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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SOME DAMPED WAVE EQUATIONS WITH SOURCE TERM

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Abstract. The idea proposed in this work is to investigate the decay estimate of the energy to the p-Laplacian wave equation with a weak nonlinear dissipation and source term. The proof is based on the multiplier techniques combined with nonlinear integral inequalities given by Martinez.

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Key words. Wave equation of p-Laplacian type, weak nonlinear dissipation, decay estimate, multiplier method.

1. INTRODUCTION

In this paper, we consider the initial-boundary value problem for the nonlinear wave equation of p-Laplacian type with a weak nonlinear dissipation of the type

$$(P)\begin{cases} \left(|u_t|^{l-2}u_t\right)_t - \operatorname{div}(|\nabla_x u|^{p-2}\nabla_x u) + \sigma(t)\ \left(u_t + |u_t|^{m-2}u_t\right) & \\ \qquad = b|u|^{r-2}u & \text{in } \Omega\times[0,+\infty[,\\ u(x,t) = 0 & \text{on } \Gamma\times[0,+\infty[,\\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{on } \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\Gamma = \partial\Omega$, $b > 0$, $l, p, m, r \geq 2$ are real numbers and σ is a positive function satisfying some conditions to be specified later.

This problem has been studied by many authors and several existence and decay results have appeared. For instance, when $l = 2$, $p = 2$, $\sigma \equiv a > 0$, the problem was treated by Benaissa and Messaoudi [2]. They showed that, for suitably chosen initial data, the problem has a global weak solution, which decays exponentially even if $m > 2$. Further they proved the global existence by using the potential well theory introduced by Sattinger [9].

Similar results have been established by Ye [12, 13]. In these works the author used the Faedo-Galerkin approximation together with compactness criteria and difference inequality introduced by Nakao [8].

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When we have $l = 2$, $\sigma \equiv 1$ and $\delta|u|^{m-2}u$ $(\delta > 0, m \geq 2)$ instead of $\big(u_t + |u_t|^{m-2}u_t\big)$ in the problem (P) and without a source term, Yao [4] proved that the energy decay rate is

$$E(t) \leq (1+t)^{\frac{-p}{mp-m-1}}$$

for $t \geq 0$, for which he used the general method of energy decay introduced by Nakao [8]. Unfortunately, the methods used by Messaoudi and Ye do not seem to be applicable in the case of $m > 2$ and for more general functions σ it is more complicated.

It is worth mentioning some other papers in connection with asymptotic behavior of solutions to the nonlinear hyperbolic equation with dissipative effects, e.g., [1, 3, 7, 11, 12] and the references therein.

The purpose of this paper is to give an energy decay estimate of the solution of problem (P). Our proof is based on the multiplier method combined with nonlinear integral inequalities given by Martinez [6] which depends on the construction of a special weight function that depends on the behavior of σ. This method is based on a new integral inequality that generalizes a result of Haraux [5].

The paper is organized as follows: in Section 2, we give our hypotheses and establish a useful lemma. In Section 3, we state and prove our main result.

Throughout this paper the functions considered are all real valued. For simplicity of notation, hereafter we denote by $\|\cdot\|_p$ the Lebesgue space $L^p(\Omega)$ norm, and $\|\cdot\|_2$ denotes $L^2(\Omega)$ norm. We also denote by $(.,.)$ the inner product of $L^2(\Omega)$. As usual, we write respectively $u(t)$ and $u_t(t)$ instead of $u(x,t)$ and $u_t(t,x)$. All along this paper we denote C various positive constants which may be different at different occurrences.

2. PRELIMINARIES AND MAIN RESULT

We begin by introducing some definition that will be used through this work. First assume that the solution exists in class

$$u \in C([0,+\infty), W_0^{1,p}(\Omega)) \cap C^1([0,+\infty), L^l(\Omega)).$$

We define the following functionals:

$$\begin{aligned} K(u) &= \|\nabla u\|_p^p - b\|u\|_r^r \\ J(u) &= \frac{1}{p}\|\nabla u\|_p^p - \frac{b}{r}\|u\|_r^r, \end{aligned}$$

for $u \in W_0^{1,p}(\Omega)$.

Then, for the problem (P), we are able to define the stable set

$$H \equiv \{u \in W_0^{1,p}(\Omega),\, K(u) > 0\}. \cup \{0\}.$$

$\sigma(t)$ and g satisfies the following hypotheses.

ASSUMPTION 2.1. • Suppose that $\sigma \in C^1(\mathbb{R}_+ \to \mathbb{R}_+)$ is a non-increasing function satisfying

$$\int_0^{+\infty} \sigma(\tau)\,\mathrm{d}\tau = +\infty. \tag{1}$$

• Consider $g : \mathbb{R} \to \mathbb{R}$ a non increasing C^0 function such that

$$g(v)v > 0 \quad \text{for all}\, v \neq 0.$$

and suppose that there exist $c_i > 0; i = 1, 2, 3, 4$ such that

$$c_1|v|^m \le |g(v)| \le c_2|v|^{\frac{1}{m}} \qquad \text{if} \quad |v| \le 1, \tag{2}$$

$$c_3|v|^p \le |g(v)| \le c_4|v|^r \qquad \text{for all} \quad |v| > 1, \tag{3}$$

where $m \ge 2$, $l \le p \le r \le \frac{n(p-1)+p}{n-p}$.

We define the total energy associated to the solution of the problem (P) by the following formula

$$E(t) = \frac{l-1}{l}\|u_t\|_l^l + \frac{1}{p}\|\nabla u\|_p^p - \frac{b}{r}\|u\|_r^r = \frac{l-1}{l}\|u_t\|_l^l + J(u),$$

for $u \in W_0^{1,p}(\Omega)$ and $t \ge 0$.

We first state some well-known lemmas.

LEMMA 2.2 (Energy identity). *Let $u(t,x)$ be a solution to the problem (P) on $[0,\infty)$. Then we have*

$$E(t) + \int_\Omega \int_0^t \sigma(s)u_t(s)g(u_t(s))\,\mathrm{d}s\,\mathrm{d}x = E(0),$$

for all $t \in [0,\infty)$, and where we set $g(u_t) = u_t + |u_t|^{m-2}u_t$.

REMARK 2.3. It is clear that $E(t)$ is a non-increasing function for $t > 0$ and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\sigma(t)\left(\|u_t\|_2^2 + \|u_t\|_m^m\right) \le 0.$$

LEMMA 2.4 (Sobolev-Poincaré inequality). *Let r be a number with $2 \le r < +\infty$ $(n = 1, 2, ..p)$ or $2 \le r \le np/(n-p)$ $(n \ge p+1)$. Then there is a constant $c_* = c_*(\Omega, r)$ such that $\|u\|_r \le c_*\|\nabla u\|_p$ for $u \in W_0^{1,p}(\Omega)$.*

LEMMA 2.5 ([6]). *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing $\mathcal{C}^2$ function such that*

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \to +\infty \quad \text{as} \quad t \to +\infty.$$

Assume that there exist $q \ge 0$ et $\gamma > 0$ such that

$$\int_S^{+\infty} E(t)^{q+1}\phi'(t)\,\mathrm{d}t \le \gamma^{-1}\,E(0)^q E(S), \quad 0 \le S < +\infty.$$

Then we have

$$\textit{if } q>0, \quad \textit{then} \quad E(t) \le E(0)\left(\frac{1+q}{1+q\,\gamma\,\phi(t)}\right)^{\frac{1}{q}}, \ \ \forall t \ge 0,$$

$$\textit{if } q=0, \quad \textit{then} \quad E(t) \le E(0)\exp(1-\gamma\,\phi(t)), \ \ \forall t \ge 0.$$

This lemma is due to Martinez and its proof can be found in [6]. Now we recall the following local existence theorem, which can be established by using the argument in [10].

Theorem 2.6. *Let* $2 < r < np/(n-p)$, $n > p$ *and* $2 < p < r < \infty$, $n \le p$ *and assume that* $2 \le m \le p$, $(u_0, u_1) \in W_0^{1,p}(\Omega) \times L^l(\Omega)$ *and* u_0 *belong the stable set* H. *Then there exists* $T > 0$ *such that the problem* (P) *has a unique local solution* $u(t)$ *in the class*

$$u \in C\left([0,T); W_0^{1,p}(\Omega)\right),$$
$$u_t \in C\left([0,T); L^l(\Omega)\right) \cap L^m\left(\Omega \times [0,T)\right).$$

We list up some useful lemmas before stating the decay property. From now on, we denote the life span of the solution $u(t)$ of the problem (P) by $T_{\max}$.

Lemma 2.7. *Assume that the hypotheses in Theorem* 2.6 *hold, then*

$$\frac{r-p}{rp}\,\|\nabla u\|_p^p \le E(t), \tag{4}$$

for $u \in H$.

Proof. The definition of $K(u)$ and $J(u)$ assume that

$$K(u) + \frac{r-p}{rp}\,\|\nabla u\|_p^p = rJ(u).$$

Since $u \in H$, we have $K(u) \ge 0$. Hence we deduce from (4) that

$$\frac{r-p}{rp}\,\|\nabla u\|_p^p \le J(u) \le E(t).$$

□

Lemma 2.8 ([13]). *Let* $u(t)$ *be a solution to problem* (P) *on* $[0, T_{\max})$. *Suppose that* $2 \le r \le np/(n-p), n \ge p$ *and* $2 < p < r < +\infty, n \le p$. *If* $u_0 \in H$ *and* $u_1 \in L^l(\Omega)$ *satisfy*

$$\theta = b\,C_*^r\left(\frac{rp}{r-p}E(0)\right)^{\frac{r-p}{p}} < 1, \tag{5}$$

then $u(t) \in H$, *for each* $t \in [0,\, T_{\max})$.

Now we are in position to state our main result.

THEOREM 2.9. *Let $u(t,x)$ be a local solution of problem (P) on $[0,T_{\max})$ with initial data $u_0 \in H$, $u_1 \in L^l(\Omega)$, and sufficiently small initial energy $E(0)$ so that*

$$b\,C_*^r\left(\frac{rp}{r-p}E(0)\right)^{\frac{r-p}{p}} < 1.$$

If the hypotheses in Theorem 2.6, and assumptions (2) and (3) are valid. Then the solution $u(x,t)$ of the problem (P) satisfies the following energy decay estimates:

- *If $m = l-1$, then there exists a positive constant ω independent of $E(0)$ such that*

$$E(t) \leq C(E(0))\exp\left(1-\omega\int_0^t \sigma(\tau)\,\mathrm{d}\tau\right) \quad \forall t > 0. \tag{6}$$

- *If $m \neq l-1$, then there exists a positive constant $C(E(0))$ depending continuously on $E(0)$ such that*

$$E(t) \leq \left(\frac{C(E(0))}{\int_0^t \sigma(\tau)\,\mathrm{d}\tau}\right)^{\frac{m(p-1)-1}{p}} \qquad \forall t > 0.$$

3. PROOF OF MAIN RESULT

Proof of energy decay. Now, we shall derive the decay estimate for the solutions in Theorem 2.9. For this we use the method of multipliers. From now on, C denotes various positive constants depending on the known constants and may be different at each appearance.

Multiplying the first equation in (P) by $E(t)^q\phi' u$ and integrating over $\Omega \times [T,S]$, where $0 \leq S \leq T \leq \infty$. We obtain that

$$\begin{aligned}
0 &= \int_S^T E(t)^q\phi'\int_\Omega u\Big[\left(|u_t|^{l-2}u_t\right)_t - \operatorname{div}(|\nabla_x u|^{p-2}\nabla_x u)\\
&\quad -\sigma(t)\,\left(u_t + u_t|^{m-2}u_t\right) - b|u|^{r-2}u\Big]\,\mathrm{d}x\,\mathrm{d}t\\
&= \left[E(t)^q\phi'\int_\Omega u|u_t|^{l-2}u_t\mathrm{d}x\right]_S^T - \int_S^T E(t)^q\phi'\int_\Omega |u_t|^l\mathrm{d}x\,\mathrm{d}t\\
&\quad -\int_S^T\left(qE'(t)E(t)^{q-1}\phi' + E(t)^q\phi''\right)\int_\Omega u|u_t|^{l-2}u_t\,\mathrm{d}x\,\mathrm{d}t\\
&\quad +\int_S^T E(t)^q\phi'\int_\Omega |\nabla u|^p\,\mathrm{d}x\mathrm{d}t - b\int_S^T E(t)^q\phi'\int_\Omega |u|^r\,\mathrm{d}x\mathrm{d}t\\
&\quad +\int_S^T E(t)^q\phi'\sigma(t)\int_\Omega u\left(u_t + |u_t|^{m-2}u_t\right)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

We deduce that

$$
\begin{aligned}
&2\int_S^T E(t)^{q+1}\phi'\,\mathrm{d}t\\
&= -\left[E(t)^q\phi'\int_\Omega u|u_t|^{l-2}u_t\mathrm{d}x\right]_S^T + \frac{3l-2}{l}\int_S^T E(t)^q\phi'\int_\Omega |u_t|^l\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_S^T \left(qE'(t)E(t)^{q-1}\phi' + E(t)^q\phi''\right)\int_\Omega u|u_t|^{l-2}u_t\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \int_S^T E(t)^q\phi'\sigma(t)\int_\Omega u\left(u_t+|u_t|^{m-2}u_t\right)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + b\left(1-\frac{2}{r}\right)\int_S^T E(t)^q\phi'\int_\Omega |u|^r\,\mathrm{d}x\mathrm{d}t\\
&\quad + \left(\frac{2}{p}-1\right)\int_S^T E(t)^q\phi'\int_\Omega |\nabla u|^p\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

From Lemma 2.4, and equations (4) and (5) we get

$$
\begin{aligned}
&b\left(1-\frac{2}{r}\right)\int_S^T E(t)^q\phi'\int_\Omega |u|^r\,\mathrm{d}x\mathrm{d}t\\
&\le b\left(1-\frac{2}{r}\right)\int_S^T E(t)^q\phi' C_*^r\|\nabla u\|_p^r\,\mathrm{d}t\\
&\le b\left(1-\frac{2}{r}\right)\int_S^T E(t)^q\phi' C_*^r\left(\frac{rp}{r-p}E(0)\right)^{\frac{r-p}{p}}\frac{rp}{r-p}E(t)\mathrm{d}t\\
&= \theta\frac{p(r-2)}{r-p}\int_S^T E(t)^{q+1}\phi'\mathrm{d}t.
\end{aligned}
$$

and

$$
\left(1-\frac{2}{p}\right)\int_S^T E(t)^q\phi'\int_\Omega |\nabla u|^p\,\mathrm{d}x\mathrm{d}t \le \frac{r(p-2)}{r-p}\int_S^T E(t)^{q+1}\phi'\mathrm{d}t.
$$

Consequently, it follows that

$$
\begin{aligned}
&\frac{4r-p[\theta(r-2)+r+2]}{r-p}\int_S^T E(t)^{q+1}\phi'\,\mathrm{d}t\\
&\le \int_S^T \left(qE'(t)E(t)^{q-1}\phi' + E(t)^q\phi''\right)\int_\Omega u|u_t|^{l-2}u_t\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \left[E(t)^q\phi'\int_\Omega u|u_t|^{l-2}u_t\,\mathrm{d}x\right]_S^T\\
&\quad + \frac{3l-2}{l}\int_S^T E(t)^q\phi'\int_\Omega |u_t|^l\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{7}
$$

$$-\int_S^T E(t)^q\phi'\sigma(t)\int_\Omega u\left(u_t+|u_t|^{m-2}u_t\right)\,\mathrm{d}x\,\mathrm{d}t.$$

Here, we have $\frac{4r-p[(r-2)\theta+r+2]}{r-p}>0$, as long as $0<\theta<1$.

We must estimate every term of the right-hand side of (7) in order to apply the results of Lemma 2.5.

Define $\phi(t)=\int_0^t\sigma(\tau)\,\mathrm{d}\tau$. So ϕ is a non-decreasing function of class $\mathcal{C}^2$ on $\mathbb{R}_+$ and the hypothesis (1) ensures that $\phi(t)\to+\infty$ for $t\to+\infty$.

Using the non-increasing property of E and that ϕ' is a bounded non negative function on $\mathbb{R}_+$ (we denote by μ its maximum). Also, using Young's inequality, Lemma 2.4 and (4) we have

$$\left|-E(t)^q\phi'\int_\Omega uu_t|u_t|^{l-2}\,\mathrm{d}x\Big|_S^T\right|\le-\mu\max\left(\frac{C_*^lrp}{l(r-p)},1\right)E(t)^{q+1}\Big|_S^T$$
$$\le CE(S)^{q+1},$$

where the above estimate follows from the fact that

$$\int_\Omega uu_t|u_t|^{l-2}\,\mathrm{d}x\le\left(\frac{C_*^lrp}{l(r-p)}\,\frac{r-p}{rp}\|\nabla u\|_p^p+\frac{l-1}{l}\|u_t\|_l^l\right)\mathrm{d}t$$
$$\le\max\left(\frac{C_*^lrp}{l(r-p)},1\right)E(t).$$

Again, exploiting Young's inequality, the Sobolev-Poincaré inequality, the definition of energy, and the previous inequality, we obtain

$$\left|\int_S^T\left(qE'(t)E(t)^{q-1}\phi'+E(t)^q\phi''\right)\int_\Omega uu_t|u_t|^{l-2}\,\mathrm{d}x\,\mathrm{d}t\right|$$
$$\le-\mu q\max\left(\frac{C_*^lrp}{l(r-p)},1\right)\int_S^TE(t)^qE'(t)\mathrm{d}t$$
$$+\max\left(\frac{C_*^lrp}{l(r-p)},1\right)\int_S^TE(t)^{q+1}(-\phi'')\mathrm{d}t$$
$$\le CE(S)^{q+1}.$$

Using these estimates we conclude from the above inequality that

$$\frac{4r-p[\theta(r-2)+r+2]}{r-p}\int_T^SE(t)^{q+1}\phi'\mathrm{d}t\le CE(S)^{q+1}\tag{8}$$
$$+\frac{3l-2}{l}\int_S^TE(t)^q\phi'\left(\int_{\Omega_1}|u_t|^l\,\mathrm{d}x+\int_{\Omega_2}|u_t|^l\,\mathrm{d}x\right)\mathrm{d}t$$
$$-\int_S^TE(t)^q\phi'\sigma(t)\left(\int_{\Omega_1}ug(u_t)\,\mathrm{d}x+\int_{\Omega_2}ug(u_t)\,\mathrm{d}x\right)\mathrm{d}t,$$

where we set $\Omega_1=\{u\in W_0^{1,p}(\Omega),|u_t|\le1\}$ and $\Omega_2=\{u\in W_0^{1,p}(\Omega),|u_t|>1\}$.

Now, we estimate the terms of the right-hand side of (8). We have two cases related to the parameters m and l.

Case 1. $m \leq l-1$. From the assumptions (2) and (3), we have

$$\begin{aligned}(9)\quad &\frac{3l-2}{l}\int_S^T E(t)^q\phi'\int_\Omega |u_t|^l \mathrm{d}x\mathrm{d}t = C\int_S^T E(t)^q\phi'\int_\Omega u_t g(u_t)\,\mathrm{d}x\mathrm{d}t\\ &\leq C\int_S^T E(t)^q\phi'\left(\frac{-E'(t)}{\sigma(t)}\right)\mathrm{d}t \leq CE(S)^{q+1}.\end{aligned}$$

Using the Hölder inequality, Lemma 2.4 and condition (2) we obtain

$$\begin{aligned}&\int_S^T E(t)^q\phi'\int_{\Omega_1}\sigma(t)ug(u_t)\mathrm{d}x\,\mathrm{d}t\\ &\leq \int_S^T E(t)^q\phi'\sigma(t)\left(\int_\Omega |u|^{\frac{m+1}{m}}\mathrm{d}x\right)^{\frac{m}{(m+1)}}\left(\int_{\Omega_1}|g(u_t)|^{m+1}\mathrm{d}x\right)^{\frac{1}{m+1}}\mathrm{d}t\\ &\leq |\Omega|^{\frac{pm-(m+1)}{pm}}\int_S^T E(t)^q\phi'\sigma(t)\left(\int_\Omega |u|^p\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{\Omega_1}|g(u_t)|^{m+1}\mathrm{d}x\right)^{\frac{1}{m+1}}\mathrm{d}t\\ &\leq C\int_S^T E(t)^{q+\frac{1}{p}}\phi'\sigma(t)^{\frac{m}{(m+1)}}\left(\int_{\Omega_1}\sigma u_t g(u_t)\mathrm{d}x\right)^{\frac{1}{m+1}}\mathrm{d}t\\ &\leq C\int_S^T E(t)^{q+\frac{1}{p}}\phi'\sigma(t)^{\frac{m}{(m+1)}}\left(-E'(t)\right)^{\frac{1}{m+1}}\mathrm{d}t.\end{aligned}$$

Setting $\varepsilon_1 > 0$, and applying Young's inequality, we obtain

$$\begin{aligned}(10)\quad &\int_S^T E(t)^q\phi'\int_{\Omega_1}\sigma(t)ug(u_t)\mathrm{d}x\,\mathrm{d}t\\ &\leq C\int_S^T\left[\frac{m}{m+1}\varepsilon_1^{\frac{m+1}{m}}\left(E(t)^{q+\frac{1}{p}}\phi'\sigma(t)^{\frac{m}{m+1}}\right)^{\frac{m+1}{m}}\right.\\ &\qquad\left.+\frac{1}{m+1}\frac{1}{\varepsilon_1^{m+1}}\left(-E'(t)\right)\right]\mathrm{d}t\\ &\leq C\mu^{\frac{m+1}{m}}\frac{m}{m+1}\varepsilon_1^{\frac{m+1}{m}}\int_S^T E(t)^{(q+\frac{1}{p})(\frac{m+1}{m})}\phi'\mathrm{d}t + C\frac{1}{m+1}\frac{1}{\varepsilon_1^{m+1}}E(S).\end{aligned}$$

In a similar way, we estimate the last term of the right-hand of (8). We have

$$
\begin{aligned}
&\int_S^T E(t)^q \phi' \int_{\Omega_2} \sigma(t) u g(u_t) \mathrm{d}x\,\mathrm{d}t \\
&\le C \int_S^T E(t)^{q+\frac{1}{p}} \phi' \sigma(t)^{\frac{1}{r+1}} \left(\int_{\Omega_2} \sigma(t) u_t g(u_t) \mathrm{d}x \right)^{\frac{r}{r+1}} \mathrm{d}t \\
&\le C \int_S^T E(t)^{q+\frac{1}{p}} \phi' \sigma(t)^{\frac{1}{r+1}} \left(-E'(t) \right)^{\frac{r}{r+1}} \mathrm{d}t \\
&\le C \frac{1}{r+1} \varepsilon_2^{r+1} \int_S^T E(t)^{(q+\frac{1}{p})(r+1)} \phi' \mathrm{d}t + C \frac{r}{r+1} \frac{1}{\varepsilon_2^{\frac{r+1}{r}}} E(S).
\end{aligned}
\tag{11}
$$

where we have also used the condition (3) for $\varepsilon_2 > 0$.

Case 2. $m > l - 1$. From Hölder's inequality, Lemma 2.2 and assumption (2), we get for $\varepsilon_1 > 0$

$$
\begin{aligned}
&\frac{3l-2}{l} \int_S^T E(t)^q \phi' \int_\Omega |u_t|^l \,\mathrm{d}x\,\mathrm{d}t \\
&\le C\, E(S)^{q+1} + C(\Omega) \int_S^T E(t)^q \phi' \left(\int_{\Omega_1} |u_t|^{m+1} \,\mathrm{d}x \right)^{\frac{l}{m+1}} \mathrm{d}t \\
&\le C\, E(S)^{q+1} + C' \int_S^T E(t)^q \phi' \left(\frac{-E'(t)}{\sigma(t)} \right)^{\frac{l}{m+1}} \mathrm{d}t \\
&\le C\, E(S)^{q+1} + C' \frac{m+1-l}{m+1} \varepsilon_1^{\frac{m+1}{m+1-l}} \int_S^T E(t)^{q\frac{m+1}{m+1-l}} \phi' \,\mathrm{d}t \\
&\quad + C' \frac{l}{m+1} \frac{1}{\varepsilon_1^{\frac{m+1}{l}}} E(S).
\end{aligned}
\tag{12}
$$

Next, we estimate the third term of the right-hand of (8). We have for $\varepsilon' > 0$

$$
\int_S^T E(t)^q \phi' \int_{\Omega_1} \sigma(t) u g(u_t) \mathrm{d}x\,\mathrm{d}t
$$

$$\leq \varepsilon^{'} \int_S^T E(t)^q \phi' \int_{\Omega 1} |u|^p \mathrm{d}x\,\mathrm{d}t + C(\varepsilon^{'}) \int_S^T E^q \phi' \int_{\Omega_1} (\sigma(t) g(u_t))^{\frac{p}{p-1}}\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq C\varepsilon^{'} \int_S^T E(t)^{q+1} \phi' \mathrm{d}t + C(\varepsilon^{'}) \int_S^T E(t)^q \phi' \int_{\Omega_1} (\sigma(t) g(u_t))^{\frac{p}{p-1}}\,\mathrm{d}x\,\mathrm{d}t.$$

We estimate the last term of the above inequality and we get that

$$\begin{aligned}
&\int_S^T E(t)^q \phi' \int_{\Omega_1} (\sigma(t) g(u_t))^{\frac{p}{p-1}}\,\mathrm{d}x\,\mathrm{d}t \\
&\leq C \int_S^T E(t)^q \phi' \int_{\Omega_1} (u_t g(u_t))^{\frac{p}{(m+1)(p-1)}}\,\mathrm{d}x\,\mathrm{d}t \\
&\leq C \int_S^T E(t)^q \phi' \frac{1}{\sigma(t)^{\frac{p}{(m+1)(p-1)}}} \int_{\Omega_1} (\sigma(t) u_t g(u_t))^{\frac{p}{(m+1)(p-1)}}\,\mathrm{d}x\,\mathrm{d}t \\
&\leq C \int_S^T E(t)^q \phi' \frac{1}{\sigma^{\frac{p}{(m+1)(p-1)}}} \left(-E'(t)\right)^{\frac{p}{(m+1)(p-1)}}\,\mathrm{d}t.
\end{aligned}$$

Setting $\varepsilon_2 > 0$, thanks to Young's inequality, we obtain

$$\begin{aligned}
\int_S^T E(t)^q \phi' \int_{\Omega_1} (\sigma g(u'))^{\frac{p}{p-1}} \mathrm{d}x\mathrm{d}t &\leq C\, \frac{p}{(m+1)(p-1)} \frac{1}{\varepsilon_2^{\frac{(m+1)(p-1)}{p}}} E(S) \\
&+ C \frac{(m+1)(p-1)-p}{(m+1)(p-1)} \varepsilon_2^{\frac{(m+1)(p-1)}{(m+1)(p-1)-p}} \int_S^T E(t)^{q\frac{(m+1)(p-1)}{(m+1)(p-1)-p}} \phi'\,\mathrm{d}t.
\end{aligned} \tag{13}$$

Finally, we estimate the last inequality in the domain Ω_2. In this case the inequality (11) is also valid, therefore we have

$$\begin{aligned}
&\int_S^T E(t)^q \phi' \int_{\Omega_2} \sigma(t) u g(u_t) \mathrm{d}x\,\mathrm{d}t \\
&\leq C \frac{1}{r+1} \varepsilon_3^{r+1} \int_S^T E(t)^{(q+\frac{1}{p})(r+1)} \phi' \mathrm{d}t + C \frac{r}{r+1} \frac{1}{\varepsilon_3^{\frac{r+1}{r}}} E(S).
\end{aligned} \tag{14}$$

We choose q such that $q\frac{(m+1)(p-1)}{(m+1)(p-1)-p} = q+1$. Giving $q = \frac{m(p-1)-1}{p}$ and thus $q\frac{m+1}{m+1-l} = q+1+\alpha$ with $\alpha = \frac{(m+1)(p(l-1)-l)}{p(m+1-l)} > 0$, and $(q+\frac{1}{p})(r+1) = q+1+\beta$ with $\beta = \frac{(rm-1)(p-1)}{p} > 0$.

Set $\varepsilon_1 = \frac{\varepsilon}{E(0)^{\frac{p(l-1)-l}{p}}}$ and $\varepsilon_3 = \frac{\varepsilon}{E(0)^{\frac{(p-1)(rm-1)}{p(r+1)}}}$.

Choosing $\varepsilon', \varepsilon_2$ small enough, then substituting the estimates (12), (13) and (14) into (8) we get for $m > l-1$

$$\begin{aligned}
&\int_S^T E(t)^{1+q}\phi' \mathrm{d}t \\
&\le CE(S) + C'E(S)^{q+1} + C''E(0)^{\frac{(p(l-1)-l)(m+1)}{pl}}E(S) \\
&\quad + C'''E(0)^{\frac{(p-1)(rm-1)}{pr}}E(S) \\
&\le \left(\frac{C + C'E(0)^q + C''E(0)^{\frac{(p(l-1)-l)(m+1)}{pl}} + C'''E(0)^{\frac{(p-1)(rm-1)}{pr}}}{E(0)^q}\right)E(0)^q E(S),
\end{aligned}$$

where C, C', C'' and C''' are different positive constants independent of $E(0)$. Hence, we deduce from Lemma 2.5 that

$$\begin{aligned}
E(t) \le &\left(C' + CE(0)^q + C''E(0)^{\frac{(m+1)(l-1)}{l} - \frac{m+1}{p}} + C'''E(0)^{(\frac{m}{p} - \frac{1}{pr})(p-1)}\right)^{\frac{1}{q}} \\
&\times \left(1 + \frac{1}{q}\right)^{\frac{1}{q}} \left(\int_0^t \sigma(s)\mathrm{d}s\right)^{-\frac{1}{q}}.
\end{aligned}$$

If $m \le l-1$, we choose q such that $\left(q + \frac{1}{p}\right)\left(\frac{m+1}{m}\right) = q+1$. Thus we take $q = \frac{pm-m-1}{p} > 0$, and hence $\left(q + \frac{1}{p}\right)(r+1) = q+1+\alpha$ with $\alpha = \frac{(p-1)(rm-1)}{p}$. Set $\varepsilon_2 = \frac{\varepsilon}{E(0)^{\frac{(p-1)(rm-1)}{p(r+1)}}}$. Choosing $\varepsilon, \varepsilon_1$ small enough, then substituting the estimates (9), (10) and (11) into (8) we get

$$\begin{aligned}
&\int_S^T E(t)^{1+q}\phi' \mathrm{d}t \\
&\le CE(S) + C'E(S)^{q+1} + C''E(0)^{\frac{(p-1)(rm-1)}{pr}}E(S) \\
&\le \left(\frac{C + C'E(0)^q + C''E(0)^{\frac{(p-1)(rm-1)}{pr}}}{E(0)^q}\right)E(0)^q E(S),
\end{aligned}$$

where C, C' and C'' are different positive constants independent of $E(0)$. Hence, we deduce from Lemma 2.5 that

$$E(t) \le \left(C' + CE(0)^q + C''E(0)^{\frac{(p-1)(rm-1)}{pr}}\right)^{\frac{1}{q}} \left(1 + \frac{1}{q}\right)^{\frac{1}{q}} \left(\int_0^t \sigma(s)\mathrm{d}s\right)^{-\frac{1}{q}}.$$

It is clear that, for $m = l - 1$, we have $q = 0$ and the energy $E(t)$ associated with the solution of the problem (P) satisfies the exponential decay of energy (6). The proof is thus finished.

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