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# RESOLVENT DYNAMICAL SYSTEMS AND MIXED GENERAL VARIATIONAL INEQUALITIES

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**Abstract.** In this paper, we introduce and study a new second order resolvent dynamical system associated with a class of mixed general variational inequalities. We suggest some new multi-step iterative methods for solving the mixed variational inequalities using the forward finite difference schemes. These methods include Mann, Ishikawa and Noor iterations as special cases. Convergence analysis is investigated under certain mild conditions. Some special cases are discussed as applications of the results. It is an interesting problem to compare these methods with other techniques for solving mixed general variational inequalities and related optimizations.

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## 1. INTRODUCTION

Variational inequality theory was introduced by Stampacchia [26] and Lions et al. [21] in the early sixties to study obstacle problems in potential theory. This theory provides us with a simple, natural, unified, novel and general framework to study an extensive range of unilateral, obstacle, free, moving and equilibrium problems arising in fluid flow through porous media, elasticity, circuit analysis, transportation, oceanography, operations research, finance, economics, and optimization.

It is known that the minimum of a differentiable convex function on convex sets can be characterized by the variational inequality. It is worth mentioning that the variational inequalities can be viewed as a significant and novel generalization of the variational principles. By variational principles,we mean maximum and minimum problems arising in game theory, mechanics, geometrical optics, general relativity theory, economics, transportation, differential geometry and related areas.

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In fact, the history of variational principles comprises of the following distinct stages. The basic search of solutions of variational problems, led through the work of Euler, Lagrange, Legendre, Jacobi and many others, to develop along the lines of differential and integral equations as well as functional analysis.

The Hamiltonian-Jacobi theory represents a general framework for the mathematical description of the propagation of actions in nature and optimal modeling of control processes in daily life. Using the ideas and techniques of Hamiltonian-Jacobi theory in mechanics, Cartan introduced differential geometry and exterior calculus in the calculus of variations.

Many basic equations of mathematical physics result from variational problems. It is known that the gauge fields theories are a continuation of Einstein's concept of describing physical effects mathematically in terms of differential geometry.

These theories play a fundamental role in the modern theory of elementary particles and are right tool of building up a unified theory of elementary particles, which includes all kind of known interactions. For example, the *Weinberg-Salam* theory unifies weak and electromagnetic interactions. It is also known that the variational formulation of field theories allows for a degree of unification absent their versions in terms of differential equations.

It is amazing that variational inequalities have influenced various areas of pure and applied sciences and are still continue to influence the recent developments, see [3, 4, 11, 13–18, 21, 22, 24–35, 38, 39, 41].

Using novel and innovative techniques and ideas, variational inequalities have been extended and generalized to tackle complicated and complex problems, which occur in various branches of mathematical and engineering sciences. An important and useful generalization is called the mixed variational inequality or the variational inequality of the second kind involving nonlinear terms.

It has been observed that the projection method and its variant forms cannot be used to study the existence of solutions, but it suggests approximation methods for solving the mixed variational inequalities.

If the nonlinear term in the mixed variational inequality is a lower semicontinuous function, then the technique of the resolvent operator is applied, the origin of which can be traced back to Brezis [4]. In this technique, the given operator is decomposed into the sum of two maximal monotone operators, whose resolvent are easier to evaluate than the resolvent of the original operator. Such a method is known as the operator splitting method. This can lead to very efficient methods, since one can treat each part of the original operator independently.

The operator splitting methods and related techniques have been analyzed and studied by many researchers including Glowinski et al [15] and Noor [24].

In the context of the general variational inequalities, Noor [25, 26] has used the projection operator technique to suggest some splitting type methods applying the approach of updating the solution. These three-step methods are also known as Noor's iterations. It is noted that these forward-backward splitting algorithms are similar to those of Glowinski et al. [15], which they suggested by using the Lagrangian technique. It is known that three-step schemes are versatile and efficient. A useful feature of the forward-backward splitting method is that the resolvent step involves the subdifferential of the proper, convex and lower-semicontinuous only and the other part facilitates the problem decomposition.

We remark that, if the nonlinear term in the mixed variational inequality is the indicator function of a closed convex set in the Hilbert space, then these splitting (forward-backward) methods reduce to the projection and extragradient methods for solving the variational inequalities. It has been established [1–3, 5, 8, 15, 20, 31, 33, 34] that Noor iterations and their modified form, perform better than two-step (Ishikawa iteration) and one step method Mann iteration. In recent years, considerable interest has been shown in developing various extensions and generalizations of Noor iterations, both for their own sake and for their applications.

Dupuis and Nagurney [13] introduced and studied the projected dynamical systems associated with variational inequalities using the equivalent fixed point formulation. The novel feature of the projected dynamical system is that its set of stationary points corresponds to the set of the corresponding set of the solutions of the variational inequality problem. It has been shown [13, 22, 26, 29, 30, 38, 39, 41] that these dynamical systems are useful in developing efficient and powerful numerical techniques for solving mixed variational inequalities and their variant forms.

In this paper, we consider a new class of mixed general variational inequalities (MGVI) involving three arbitrary operators.

Using the resolvent operator technique, we establish the equivalence between the mixed general variational inequalities and the fixed point problem. This equivalent fixed point formulation is used to suggest a second order resolvent system, which is in fact a second order boundary value problem. It is shown that the second boundary value problems can be exploited to suggest and analyze multi step methods for finding the approximate solutions of mixed general variational inequalities and related optimization problem. This is a new approach.

Using the finite difference schemes, we suggest and analyze some new multi step iterative methods for solving variational inequalities. Some special cases are also pointed as potential applications of the obtained results. These multi step methods include Mann iteration, Ishikawa iterations and Noor iterations as special cases.

We have only considered theoretical aspects of the suggested methods. It is an interesting problem to implement these methods and to illustrate the their efficiency. Comparison with other methods need further research efforts. The ideas and techniques of this paper may be used to explore other classes of mixed quasi variational inequalities and related optimization problems.

## 2. BASIC DEFINITIONS AND RESULTS

Let $\Omega$ be a set in a real Hilbert space $\mathcal{H}$ with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $\mathtt{T}, g, h : \mathcal{H} \longrightarrow \mathcal{H}$ be nonlinear operators and let $\phi : \mathcal{H} \longrightarrow \mathcal{H}$ be a lower semi-continuous function.

We consider the problem of finding $\mu \in \mathcal{H}$, such that

$$\langle \mathcal{T}\mu + \mu - g(\mu), h(\nu) - \mu\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall \nu \in \mathcal{H}, \tag{1}$$

which is called the mixed general variational inequality involving three operators. This is the main motivation of this paper.

It has been shown [28, 29] that the optimality conditions of the sum of one differentiable nonconvex function and convex function can be characterized via the general variational inequalities of the type (1).

Let $F : H \leftrightarrow R$ be a differentiable convex function and $\phi$ be a lower semi-continuous convex function. If $T = \nabla F$ and $g = I$, then problem (1) is equivalent to finding $u \in H$ such that

$$0 \in \nabla F(u) + \partial\phi(u), \tag{2}$$

which is known as the variational inclusion problem. Problem (2) is nothing else than the convex optimization optimization problem:

$$\min_{u \in H}\{J(u) + \phi(u)\}.$$

### SPECIAL CASES

We now discuss some special cases of general variational inequalities (1).

(i) $h = I$, the identity operator, then problem (1) collapses to the problem of finding $u \in H$ such that

$$\langle \mathcal{T}\mu + \mu - g(\mu), h(\nu) - \mu\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall \nu \in \mathcal{H}, \tag{3}$$

which is called the mixed the general variational inequality, and it was introduced and studied by Noor et al [29].

(ii) If $g = I$, the identity operator, and $\phi$ is the indicator function of a closed convex set $\Omega \subseteq \mathcal{H}$, then problem (1) reduces to finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}\mu + \mu - g(\mu), h(\nu) - \mu\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{4}$$

which is called the general variational inequality, and it was introduced and studied by Noor and Noor 24.

It can be shown [39] that the optimality conditions of the differentiable nonconvex functions can be characterized via the general variational inequalities of the type (4).

(iii) If $g = I$ and $h = I$, then problem (1) reduces to finding $\mu \in \mathcal{H}$, such that

$$\langle \mathcal{T}\mu, \nu - \mu \rangle + \phi(\nu) - \phi(\mu) \geq 0, \quad \forall\, \nu \in \mathcal{H}, \tag{5}$$

which is called the mixed variational inequality.

It has been shown that a wide class of obstacle boundary value and initial value problems can be studied in the general framework of variational inequalities. For the applications, motivation, numerical methods, sensitivity analysis, dynamical system, merit functions and other aspects of variational inequalities, see [3, 11, 13–18, 21, 22, 24–29, 32] and the references therein.

Kikuchi et al [17] have proved that the mixed variational inequality (5) characterizes the Signorini problem with non-local friction.

If $S$ is an open bounded domain in $R^n$ with regular boundary $\partial S$, representing the interior of an elastic body subject to external forces and if a part of the boundary may come into contact with a rigid foundation, then (5) is simply a statement of the virtual work for an elastic body restrained by friction forces, assuming that a non-local law of friction holds. The strain energy of the body corresponding to an admissible displacement $v$ is $\langle Tv, v\rangle$. Thus $\langle Tu, v - u\rangle, \forall \mu, \nu \in \mathcal{H}$ is the work produced by the stresses through strains caused by the virtual displacement $v - u$. The friction forces are represented by the function $\phi(\cdot)$. Similar problems arise in the study of the fluid flow through porous media.

(iv) If $\mu = g(\mu)$, then problem (1) is equivalent to finding $\mu \in \mathcal{H}$, such that

$$\langle T(g(u)), h(\nu) - g(u)\rangle + \phi(\nu) - \phi(g(\mu)) \geq 0, \quad \forall \nu \in \mathcal{H}, \tag{6}$$

which is called the mixed general variational inequalities.

(v) If $g = I, \quad h = I$, and $\Omega^* = \{\mu \in \mathcal{H} : \langle \mu, \nu\rangle \geq 0, \forall \nu \in \Omega\}$, is a polar (dual) cone, then problem (4) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$h(\mu) \in \Omega, \quad \mathcal{T}\mu \in \Omega^*, \quad \langle \mathcal{T}\mu, h(\mu)\rangle = 0, \tag{7}$$

which is called the general complementarity problem, and it was introduced and studied by Noor [25, 26].

For the applications, motivations, generalization, numerical methods and other aspects of the complementarity problems in engineering and applied sciences, see [11, 25, 26, 30] and the references therein.

(vi) If $\Omega = \mathcal{H}$, then problem (4) collapses to finding $\mu \in \mathcal{H}$ such that

$$\langle \rho\mathcal{T}\mu + \mu - g(\mu), h(\nu) - \mu\rangle = 0, \quad \forall \nu \in \mathcal{H}.$$

Consequently, it follows that $\mu \in \mathcal{H}$ satisfies

$$\mu = g(\mu) - \rho \mathcal{T}\mu, \tag{8}$$

which is called the general equation and appears to be a new one.

For a different and appropriate choice of the operators and spaces, one can obtain several known and new classes of mixed variational inequalities and related problems. We would like to point out that problems (1) and (3) are quite different from each other and have different applications. There is a delicate difference between these problems. This clearly shows that problem (1) considered in this paper is more general and it is a unifying one.

We need the following well-known definitions and results in obtaining our results.

Definition 2.1. Let $\mathtt{T} : \mathcal{H} \longrightarrow \mathcal{H}$ be a given mapping.

(i) The mapping $\mathcal{T}$ is called strongly monotone, if there exists a constant $\alpha \geq 0$ such that

$$\left\langle \mathcal{T}\mu - \mathcal{T}\nu\, ,\, \mu - \nu \right\rangle \geq \alpha \, \|\mu - \nu\|^2, \qquad \forall\, \mu,\, \nu \,\in\, \mathcal{H}.$$

(ii) The mapping $\mathcal{T}$ is called monotone, if

$$\left\langle \mathcal{T}\mu - \mathcal{T}\nu\, ,\, \mu - \nu \right\rangle \geq 0, \quad \mu,\, \nu \,\in\, \mathcal{H}.$$

(iii) The mapping $\mathcal{T}$ is called $\eta$-Lipschitz continuous, if there exists a constant $\eta > 0$ such that

$$\|\, \mathcal{T}\mu - \mathcal{T}\nu \,\| \leq \, \eta \, \|\mu - \nu\|, \qquad \forall\, \mu,\, \nu \,\in\, \mathcal{H}.$$

Definition 2.2 ([4,24]). If $T$ is a maximal monotone operator on $\mathcal{H}$, then, for a constant $\rho > 0$, the resolvent operator associated with $T$ is defined by

$$J_T(\mu) = (I + \rho T)^{-1}(\mu), \quad \forall \mu \in \mathcal{H},$$

where $I$ is the identity operator. It is known that a monotone operator $T$ is maximal monotone, if and only if, its resolvent operator $J_T$ is defined everywhere. Furthermore, the resolvent operator $J_T$ is nonexpansive, that is,

$$\|J_T(\mu) - J_T(\nu)\| \leq \|\mu - \nu\|, \quad \forall \mu, \nu \in \mathcal{H}.$$

Remark 2.3. Since the subdifferential $\partial\phi$ of a proper, convex and lower-semicontinuous $\phi : \mathcal{H} \leftrightarrow R \cup \{+\infty\}$ is a maximal monotone operator, we define by

$$J_\varphi \equiv (I + \rho\partial\phi)^{-1},$$

the resolvent operator associated with $\partial\phi$ and $\rho > 0$ is a constant.

We also need the following result, which plays a crucial part in establishing the equivalence between the mixed variational inequalities and the fixed point problem.

LEMMA 2.4 ([4, 24]). *For a given $z \in H$, $u \in H$ satisfies the inequality*

$$\langle \mu - z, \nu - \mu \rangle + \phi(\nu) - \phi(\mu) \geq 0, \quad \forall \nu \in \mathcal{H}, \tag{9}$$

*if and only if,*

$$\mu = J_\phi(z),$$

*where $J_\phi$ is the resolvent operator.*

It is well known that the resolvent operator $J_\phi$ is nonexpansive, that is,

$$\|J_\phi(\mu) - J_\phi(\nu)\| \leq \|\mu - \nu\|, \forall \mu, \nu \in \mathcal{H}.$$

One can prove the equivalence between problem (1) and the fixed point problems using Lemma 2.4. This alternative formulation is used to consider the resolvent dynamical systems associated with the general mixed variational inequalities.

LEMMA 2.5. *The function $\mu \in \mathcal{H}$ is a solution of the general mixed variational inequality* (1)*, if and only if, $\mu \in \mathcal{H}$, satisfies the relation*

$$\mu = J_\varphi[g(\mu) - \rho \mathcal{T} \mu] \tag{10}$$

*where $J_\varphi$ is the resolvent operator and $\rho$ is a constant.*

It is clear that $\mu \in \mathcal{H}$ is the solution of problem (1), if and only if, $\mu \in \mathcal{H}$ satisfies the equation

$$\mathcal{R}(\mu) = J_\varphi[g(\mu) - \rho \mathcal{T} \mu] - \mu = 0, \tag{11}$$

where $\mathcal{R}(\mu)$ is called the residue vector.

## 3. DYNAMICAL SYSTEMS AND ITERATIVE METHODS

In this section, we introduce and investigate some resolvent dynamical system associated with the mixed general variational inequalities. The innovative and novel feature of a projected dynamical system is that its set of stationary points corresponds to the set of solutions of the corresponding variational inequality problem.

It has been shown [13, 22, 26, 29, 30, 38, 39, 41] that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems. We again use these resolvent dynamical systems to suggest and analyze some iterative methods for approximating solutions of the mixed general variational inequalities (1) and their special cases, essentially following the technique of Noor et al [29],

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \lambda\{J_\phi[g(\mu) - \rho T \mu] - \mu\}, \quad \mu(t_0) = \mu_0 \in \mathcal{H}, \tag{12}$$

where $\lambda$ is a parameter.

The system of type (12) is called the resolvent general dynamical system, which is the first order initial value problem. Here the right hand side is related to the projection operator and is discontinuous on the boundary.

It is clear from the definition that the solution to (12) always stays in the constraint set. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution on the given data can be studied. All the basic concepts and results are mainly due to Noor at al. [29].

The equilibrium points of the dynamical system (12) are naturally defined as follows.

Definition 3.1. An element $\mu \in \mathcal{H}, g$ is an equilibrium point of the dynamical system (12), if, $\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0$, that is,

$$J_{\phi}[g(\mu) - \rho T\mu] - \mu = 0,$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the general variational inequality (1), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.

Definition 3.2. The dynamical system is said to converge to the solution set $S^*$ of (12), if, irrespective of the initial point, the trajectory of the dynamical system satisfies

$$\lim_{t\to\infty} \operatorname{dist}(\mu(t), S^*) = 0, \tag{13}$$

where

$$\operatorname{dist}(\mu, S^*) = \inf_{\nu\in S^*} \|\mu - \nu\|.$$

It is easy to see that if the set $S^*$ has a unique point $\mu^*$, then (13) implies that

$$\lim_{t\to\infty} \mu(t) = \mu^*.$$

If the dynamical system is still stable at $\mu^*$ in the Lyapunov sense, then the dynamical system is globally asymptotically stable at $\mu^*$.

Definition 3.3. The dynamical system is said to be globally exponentially stable with degree $\eta$ at $\mu^*$, if, irrespective of the initial point, the trajectory of the system satisfies

$$\|\mu(t) - \mu^*\| \leq \eta_1 \|\mu(t_0) - \mu^*\| \exp(-\eta(t - t_0)), \quad \forall t \geq t_0,$$

where $\mu_1$ and $\eta$ are positive constants independent of the initial point.

It is clear that the globally exponentially stability is necessarily globally asymptotically stable and the dynamical system converges arbitrarily fast.

LEMMA 3.4 (Gronwall lemma [13, 22]). *Let $\hat{\mu}$ and $\hat{\nu}$ be real-valued nonnegative continuous functions with domain $\{t : t \leq t_0\}$ and let $\alpha(t) = \alpha_0(|t - t_0|)$, where $\alpha_0$ is a monotone increasing function. If for $t \geq t_0$,*

$$\hat{\mu} \leq \alpha(t) + \int_{t_0}^{t} \hat{\mu}(s)\hat{\nu}(s)\mathrm{d}s,$$

*then*

$$\hat{\mu}(s) \leq \alpha(t) \exp\left\{\int_{t_0}^{t} \hat{\nu}(s)\mathrm{d}s\right\}.$$

We now show that the trajectory of the solution of the general dynamical system (12) converges to the unique solution of the mixed general variational inequality (1). The analysis is in the spirit of Noor [29] and Xia and Wang [38, 39].

THEOREM 3.5. *Let the operators $T, g : H \longrightarrow H$ be both Lipschitz continuous with constants $\beta > 0$ and $\mu > 0$ respectively. Then, for each $\mu_0 \in \mathcal{H}$, there exists a unique continuous solution $\mu(t)$ of the dynamical system* (12) *with $\mu(t_0) = \mu_0$ over $[t_0, \infty)$.*

*Proof.* Let

$$G(\mu) = \lambda\{J_\phi[g(\mu) - \rho T\mu] - \mu\}.$$

where $\lambda > 0$ is a constant and $G(\mu) = \frac{\mathrm{d}\mu}{\mathrm{d}t}$. $\forall \mu, \nu \in \mathcal{H}$, we have

$$\begin{aligned}
\|G(\mu) - G(\nu)\| &\leq \lambda\{\|J_\phi[g(\mu) - \rho T\mu] - J_\phi[g(\nu) - \rho T\nu]\| + \|\mu - \nu\|\} \\
&\leq \lambda\|\mu - \nu\| + \lambda\|g(\mu) - g(\nu)\| + \lambda\rho\|T\mu - T\nu\| \\
&\leq \lambda\{1 + \mu + \beta\rho\}\|\mu - \nu\|.
\end{aligned}$$

This implies that the operator $G(\mu)$ is a Lipschitz continuous in $\mathcal{H}$, and for each $\mu_0 \in \mathcal{H}$, there exists a unique and continuous solution $\mu(t)$ of the dynamical system (12), defined on an interval $t_0 \leq t < T_1$ with the initial condition $\mu(t_0) = \mu_0$. Let $[t_0, T_1)$ be its maximal interval of existence. Then we have to show that $T_1 = \infty$. Consider , for any $\mu \in \mathcal{H}$,

$$\begin{aligned}
\|G(\mu)\| = \|\frac{\mathrm{d}\mu}{\mathrm{d}t}\| &= \lambda\|J_\phi[g(\mu) - \rho T\mu] - \mu\| \\
&\leq \lambda\{\|J_\phi[g(\mu) - \rho T\mu] - J_\phi[0]\| + \|J_\phi[0] - \mu\|\} \\
&\leq \lambda\{\rho\|T\mu\| + \|J_\phi[u] - J_\phi[0]\| + \|J_\phi[0] - \mu\|\} \\
&\leq \lambda\{(\rho\beta + 1 + \eta)\|\mu\| + \|J_\phi[0]\|\}
\end{aligned}$$

Then

$$\begin{aligned}
\|\mu(t)\| &\leq \|\mu_0\| + \int_{t_0}^{t} \|T\mu(s)\|\mathrm{d}s \\
&\leq (\|\mu_0\| + k_1(t - t_0)) + k_2 \int_{t_0}^{t} \|\mu(s)\|\mathrm{d}s,
\end{aligned}$$

where $k_1 = \lambda\|J_\phi[0]\|$ and $k_2 = \lambda(\rho\beta + 1 + \mu)$. Hence by the Gronwall lemma, Lemma 3.4, we have

$$\|u(t)\| \leq \{\|\mu_0\| + k_1(t - t_0)\}e^{k_2(t-t_0)}, \quad t \in [t_0, T_1).$$

This shows that the solution is bounded on $[t_0, T_1)$. So $T_1 = \infty$. □

THEOREM 3.6. *Let the operators $T, g : \mathcal{H} \longrightarrow H$ be Lipschitz continuous with constants $\beta > 0$ and $\mu > 0$ respectively. If the operator $g : \mathcal{H} \longrightarrow \mathcal{H}$ is strongly monotone with constant $\gamma > 0$ and $\lambda > 0$, then the dynamical system* (12) *converges globally exponentially to the unique solution of the mixed general variational inequality* (1).

*Proof.* Since the operators $T, g$ are both Lipschitz continuous, it follows from Theorem 3.5 that the dynamical system (12) has unique solution $\mu(t)$ over $[t_0, T_1)$ for any fixed $\mu_0 \in \mathcal{H}$. Let $\mu(t)$ be a solution of the initial value problem (12). For a given $\mu^* \in \mathcal{H}$ satisfying (1), consider the Lyapunov function

$$L(\mu) = \lambda\|\mu(t) - \mu^*\|^2, \quad \mu(t) \in \mathcal{H}. \tag{14}$$

From (12) and (14), we have

$$\begin{aligned} \frac{\mathrm{d}L}{\mathrm{d}t} &= 2\lambda\langle\mu(t) - \mu^*, J_\phi[g(\mu(t)) - \rho T\mu(t)] - \mu(t)\rangle \\ &= -2\lambda\langle\mu(t) - \mu^*, \mu(t) - \mu^*\rangle \\ &\quad + 2\lambda\langle\mu(t) - \mu^*, J_\phi[g(\mu(t)) - \rho T\mu(t)] - \mu^*\rangle \\ &\leq -2\lambda\|\mu(t) - \mu^*\|^2 \\ &\quad + 2\lambda\langle\mu(t) - \mu^*, J_\phi[g(\mu(t)) - \rho T\mu(t)] - \mu^*\rangle, \end{aligned} \tag{15}$$

where $u^* \in \mathcal{H}$ is a solution of (1). Thus

$$\mu^* = J_\phi[g(\mu^*) - \rho T\mu^*].$$

Using the Lipschitz continuity of the operators $T, g$, we have

$$\begin{aligned} &\|J_\phi[g(\mu) - \rho T\mu] - J_\phi[g(\mu^*) - \rho T\mu^*]\| \\ &\leq \|g(\mu) - g(\mu^*) - \rho(T\mu - T\mu^*)\| \\ &\leq (\mu + \rho\beta)\|\mu - \mu^*\|. \end{aligned} \tag{16}$$

From (15) and (16), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mu(t) - \mu^*\| \leq 2\alpha\lambda\|\mu(t) - \mu^*\|,$$

where $\alpha = \mu + \rho\beta\lambda$. Thus, for $\lambda = -\lambda_1$, where $\lambda_1$ is a positive constant, we have

$$\|\mu(t) - \mu^*\| \leq \|\mu(t_0) - \mu^*\|e^{-\alpha\lambda_1(t-t_0)},$$

which shows that the trajectory of the solution of the dynamical system (12) converges globally exponentially to the unique solution of the general variational inequality (1). □

We use the resolvent dynamical system (12) to suggest some iterative methods for solving general mixed variational inequalities (1). These methods can be viewed in the sense of Korpelevich [19] and Noor [25, 26] involving the double projection operator.

For simplicity, we take $\lambda = 1$. Thus the dynamical system (12) becomes

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} + \mu = J_\phi[g(\mu) - \rho T\mu], \quad \mu(t_0) = \alpha. \tag{17}$$

We construct the implicit iterative method using the forward difference scheme. Discretizing (12), we have

$$\frac{\mu_{n+1} - \mu_n}{h_1} + \mu_{n+1} = J_\phi[g(\mu_{n+1}) - \rho T\mu_{n+1}], \tag{18}$$

where $h_1 > 0$ is the step size. Now, we can suggest the following implicit iterative method for solving the mixed general variational inequality (1).

ALGORITHM 3.7. *For a given $\mu_0 \in \mathcal{H}$, compute $\mu_{n+1}$ by the iterative scheme*

$$\mu_{n+1} = J_\phi\left[g(\mu_{n+1}) - \rho T\mu_{n+1} - \frac{\mu_{n+1} - \mu_n}{h_1}\right], \quad n = 0, 1, 2, \ldots.$$

This is an implicit method and is quite different from the implicit method of [4]. Using Lemma 2.4, Algorithm 3.7 can be rewritten in the equivalent form as:

ALGORITHM 3.8. *For a given $\mu_0 \in \mathcal{H}$, compute $\mu_{n+1}$ by the iterative scheme*

$$\begin{aligned} &\left\langle \rho T\mu_{n+1} + \mu_{n+1} - g(\mu_{n+1}) + \frac{\mu_{n+1} - \mu_n}{h_1}, H(\nu) - \mu_{n+1} \right\rangle \\ &\quad + \phi(\nu) - \phi(\mu_{n+1}) \geq 0, \quad \forall \nu \in \mathcal{H}. \end{aligned} \tag{19}$$

We now study the convergence analysis of Algorithm 3.7 using the technique of Noor et al [29], which is the subject of the following result.

THEOREM 3.9. *Let $\mu \in \mathcal{H}$ be a solution of mixed general variational inequality* (1). *Let $\mu_{n+1}$ be the approximate solution obtained from* (17). *If $T$ is pseudo gh-monotone, then*

$$\|\mu - \mu_{n+1}\|^2 \leq \|\mu - \mu_n\|^2 - \|\mu_n - \mu_{n+1}\|^2. \tag{20}$$

*Proof.* Let $\mu \in \mathcal{H}$ be a solution of (1). Then

$$\langle \rho T\nu + \nu - g(\nu), h(\nu) - \mu \rangle + \phi(\nu) - \phi(\mu) \geq 0, \quad \forall \nu \in \mathcal{H}, \tag{21}$$

since $T$ is a pseudo $gh$-monotone operator.

Set $\nu = \mu_{n+1}$ in (21), to have

$$\langle \rho T\mu_{n+1} + \mu_{n+1} - g(\mu_{n+1}), h(\mu_{n+1}) - \mu \rangle + \phi(\mu_{n+1}) - \phi(\mu) \geq 0. \tag{22}$$

Take $\nu = \mu$ in equation (19), we have

$$(23) \quad \begin{aligned} &\langle \rho T\mu_{n+1} + u_{n+1} - g(u_{n+1}) + \frac{\mu_{n+1} - \mu_n}{h_1}, h(\mu) - \mu_{n+1}\rangle \\ &+ \phi(\mu) - \phi(u_{n+1}) \geq 0. \end{aligned}$$

From (22) and (23), we have

$$(24) \quad \langle \mu_{n+1} - \mu_n, \mu - \mu_{n+1}\rangle \geq 0.$$

From (24) and using $2\langle a, b\rangle = \|a+b\|^2 - \|a\|^2 - \|b\|^2, \quad \forall a, b \in \mathcal{H}$, we obtain

$$(25) \quad \|\mu_{n+1} - \mu\|^2 \leq \|\mu - \mu_n\|^2 - \|\mu_{n+1} - \mu_n\|^2,$$

which is the required result. □

THEOREM 3.10. *Let $\mu \in \mathcal{H}$ be the solution of the mixed general variational inequality* (1). *Let $\mu_{n+1}$ be the approximate solution obtained from* (17). *If $T$ is a pseudo $g$-monotone operator, then $\mu_{n+1}$ converges to $\mu \in \mathcal{H}$ satisfying* (1).

*Proof.* Let $T$ be a pseudo $g$-monotone operator. Then, from (20), it follows the sequence $\{\mu_i\}_{i=1}^{\infty}$ is a bounded sequence and

$$\sum_{i=1}^{\infty} \|\mu_n - \mu_{n+1}\|^2 \leq \|\mu - \mu_0\|^2,$$

which implies that

$$(26) \quad \lim_{n\to\infty} \|\mu_{n+1} - \mu_n\|^2 = 0.$$

Since sequence $\{u_i\}_{i=1}^{\infty}$ is bounded, there exists a cluster point $\hat{\mu}$ to which the subsequence $\{u_{ik}\}_{k=1}^{\infty}$ converges. Taking limit in (19) and using (26), it follows that $\hat{\mu} \in \mathcal{H}$ satisfies

$$\langle T\hat{\mu} + \hat{\mu} - g(\hat{\mu}), h(\nu) - \hat{\mu}\rangle + \phi(\nu) - \phi(\hat{u}) \geq 0, \quad \forall \nu \in \mathcal{H},$$

and

$$\|\mu_{n+1} - \mu\|^2 \leq \|\mu - \mu_n\|^2.$$

Using this inequality, one can show that the cluster point $\hat{u}$ is unique and $\lim_{n\to\infty} \mu_{n+1} = \hat{\mu}$. □

We now suggest an other implicit iterative method for solving (1). Discretizing (12), we have

$$(27) \quad \frac{\mu_{n+1} - \mu_n}{h_1} + \mu_n = J_\phi[g(\mu_{n+1}) - \rho T\mu_{n+1}],$$

where $h_1$ is the step size.

For $h_1 = 1$, this formulation enable us to suggest the following iterative method.

Algorithm 3.11. *For a given $\mu_0 \in \mathcal{H}$, compute $\mu_{n+1}$ by the iterative scheme*

$$\mu_{n+1} = J_\phi\Big[g(\mu_{n+1}) - \rho T\mu_{n+1}\Big], \quad n = 0, 1, 2, \ldots.$$

Using Lemma 2.4, Algorithm 3.11 can be rewritten in the equivalent form as follows.

Algorithm 3.12. *For a given $\mu_0 \in \mathcal{H}$, compute $\mu_{n+1}$ by the iterative scheme*

$$\langle \rho T\mu_{n+1} + \mu_{n+1} - g(\mu_{n+1}), h(\nu) - \mu_{n+1}\rangle + \phi(\nu) - \phi(\mu_{n+1}) \geq 0, \quad \forall \nu \in \mathcal{H}.$$

Remark 3.13. For appropriate and suitable choice of the discretization of (12), one can suggest and analyze several new classes of iterative methods for solving mixed general variational inequalities. This is an interesting problem for future research.

We now introduce the second order resolvent dynamical system associated with the variational inequality (1), which is the main aim of this paper. To be more precise, we consider the second order boundary value problem of finding $\mu \in \mathtt{H}$ such that

$$\gamma\frac{\mathrm{d}^2\mu}{\mathrm{d}x^2} + \frac{\mathrm{d}\mu}{\mathrm{d}x} = \lambda\big\{J_\phi[g(\mu) - \rho\mathcal{T}\mu] - \mu]\big\}, \quad \mu(a) = \alpha, \quad \mu(b) = \beta, \tag{28}$$

where $\gamma > 0, \lambda > 0$ and $\rho > 0$ are constants.

The equilibrium point of the dynamical system (28) is naturally defined as follows.

Definition 3.14. An element $\mu \in \mathcal{H}$ is an equilibrium point of the dynamical system (28), if $\gamma\frac{\mathrm{d}^2\mu}{\mathrm{d}x^2} + \frac{\mathrm{d}\mu}{\mathrm{d}x} = 0$, that is,

$$\mu = J_\phi[g(\mu) - \rho\mathcal{T}\mu].$$

Consequently, from (28), we have

$$\mu = J_\phi\big[g(\mu) - \rho\mathcal{T}\mu + \gamma\frac{\mathrm{d}^2\mu}{\mathrm{d}x^2} + \frac{\mathrm{d}\mu}{\mathrm{d}x}\big]. \tag{29}$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the variational inequality (1), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.

For simplicity, we take $\lambda = 1$. Thus the problem (28) is equivalent to finding $\mu \in \phi$ such that

$$\gamma\ddot{\mu} + \dot{\mu} + \mu = J_\phi\big[g(\mu) - \rho\mathcal{T}\mu\big], \quad \mu(a) = \alpha, \quad \mu(b) = \beta. \tag{30}$$

Problem (30) is called the resolvent dynamical system, which is a second order boundary value problem. This interlink can be used to explore the ideas and techniques of the differential equations to suggest and propose hybrid proximal point methods for solving the variational inequalities and related optimization problems.

We discretize the second-order dynamical systems (30) using the central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1}-2\mu_n+\mu_{n-1}}{h_1^2}+\frac{\mu_n-\mu_{n-1}}{h_1}+\mu_n=J_\phi[g(\mu_n)-\rho(\mathcal{T}\mu_{n+1}], \tag{31}$$

where $h_1$ is the step size.

If $\gamma=1$, $h_1=1$, then, from equation (31) we have the following algorithm.

ALGORITHM 3.15. *For a given $\mu_0\in\mathcal{H}$, compute $\mu_{n+1}$ by the iterative scheme*

$$\mu_{n+1}=J_\phi[g(\mu_n)-\rho\mathcal{T}\mu_{n+1}].$$

Algorithm 3.15 is an implicit method. To implement the implicit method, we use the predictor-corrector technique to suggest the method.

ALGORITHM 3.16. *For given $\mu_0,\mu_1\in\mathcal{H}$, compute $\mu_{n+1}$ by the iterative scheme*

$$\begin{aligned} y_n &= J_\phi[g(\mu_n)-\rho\mathcal{T}\mu_n] \\ \mu_{n+1} &= J_\phi[g(\mu_n)-\rho\mathcal{T}y_n]. \end{aligned}$$

*This is called the extraresolvent method for solving the mixed general variational inequalities.*

Problem (30) can be rewritten as

$$\begin{aligned} &\gamma\frac{\mathrm{d}^2\mu}{\mathrm{d}x^2}+\frac{\mathrm{d}\mu}{\mathrm{d}x}=J_\phi[g((1-\theta_n)\mu+\theta_n\mu)-\rho\mathcal{T}((1-\theta_n)\mu+\theta_n\mu)], \\ &\mu(a)=\alpha, \qquad \mu(b)=\beta, \end{aligned} \tag{32}$$

where $\gamma>0,\theta_n$ and $\rho>0$ are constants.

Discretising the system (32), we have

$$\begin{aligned} &\gamma\frac{\mu_{n+1}-2\mu_n+\mu_{n-1}}{h_1^2}+\frac{\mu_{n+1}-\mu_n}{h_1}+\mu_n \\ &=J_\phi[g((1-\theta_n)\mu_n+\theta_n\mu_{n-1})-\rho\mathcal{T}((1-\theta_n)\mu_n+\theta_n\mu_{n-1})] \end{aligned}$$

from which, for $\gamma=0$, $h_1=1$, we have the following algorithm.

ALGORITHM 3.17. *For a given $\mu_0,\mu_1\in\mathcal{H}$, compute $\mu_{n+1}$ by the iterative scheme*

$$\mu_{n+1}=J_\phi[g((1-\theta_n)u_n+\theta_n\mu_{n-1})-\rho\mathcal{T}((1-\theta_n)\mu_n+\theta_n\mu_{n-1})].$$

Using the predictor corrector technique, Algorithm 3.17 can be written as follows.

ALGORITHM 3.18. *For a given $\mu_0, \mu_1 \in \mathcal{H}$, compute $\mu_{n+1}$ by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta_n \mu_{n-1} \\ \mu_{n+1} &= J_\phi[g(y_n) - \rho\mathcal{T} y_n], \end{aligned}$$

*which is called the new two step inertial iterative method for solving the variational inequalities.*

We discretize the second-order resolvent dynamical systems (30) using the central finite difference and backward difference schemes to have

$$\gamma \frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h_1^2} + \frac{\mu_n - \mu_{n-1}}{h_1} + \mu_{n+1} = J_\phi[g(\mu_n) - \rho\mathcal{T}\mu_{n+1}],$$

where $h_1$ is the step size. Using this discrete form, we can suggest the following iterative method for solving the mixed general variational inequalities (1).

ALGORITHM 3.19. *For given $\mu_0, \mu_1 \in \mathcal{H}$, compute $\mu_{n+1}$ by the iterative scheme*

$$\mu_{n+1} = J_\phi\left[g(\mu_n) - \rho\mathcal{T}\mu_{n+1} - \frac{\gamma\mu_{n+1} - (2\gamma - h_1)\mu_n + (\gamma - h)\mu_{n-1}}{h_1^2}\right].$$

Algorithm 3.19 is called the inertial proximal method for solving the mixed general variational inequalities and related optimization problems. This is a new proposed method. We can rewrite Algorithm 3.19 in the equivalent form as follows.

ALGORITHM 3.20. *For a given $\mu_0, \mu_1 \in \mathcal{H}$, compute $\mu_{n+1}$ by the iterative scheme*

$$\begin{aligned} &\left\langle \rho\mathcal{T}\mu_{n+1} + \frac{(\gamma + h_1^2)\mu_{n+1} - (2\gamma - h_1)\mu_n + (\gamma - h_1)\mu_{n-1}}{h_1^2} - g(\mu_n), \right. \\ &\left. h(\nu) - \nu_{n+1} \right\rangle + \rho(\phi(\nu) - \phi(\mu_{n+1})) \geq 0, \qquad \forall \nu \in \mathcal{H}. \end{aligned}$$

We note that, for $\gamma = 0, h_1 = 1$, Algorithm 3.20 reduces to the following iterative method for solving variational inequalities (1).

ALGORITHM 3.21. *For given $\mu_0, \mu_1 \in \mathcal{H}$, compute $\mu_{n+1}$ by the iterative scheme*

$$\mu_{n+1} = J_\phi[g(\mu_n) + (\mu_n - \mu_{n-1}) - \rho\mathcal{T}\mu_{n+1})].$$

We again discretize the second-order resolvent dynamical systems (30) using the central difference scheme and the forward difference scheme to suggest the following inertial proximal method for solving (1).

ALGORITHM 3.22. *For a given $\mu_0, \mu_1 \in \mathcal{H}$, compute $\mu_{n+1}$ by the iterative scheme*

$$\mu_{n+1} = J_\phi\left[g(\mu_{n+1}) - \rho\mathcal{T}\mu_{n+1} - \frac{(\gamma + h_1)\mu_{n+1} - (2\gamma + h_1)\mu_n + \gamma\mu_{n-1}}{h_1^2}\right].$$

Algorithm 3.22 is quite different from other inertial proximal methods for solving the variational inequalities. If $\gamma = 0$, then Algorithm 3.22 collapses to the following algorithm.

Algorithm 3.23. *For a given* $\mu_0 \in \mathcal{H}$, *compute* $\mu_{n+1}$ *by the iterative scheme*

$$\mu_{n+1} = J_\phi \left[ g(\mu_{n+1}) - \rho \mathcal{T} \mu_{n+1} - \frac{\mu_{n+1} - \mu_n}{h_1} \right].$$

Algorithm 3.22 is a proximal method for solving the variational inequalities. Such type of proximal methods were suggested by Noor [29] using the fixed point problems. Briefly, by suitable discretization of the second-order dynamical systems (30), one can construct a wide class of explicit and implicit methods for solving inequalities. Rewriting problem (30) in the following form

$$\gamma \frac{\mathrm{d}^2\mu}{\mathrm{d}x^2} + \frac{\mathrm{d}\mu}{\mathrm{d}x} + \mu = J_\phi \left[ g\left(\frac{\mu + \mu}{2}\right) - \rho \mathcal{T} \left(\frac{\mu + \mu}{2}\right) \right], \tag{33}$$

and discretizing, taking $\lambda = 1, h_1 = 1$, we obtain the following algorithm.

Algorithm 3.24. *For given* $\mu_0 \in \mathcal{H}$, *compute the approximate solution* $u_{n+1}$ *by the iterative scheme*

$$\mu_{n+1} = J_\phi \left[ g\left(\frac{\mu_n + \mu_{n+1}}{2}\right) - \rho \mathcal{T} \left(\frac{\mu_n + \mu_{n+1}}{2}\right) \right],$$

*which is an implicit iterative method.*

Using the predictor and corrector technique, we suggest the following two-step iterative method for solving the variational inequalities.

Algorithm 3.25. *For given* $\mu_0 \in \mathcal{H}$, *compute the approximate solution* $\mu_{n+1}$ *by the iterative scheme*

$$\begin{aligned} y_n &= J_\phi \big[ g(\mu_n) - \rho \mathcal{T} \mu_n \big] \\ \mu_{n+1} &= J_\phi \left[ g\left(\frac{\mu_n + y_n}{2}\right) - \rho \mathcal{T} \left(\frac{\mu_n + y_n}{2}\right) \right]. \end{aligned}$$

Algorithm 3.25 is a two step iterative method. Clearly Algorithm 3.24 and Algorithm 3.25 are equivalent. Using the ideas and techniques of Noor et al. [29] one can prove the convergence of Algorithm 3.24. However, for the sake of completeness and to convey the main ideas, we include all the details.

Theorem 3.26. *Let the operators* $T, g$ *be Lipschitz continuous with constants* $\beta > 0, \sigma > 0$, *respectively. Let* $u \in \mathcal{H}$ *be solution of* (1) *and* $\mu_{n+1}$ *be an approximate solution obtained from Algorithm* 3.24. *If there exists a constant* $\rho > 0$, *such that*

$$\rho < \frac{1 - \sigma}{\beta}, \quad \sigma < 1, \tag{34}$$

*then the approximate solution* $\mu_{n+1}$ *converges to the exact solution* $\mu \in \Omega$.

*Proof.* Let $\mu \in \mathcal{H}$ be a solution of (1) and $\mu_{n+1}$ be the approximate solution obtained from Algorithm 3.24. Then, using the Lipschitz continuity of the operators $T$ and $g$ with constants $\beta, \sigma$, we obtain

$$\begin{aligned}
\|\mu_{n+1} - \mu\| &= \Big\| J_\phi \left[ g\left(\frac{\mu_n + \mu_{n+1}}{2}\right) - \rho T\left(\frac{\mu_n + \mu_{n+1}}{2}\right)\right] \\
&\qquad - J_\phi \left[ g\left(\frac{\mu + \mu}{2}\right) - \rho T\left(\frac{\mu + \mu}{2}\right)\right] \Big\| \\
&\leq \left\| g\left(\frac{\mu_n + \mu_{n+1}}{2}\right) - g\left(\frac{\mu + \mu}{2}\right) - \rho\left(T\left(\frac{\mu_{n+1} + \mu_n}{2}\right) - T\left(\frac{\mu + \mu}{2}\right)\right)\right\| \\
&\leq \left\| g\left(\frac{\mu_n + \mu_{n+1}}{2}\right) - g\left(\frac{\mu + \mu}{2}\right) + \rho\left(T\left(\frac{\mu_{n+1} + \mu_n}{2}\right) - T\left(\frac{\mu + \mu}{2}\right)\right)\right\| \\
&\leq (\sigma + \rho\beta) \left\| \left(\frac{\mu_n + \mu_{n+1}}{2}\right) - \left(\frac{\mu + \mu}{2}\right)\right\| \\
&\leq \frac{\sigma + \rho\beta}{2} \left\{ \|\mu_{n+1} - \mu\| + \|\mu_n - \mu\| \right\},
\end{aligned}$$

from which, we obtain

$$\|\mu_{n+1} - \mu\| \leq \frac{\sigma + \rho\beta}{2 - \sigma - \rho\beta} \|\mu_n - \mu\| = \theta \|\mu_n - \mu\|,$$

where $\theta = \frac{\sigma + \rho\beta}{2 - \sigma - \rho\beta}$.

From (34), we have that $\theta < 1$. This shows that the approximate solution $\mu_{n+1}$ obtained from Algorithm 3.24 converges to the exact solution $\mu \in \mathcal{H}$ satisfying the general variational inequality (1). □

To implement the implicit Algorithm 3.24, one uses the predictor-corrector technique. Thus, we obtain new multi step method for solving variational inequalities.

ALGORITHM 3.27. *For given $\mu_0 \in \Omega$, compute the approximate solution $u_{n+1}$ by the iterative schemes*

$$\begin{aligned}
y_n &= (1 - \alpha_n)\mu_n + \alpha_n J_\phi \left[ g(\mu_n) - \rho \mathcal{T} \mu_n \right] \\
w_n &= (1 - \eta_n) y_n + \eta_n J_\phi \left[ g\left(\frac{\mu_n + y_n}{2}\right) - \rho \mathcal{T}\left(\frac{\mu_n + y_n}{2}\right)\right] \\
\mu_{n+1} &= (1 - \beta_n) w_n + \beta_n J_\phi \left[ g\left(\frac{w_n + y_n}{2}\right) - \rho \mathcal{T}\left(\frac{w_n + y_n}{2}\right)\right],
\end{aligned}$$

which is a three step method, where $\alpha_n, \eta_n, \beta_n$ are constants.

Algorithm 3.28. *For given $\mu_0, \mu_1 \in \mathcal{H}$, compute the approximate solution $u_{n+1}$ by the iterative schemes*

$$\begin{aligned}
t_n &= (1-\theta_n)\mu_n + \theta_n \mu_{n-1} \\
y_n &= (1-\alpha_n)t_n + \alpha_n J_\phi\left[g\left(\frac{\mu_n + t_n}{2}\right) - \rho\mathcal{T}\left(\frac{\mu_n + t_n}{2}\right)\right] \\
w_n &= (1-\beta_n)y_n + \beta_n J_\phi\left[g\left(\frac{t_n + y_n}{2}\right) - \rho\mathcal{T}\left(\frac{t_n + y_n}{2}\right)\right] \\
\mu_{n+1} &= (1-\zeta_n)w_n + \zeta_n J_\phi\left[g\left(\frac{w_n + y_n + t_n}{3}\right) - \rho\mathcal{T}\left(\frac{w_n + y_n + t_n}{3}\right)\right],
\end{aligned}$$

*which is a four step inertial iterative method, where $\theta_n, \alpha_n, \beta_n, \zeta_n$ are constants.*

Remark 3.29. These multi-step methods contain Mann (one-step) iterations, Ishikawa (two-step) iterations and Noor (three-step) iterations as special cases. Noor [25, 26] has proposed and suggested three step forward-backward iterative methods for approximating solutions of general variational inequalities using the technique of updating the solution and the auxiliary principle.

These three-step methods are known as Noor iterations, and have been used to consider some novel forward-backward algorithms for optimization, with applications to compressive sensing and image inpainting, see [1,2,5–7,9, 10,12,20,23,36,37,40]. In this paper, we have shown that these multi step can be proposed and suggested using dynamical systems coupled with boundary value problems, which can be considered as an entirely new approach.

Zeng et al. [41] have investigated the fractional dynamical systems associated with variational inequalities. They have investigated the criteria for the asymptotic stability of the equilibrium points. We would like to point out that our results are more general than the results of Zeng et al. [41]. These ideas and techniques may inspire further research in this area.

We now suggest a new fractional resolvent dynamical system associated with mixed general variational inequalities:

$$\begin{aligned}
&D_t^\alpha \mu = \gamma\{-R(\mu) - \rho T J_\phi[g(\mu) - \rho\mathcal{T}\mu] + \rho\mathcal{T}\mu\}, \\
&\mu(0) = \eta, \quad \mu \in H,
\end{aligned} \tag{35}$$

where $0 < \alpha < 1$ and $\gamma, \eta$ are constants associated with general mixed variational inequality. For more applications and motivation, see [29].

For $\alpha = 1$, problem (35) reduces to finding $u \in H$ such that

$$\frac{\mathrm{d}\mu}{\mathrm{d}x} = \gamma\{-R(\mu) - \rho T J_\phi[g(\mu) - \rho\mathcal{T}u] + \rho\mathcal{T}\mu\}, \quad \mu(0) = \eta, \quad \mu \in \mathcal{H},$$

which is called the resolvent dynamical system, and it appears to be a new one.

Using the techniques of this section, one can investigate the asymptotic stability, iterative methods and other aspects of the second order fractional dynamical systems.

### 4. GENERALIZATIONS AND FUTURE RESEARCH

We would like to mention that some of the results obtained and presented in this paper can be extended for mixed multivalued variational inequalities.

To be more precise, let $C(H)$ be a family of nonempty compact subsets of $H$. Let $T, V : H \longrightarrow C(H)$ be multivalued operators. For a given nonlinear bifunction $N(\cdot,\cdot) : H \times H \longrightarrow H$, consider the problem of finding $u \in \mathcal{H}, w \in T(\mu), y \in V(\mu)$ such that

$$\langle N(w,y) + \mu - g(\mu), h(\nu) - \mu\rangle + \phi(\nu) - \phi(\mu) \geq 0, \quad \forall \nu \in \mathcal{H}, \tag{36}$$

which is called the multivalued mixed general variational inequality.

We would like to mention that one can obtain various classes of variational inequalities for appropriate and suitable choices of the bifunction $N(\cdot,\cdot)$, and the operators.

(i) For $g(\mu) = \mu$, the problem (36) reduces to finding $\mu \in \mathcal{H}$, such that

$$\langle N(w,y), h(\nu) - \mu\rangle + \phi(\nu) - \phi(g(\mu)) \geq 0, \quad \forall \quad \nu \in \mathcal{H}, \tag{37}$$

which is called the multivalued mixed general variational inequality, and which appears to be a new one.

(ii) If $N(w,y) = T\mu$, then the problem (36) is equivalent to finding $\mu \in \mathcal{H}$, such that

$$\langle T\mu + \mu - g(\mu), h(\nu) - \mu\rangle + \phi(\nu) - \phi(\mu) \geq 0 \quad \forall \nu \in \mathcal{H},$$

which is the mixed general variational inequality and also appears to be a new one.

Using Lemma 2.5, one can prove that problem (36) is equivalent to finding $u \in \mathcal{H}$ such that

$$\mu = J_\phi[g(\mu) - \rho N(w,y)]. \tag{38}$$

This shows that problem (36) is equivalent to the fixed point problem (38).

This equivalent formulation can be applied to consider the second order resolvent dynamical system associated with problem (36) as

$$\gamma \frac{\mathrm{d}^2\mu}{\mathrm{d}x^2} + \frac{\mathrm{d}\mu}{\mathrm{d}x} + \mu = J_\phi[g(\mu) - \rho N(w,y)], \quad \mu(a) = \alpha, \quad \mu(b) = \beta,$$

which is the second order boundary value problem and may be applied to suggest and investigate proximal point methods for solving the multivalued mixed variational inequality (36) by applying the techniques developed in this paper.

Consequently, all results obtained for the problem (1) continue to hold for problem (36), with suitable modifications and adjustments.

The development of efficient implementable numerical methods for solving the multivalued mixed general variational inequalities, random elastic traffic equilibrium problems and optimization problems requires further efforts.

Despite current research activities, very few results are available. The development of efficient implementable numerical methods for solving the mixed general variational inequalities and non optimization problems requires further efforts.

## 5. CONCLUSION

In this paper, we have used the technique of the dynamical systems coupled with the second order boundary value problem to suggest some multi step inertial proximal methods for solving variational inequalities.

The convergence analysis of these methods has been considered under some weaker conditions. Our method of convergence criteria is very simple as compared with other techniques.

Comparison and implementation of these new methods need further efforts. We have only discussed the theoretical aspects of the proposed iterative methods.

It is an interesting problem to discuss the implementation and performance of these new methods with other methods.

Applications of the fuzzy set theory, stochastic, quantum calculus, fractal, fractional and random can be found in many branches of mathematical and engineering sciences including artificial intelligence, computer science, control engineering, management science, operations research and variational inequalities.

For the novel and innovative applications of Noor iterations and Noor orbits, see [1, 2, 5–7, 9, 10, 12, 20, 23, 36, 37, 40] and the references therein.

Similar methods can be suggested for stochastic, fuzzy, quantum, random and fractional variational inequalities, which is an interesting and challenging problem. Despite recent research activities, very few results are available. The development of efficient numerical methods requires further efforts. The ideas and techniques presented in this paper may be a starting point for further developments.

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