or otherwise $\mathcal{V}_n$ will be a $\overline{c}_f$-cover of X.

We claim that $\underline{0} \in \overline{\bigcup_{n\in\mathbb{N}} T_n}$. Let $\mathcal{S}' = \{\overline{S}_1, ..., \overline{S}_k\}$ be a finite subcollection of $\overline{\mathcal{S}}(X)$ and choose $\epsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \epsilon$. Since $\mathcal{V}_n$ is not a $\overline{c}_f$-cover of X, for $n \geq n_0$, the sets $\bigcup_{n\geq n_0} \mathcal{V}_n$ is a $\overline{c}_f$-cover of X. Hence there exists $n \geq n_0$, $f \in T_n$ such that $\bigcup_{i=1}^k \overline{S}_i \subset f^{-1}(\frac{-1}{n}, \frac{1}{n})$. Hence for all $n \geq n_0$, $f \in V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_k\}, \epsilon) \cap T_n$. □

4. MAIN RESULTS II

In this section we first introduce $\gamma_{\overline{c}_f}$-shrinkable covers of a space X. Then using this covers investigations are made to study the selection principles of X and that of $C(X)$ with the Cauchy convergence topology for different collections of subsets $\mathcal{A}$ and $\mathcal{B}$ of X.

Definition 4.1. A $\gamma_{\overline{c}_f}$-cover $\mathcal{U}$ of cozero sets of X is said to be $\gamma_{\overline{c}_f}$-shrinkable if there exists a $\gamma_{\overline{c}_f}$-cover $\{F(U) : U \in \mathcal{U}\}$ of zero sets of X such that $F(U) \subset U$, for every $U \in \mathcal{U}$.

The collection of all $\gamma_{\overline{c}_f}$-shrinkable covers of X is denoted by $\Gamma^{\mathrm{sh}}_{\overline{c}_f}$.

Theorem 4.2. *For a Tychonoff space X, the following are equivalent:*

(i) $(C(X), \tau_{\mathrm{ch}(d)})$ *satisfies* $S_1(\Gamma_{\underline{0}}, \Gamma_{\underline{0}})$.

(ii) X *satisfies* $S_1(\Gamma^{\mathrm{sh}}_{\overline{c}_f}, \Gamma_{\overline{\mathbb{C}}_f})$.

Proof. (i) $\Rightarrow$ (ii) Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $\gamma_{\overline{c}_f}$-shrinkable covers of X and let $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$, for $n \in \mathbb{N}$. Also let $\{F(U_{n,m}) : U_{n,m} \in \mathcal{U}_n\}$ be a sequence of $\gamma_{\overline{c}_f}$-covers of zero sets of X with $F(U_{n,m}) \subset U_{n,m}$, for every $U_{n,m} \in \mathcal{U}_n$, $n \in \mathbb{N}$. For each $n, m \in \mathbb{N}$, we fix $f_{n,m} \in C(X)$ such that $f_{n,m}(F(U_{n,m})) = \{0\}$ and $f_{n,m}(X \setminus U_{n,m}) = \{1\}$.

For each $n \in \mathbb{N}$, consider $S_n = \{f_{n,m} : m \in \mathbb{N}\}$. We claim that $S_n \in \Gamma_{\underline{0}}$, for each $n \in \mathbb{N}$. Let $V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_k\}, \epsilon)$ be a $\tau_{\mathrm{ch}(d)}$-neighbourhood of $\underline{0}$, where $\{\overline{S}_1, ..., \overline{S}_k\}$ is a finite subcollection of $\overline{\mathcal{S}}(X)$ and $\epsilon > 0$. Then as $\{F(U_{n,m}) : U_{n,m} \in \mathcal{U}_n\}$ is a sequence of $\gamma_{\overline{c}_f}$-covers of X, there exists $n_0 \in \mathbb{N}$ such that $\bigcup_{i=1}^k \overline{S_i} \subset F(U_{n,m})$, for all $n \geq n_0$, i.e. $\bigcup_{i=1}^k \overline{S_i} \subset U_{n,m}$, for all $n \geq n_0$. Then there exists $f_{n,m}$ such that $f_{n,m}(\bigcup_{i=1}^k \overline{S_i}) = \{0\}$ and $f_{n,m}(X \setminus U_{n,m}) = \{1\}$. Clearly $f_{n,m} \in S_n$ and $f_{n,m} \in V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_k\}, \epsilon)$, for all $n \geq n_0$.

By (i), there exists $f_{n,m(n)} \in S_n$, for each $n \in \mathbb{N}$ such that $\{f_{n,m(n)} : n \in \mathbb{N}\} \in \Gamma_{\underline{0}}$.

We now claim that $\{U_{n,m(n)} : n \in \mathbb{N}\}$ is a $\gamma_{\overline{c}_f}$-cover of X. Let $\mathcal{S}' = \{\overline{S}_1, ..., \overline{S}_p\}$ be a finite subcollection of $\overline{\mathcal{S}}(X)$. As $\{f_{n,m(n)} : n \in \mathbb{N}\} \in V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_p\}, 1)$, for all $n \geq n_0$, it follows that $\bigcup_{i=1}^p \overline{S_i} \subset U_{n,m(n)}$, for all $n \geq n_0$. Thus X satisfies $S_1(\Gamma^{\mathrm{sh}}_{\overline{c}_f}, \Gamma_{\overline{\mathbb{C}}_f})$.

(ii) $\Rightarrow$ (i) Let $\{S_n : n \in \mathbb{N}\}$ be such that $S_n \in \Gamma_{\underline{0}}$, for each $n \in \mathbb{N}$ and let for each $n \in \mathbb{N}$, $S_n = \{f_{n,j} : j \in \mathbb{N}\}$. Consider $\mathcal{V}_n = \{f_{n,j}^{-1}((\frac{-1}{n}, \frac{1}{n})) : f_{n,j} \in S_n\}$, for each $n \in \mathbb{N}$. Let $M = \{n \in \mathbb{N} : f_{n,j}^{-1}((\frac{-1}{n}, \frac{1}{n})) = X$ for some $j \in \mathbb{N}\}$.

If M is finite, then we can ignore such finitely many n. If M is infinite, then for some j_n $(n \in M)$, $f_{n,j_n} \to \underline{0}$ uniformly. Thus without loss of generality, we may assume that $f_{n,j}^{-1}((\frac{-1}{n}, \frac{1}{n})) \neq X$ for each $n, j \in \mathbb{N}$. Then $\mathcal{W}_n = \{f_{n,j}^{-1}([\frac{-1}{n+1}, \frac{1}{n+1}]) : f_{n,j} \in S_n\}$ is a $\gamma_{\overline{c}_f}$-cover of zero-sets of X, for each $n \in \mathbb{N}$. Hence $\mathcal{V}_n \in \Gamma^{\mathrm{sh}}_{\overline{c}_f}$, for each $n \in \mathbb{N}$.

By (ii), there is $\{f_{n,j(n)} : n \in \mathbb{N}\}$ such that $\{f_{n,j(n)}^{-1}((\frac{-1}{n}, \frac{1}{n})) : n \in \mathbb{N}\} \in \Gamma_{\overline{\mathbb{C}}_f}$.

We show that $\{f_{n,j(n)} : n \in \mathbb{N}\} \in \Gamma_{\underline{0}}$. Let $V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_k\}, \epsilon)$ be a basic $\tau_{\mathrm{ch}(d)}$-neighbourhood of $\underline{0}$, where $\overline{S}_1, ..., \overline{S}_k \in \overline{\mathcal{S}}(X)$ and $\epsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $\bigcup_{i=1}^k \overline{S_i} \subset f_{n,j(n)}^{-1}((\frac{-1}{n}, \frac{1}{n}))$, for each $n > n_0$. Now there is $n_1 > n_0$ such that $\frac{1}{n_1} < \epsilon$. Thus $f_{n,j(n)} \in V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_k\}, \epsilon)$, for each $n > n_1$. Thus $(C(X), \tau_{\mathrm{ch}(d)})$ satisfies $S_1(\Gamma_{\underline{0}}, \Gamma_{\underline{0}})$. □

Theorem 4.3. *For a Tychonoff space X, the following statements are equivalent:*

(i) *$(C(X), \tau_{\mathrm{ch}(d)})$ satisfies $S_1(\Gamma_{\underline{0}}, \Omega_{\underline{0}})$.*

(ii) *X satisfies $S_1(\Gamma^{\mathrm{sh}}_{\overline{c}_f}, \overline{\mathbb{C}}_f(X))$.*

Proof. (i) $\Rightarrow$ (ii) Let $\{\mathcal{U}_n : n \in \mathbb{N}\} \subset \Gamma^{\mathrm{sh}}_{\overline{c}_f}$ and $\{F(U) : U \in \mathcal{U}_n\}$ be a collection of $\gamma_{\overline{c}_f}$-covers of zero sets of X with $F(U) \subset U$, for every $U \in \mathcal{U}_n$, $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, consider a set $A_n = \{f_{F(U),U,n} \in C(X) : f_{F(U),U,n}(F(U)) = \{0\}$ and $f_{F(U),U,n}(X \setminus U) = \{1\}$, for $U \in \mathcal{U}_n\}$. Since $\{F(U) : U \in \mathcal{U}_n\}$ is a $\gamma_{\overline{c}_f}$-cover of X, we have that A_n converges to $\underline{0}$, for each $n \in \mathbb{N}$.

Since $(C(X),\tau_{\mathrm{ch}(d)})$ satisfies $S_1(\Gamma_{\underline{0}},\Omega_{\underline{0}})$, there is a sequence $\{f_{F(U_n),U_n,n}: n\in\mathbb{N}\}$ such that for each n, $f_{F(U_n),U_n,n}\in A_n$ and $\{f_{F(U_n),U_n,n}:n\in\mathbb{N}\}$ is an element of $\Omega_{\underline{0}}$. Consider $\{U_n:n\in\mathbb{N}\}$. Then $U_n\in\mathcal{U}_n$, for each $n\in\mathbb{N}$.

We claim that $\{U_n:n\in\mathbb{N}\}$ is a $\overline{c}_f$-cover of X. Let $\{\overline{S}_1,...,\overline{S}_k\}$ be a finite subcollection of $\overline{\mathcal{S}}(X)$. Then $V^*(f,\{\overline{S}_1,...,\overline{S}_k\},\frac{1}{2})$ is a $\tau_{\mathrm{ch}(d)}$-basic neighbourhood of f. Then there is $f_{F(U_{n'}),U_{n'},n'}\in V^*(f,\{\overline{S}_1,...,\overline{S}_k\},\frac{1}{2})$, for some $n'\in\mathbb{N}$. Hence $\bigcup_{i=1}^k\overline{S}_i\subset U_{n'}$, $n'\in\mathbb{N}$. Thus X satisfies $S_1(\Gamma^{\mathrm{sh}}_{\overline{c}_f},\overline{\mathbb{C}}_f(X))$.

(ii) $\Rightarrow$ (i) Let $\{f_{k,n}:k\in\mathbb{N}\}$ be a sequence converging to $\underline{0}$, for each $n\in\mathbb{N}$. Let $W_k^n=\{x\in X:\frac{-1}{n}<f_{k,n}(x)<\frac{1}{n}\}\neq X$, for any $n\in\mathbb{N}$ and $S_k^n=\{x\in X:\frac{-1}{n}\le f_{k,n}(x)\le\frac{1}{n}\}\neq X$, for any $n\in\mathbb{N}$, $k\in\mathbb{N}$. Consider $\mathcal{V}_n=\{W_k^n:k\in\mathbb{N}\}$ and $\mathcal{S}_n=\{S_k^n:k\in\mathbb{N}\}$, for each $n\in\mathbb{N}$.

We claim that $\mathcal{V}_n$ is a $\gamma_{\overline{c}_f}$-cover of X, for each $n\in\mathbb{N}$. Since $\{f_{k,n}:k\in\mathbb{N}\}$ converges to $\underline{0}$, for each finite subcollection $\{\overline{S}_1,...\overline{S}_p\}$ of $\overline{\mathcal{S}}(X)$, there is $k_0\in\mathbb{N}$ such that $f_{k,n}\in V^*(f,\{\overline{S}_1,...,\overline{S}_p\},\frac{1}{n})$, for $k>k_0$. It then follows that $\bigcup_{i=1}^p\overline{S}_i\subset W_k^n$, for any $k>k_0$. Since $\mathcal{V}_{n+1}$ is a $\gamma_{\overline{c}_f}$-cover, $\mathcal{S}_{n+1}$ is a $\gamma_{\overline{c}_f}$-cover too.

Also $\mathcal{S}_{n+1}$ is a refinement of the family $\mathcal{V}_n$, hence $\mathcal{V}_n\in\Gamma^{\mathrm{sh}}_{\overline{c}_f}$, for each $n\in\mathbb{N}$. As X satisfies $S_1(\Gamma^{\mathrm{sh}}_{\overline{c}_f},\overline{\mathbb{C}}_f(X))$, there is a sequence $\{W^n_{k(n)}:n\in\mathbb{N}\}$ such that $W^n_{k(n)}\in\mathcal{V}_n$, for each n and $\{W^n_{k(n)}:n\in\mathbb{N}\}$ is an element of $\overline{\mathbb{C}}_f(X)$.

We claim that $f\in\overline{\{f_{k(n),n}:n\in\mathbb{N}\}}$. Let $U=V^*(f,\{\overline{S}_1,...,\overline{S}_n\},\epsilon)$ be a $\tau_{\mathrm{ch}(d)}$-basic neighbourhood of f, where $\epsilon>0$ and $\{\overline{S}_1,...,\overline{S}_n\}$ is a finite subcollection of $\overline{\mathcal{S}}(X)$. Then there exists $n_1\in\mathbb{N}$ such that $\frac{1}{n_1}<\epsilon$ and $\bigcup_{i=1}^n\overline{S}_i\subset W^{n_1}_{k(n_1)}$. Then $f_{k(n_1),n_1}\in V^*(f,\{\overline{S}_1,...,\overline{S}_n\},\epsilon)$. Hence $f\in\overline{\{f_{k(n),n}:n\in\mathbb{N}\}}$, so that $(C(X),\tau_{\mathrm{ch}(d)})$ satisfies $S_1(\Gamma_{\underline{0}},\Omega_{\underline{0}})$. □

Definition 4.4 ([6]). A subset D of a topological space X is said to be sequentially dense in X if for each $x\in X$, there exists a sequence in D converging to x.

Lemma 4.5. *Let $\mathcal{U}=\{U_n:n\in\mathbb{N}\}$ be a $\gamma_{\overline{c}_f}$-shrinkable covers of a space X. Then the set*

$$A=\{f\in C(X):f(X\setminus U)=\{1\},\ \textit{for some}\ n\in\mathbb{N}\}$$

is sequentially dense in $(C(X),\tau_{\mathrm{ch}(d)})$.

Proof. Let $h\in C(X)$. For each $n\in\mathbb{N}$, take $f_n\in C(X)$ such that

$$f_n(F(U_n))=h(F(U_n))\qquad\text{and}\qquad f_n(X\setminus U_n)=\{1\}.$$

Obviously $f_n\in A$, and the sequence $\{f_n:n\in\mathbb{N}\}$ converges to h, because $\{F(U_n):n\in\mathbb{N}\}$ is a $\gamma_{\overline{c}_f}$-cover of X. □

THEOREM 4.6. *For a Tychonoff space X with* $\mathrm{iw}(X) = \aleph_0$, *the following statements are equivalent:*

(i) $(C(X), \tau_{\mathrm{ch}(d)})$ *satisfies* $S_1(\mathcal{S}, \mathcal{D})$.

(ii) $(C(X), \tau_{\mathrm{ch}(d)})$ *satisfies* $S_1(\mathcal{S}, \Omega_{\underline{0}})$.

(iii) $(C(X), \tau_{\mathrm{ch}(d)})$ *satisfies* $S_1(\Gamma_{\underline{0}}, \Omega_{\underline{0}})$.

(iv) X *satisfies* $S_1(\Gamma^{\mathrm{sh}}_{\overline{c}_f}, \overline{\mathbb{C}}_f(X))$.

Proof. (i) $\Rightarrow$ (iv) Let $\{\mathcal{U}_n : n \in \mathbb{N}\} \in \Gamma^{\mathrm{sh}}_{\overline{c}_f}$. Then by Lemma 4.5, $A_n = \{f \in C(X) : f(X \setminus U^n_k) = \{1\}$ for some $U^n_k \in \mathcal{U}_n\}$ is a sequentially dense subset of $(C(X), \tau_{\mathrm{ch}(d)})$, for each $n \in \mathbb{N}$. Now by (i), there exists $\{f_n : n \in \mathbb{N}\}$ such that $f_n \in A_n$, for $n \in \mathbb{N}$ and $\{f_n : n \in \mathbb{N}\} \in \mathcal{D}$. Now consider the sequence $\{U^n_{k(n)} : n \in \mathbb{N}\}$. Then $U^n_{k(n)} \in \mathcal{U}_n$, for $n \in \mathbb{N}$. We claim that $\{U^n_{k(n)} : n \in \mathbb{N}\}$ is a $\overline{c}_f$-cover of X. Let $U = V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_n\}, \epsilon)$ be a $\tau_{\mathrm{ch}(d)}$-basic neighbourhood of $\underline{0}$, where $\epsilon > 0$ and $\{\overline{S}_1, ..., \overline{S}_n\}$ is a finite subcollection of $\overline{\mathcal{S}}(X)$. Then there exists $n_1 \in \mathbb{N}$ such that $f_{n_1} \in V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_n\}, \epsilon)$. Hence $\bigcup_{i=1}^n \overline{S}_i \subset U^{n_1}_{k(n_1)}$.

(iv) $\Rightarrow$ (iii) Let $\{f_{n,m} : m \in \mathbb{N}\}$ converge to $\underline{0}$ for each $n \in \mathbb{N}$. Consider $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\} = \{f^{-1}_{n,m}(\frac{-1}{n}, \frac{1}{n}) : m \in \mathbb{N}\}$, for each $n \in \mathbb{N}$.

Without loss of generality we can assume that a set $U_{n,m} \neq X$, for any $n, m \in \mathbb{N}$, for otherwise there exists a sequence $\{f_{n_k,m_k} : k \in \mathbb{N}\}$ such that $\{f_{n_k,m_k} : k \in \mathbb{N}\}$ uniformly converges to $\underline{0}$ and $\{f_{n_k,m_k} : k \in \mathbb{N}\} \in \Omega_{\underline{0}}$. Also each $\mathcal{U}_n$ is $\gamma_{\overline{c}_f}$-shrinkable, for $n \in \mathbb{N}$.

Now by (iv), there exists a sequence $\{U_{n,m(n)} : n \in \mathbb{N}\}$ such that for each n, $U_{n,m(n)} \in \mathcal{U}_n$ and $\{U_{n,m(n)} : n \in \mathbb{N}\}$ is an element of $\overline{\mathbb{C}}_f(X)$.

We claim that $\underline{0} \in \overline{\{f_{n,m(n)} : n \in \mathbb{N}\}}$. Let $U = V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_n\}, \epsilon)$ be a $\tau_{\mathrm{ch}(d)}$-basic neighbourhood of $\underline{0}$, where $\epsilon > 0$ and $\{\overline{S}_1, ..., \overline{S}_n\}$ is a finite subcollection of $\overline{\mathcal{S}}(X)$. Then there exists $n_1 \in \mathbb{N}$ such that $\frac{1}{n_1} < \epsilon$ and $\bigcup_{i=1}^n \overline{S}_i \subset U_{n_1,m(n_1)}$. Then $f_{n_1,m(n_1)} \in V^*(\underline{0}, \{\overline{S}_1, ..., \overline{S}_n\}, \epsilon)$. Hence $\underline{0} \in \overline{\{f_{n,m(n)} : n \in \mathbb{N}\}}$. Thus $(C(X), \tau_{\mathrm{ch}(d)})$ satisfies $S_1(\Gamma_{\underline{0}}, \Omega_{\underline{0}})$.

(iii) $\Rightarrow$ (ii) Obvious.

(ii) $\Rightarrow$ (i) Let $D = \{d_n : n \in \mathbb{N}\}$ be a dense subspace of $(C(X), \tau_{\mathrm{ch}(d)})$. Given a sequence of sequentially dense subspaces of $(C(X), \tau_{\mathrm{ch}(d)})$, enumerate it as $\{S_{n,m} : n, m \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, pick $d_{n,m} \in S_{n,m}$ so that $d_n \in \overline{\{d_{n,m} : m \in \mathbb{N}\}}$. Then $\{d_{n,m} : n, m \in \mathbb{N}\}$ is dense in $(C(X), \tau_{\mathrm{ch}(d)})$. □

Similarly we can prove the following.

THEOREM 4.7. *For a Tychonoff space X, the following statements are equivalent:*

(i) $(C(X), \tau_{\mathrm{ch}(d)})$ *satisfies* $S_{\mathrm{fin}}(\Gamma_{\underline{0}}, \Omega_{\underline{0}})$.

(ii) X *satisfies* $S_{\mathrm{fin}}(\Gamma^{\mathrm{sh}}_{\overline{c}_f}, \overline{\mathbb{C}}_f(X))$.

THEOREM 4.8. *For a Tychonoff space X with* $\mathrm{iw}(X) = \aleph_0$*, the following statements are equivalent:*

(i) $(C(X), \tau_{\mathrm{ch}(d)})$ *satisfies* $S_{\mathrm{fin}}(\mathcal{S}, \mathcal{D})$.

(ii) $(C(X), \tau_{\mathrm{ch}(d)})$ *satisfies* $S_{\mathrm{fin}}(\mathcal{S}, \Omega_{\underline{0}})$.

(iii) $(C(X), \tau_{\mathrm{ch}(d)})$ *satisfies* $S_{\mathrm{fin}}(\Gamma_{\underline{0}}, \Omega_{\underline{0}})$.

(iv) X *satisfies* $S_{\mathrm{fin}}(\Gamma^{\mathrm{sh}}_{\overline{\mathbb{C}}_f}, \overline{\mathbb{C}}_f(X))$.

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Presidency University
Department of Mathematics
Kolkata, India
E-mail: ritu_sen29@yahoo.co.in
https://orcid.org/0000-0002-3804-8355