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WEAK SOLUTIONS FOR A NONLINEAR PROBLEM FOR $p(x)$-LAPLACIAN-LIKE VIA TOPOLOGICAL DEGREE METHODS

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Abstract. Based on the topological degree method for a class of bounded and demicontinuous operators of type (S_+) and properties of variable exponent Sobolev spaces, we study the existence of weak solutions for a nonlinear Dirichlet problem of capillary phenomena involving an equation driven by the $p(x)$-Laplacian-like operator.

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be an open bounded domain with smooth boundary $\partial\Omega$. In this paper we study the following nonlinear Dirichlet boundary value problem:

$$\begin{cases} -\Delta^l_{p(x)} u + a(x)|u|^{p(x)-2}u = \lambda f(x, u, \nabla u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Delta^l_{p(x)} u$ denotes the $p(x)$-Laplacian-like operator defined by

$$\Delta^l_{p(x)} u := \operatorname{div}\Big(\Big(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1 + |\nabla u|^{2p(x)}}}\Big)|\nabla u|^{p(x)-2}\nabla u\Big),$$

where $p \in C(\overline{\Omega})$ is a function with some regularity satisfying

$$1 < p^- := \inf_{x\in\Omega} p(x) \leq p(x) \leq p^+ := \sup_{x\in\Omega} p(x),$$

and λ is a positive parameter. The potential function a is assumed to be in $L^\infty(\Omega)$ with $\operatorname{essinf}_\Omega a > 0$ and the reaction term $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function verifying only a growth condition. Our approach is the topological degree theory for a class of bounded and demicontinuous operators of (S_+) type.

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In recent years, the study of differential problems driven by the $p(x)$-Laplacian and the $(p(z), q(z))$-Laplacian with non-standard growth has received increasing attention. As recent references related to such problems, we can cite [6, 15, 18–20]

We point out that differential equations driven by p-Laplacian-like or $p(x)$-Laplacian-like operators have recently attracted considerable interest. This type of operator is used to model the capillary phenomenon, which can be briefly explained by considering the effects of two opposing forces: adhesion, i.e. the force of attraction (or repulsion) between the molecules of the liquid and those of the container; and cohesion, i.e. the force of attraction between the molecules of the liquid and those in the container (for more details see [10, 12, 13, 17]).

There are many works on this type of equation using different approaches. We cite the paper of Vetro [24] where he proved, by applying the energy functional method and weaker compactness conditions, the existence of at least one non-trivial weak solution to problem (1) (with $\lambda = 1$ and p is independent of x), when the reaction term, independent of ∇u, satisfies a subcritical growth condition and the potential term exhibits certain regularities.

Rodrigues in [21], using the mountain pass lemma and Fountain's theorem, established the existence of non-trivial solutions for problem (1) without the term $a(x)|u|^{q(x)-2}u$ and without f depending on ∇u. The same problem is treated later by Zhou [26], when f is superlinear and does not satisfy the Ambrosetti and Rabinowitz type condition. In [5], the authors have established new sufficient conditions under which problem

$$\begin{cases} -\Delta^l_{p(x)} u + a(x)|u|^{p(x)-2}u = \lambda f(x,u) + \mu g(x,u), & x \in \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

has infinitely many weak solutions. Their approach is a fully variational method and the main tool is a general critical point theorem. The same problem was studied by Shokooh in [23] but with a constant exponent p, $a(x) \equiv 1$ and $\mu = 0$. He established existence results for positive solutions to this problem and also reported multiplicity results. In the same context, we also cite [9] where the authors discussed the eigenvalue problem for the following p-Laplacian-like equation:

$$\begin{cases} -\mathrm{div}(a(|Du|^p)|Du|^{p-2}Du) = \lambda f(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

and proved the existence of two eigenfunctions which have very different asymptotic behaviors. When

$$a(|Du|^p) = 1 + \frac{|Du|^p}{\sqrt{1 + |Du|^{2p}}} \text{ and } f(x,u) = |u|^{q-2}u + |u|^{r-2}u,$$

where $1 < q < p$ and $2p < r < p^* = \frac{Np}{N-p}$, this is the general capillarity equation (see [9, Example 4.2]). In [16], the authors proved the existence of one or infinitely many nontrivial solutions to problem (2) with $\lambda = 1$ and f is a Carathéodory function involving the Hardy potential or the critical exponent.

As a generalization of the previous work by using another approach with fewer conditions, we will treat problem (1) using the topological degree method in the framework of variable exponent Sobolev spaces. For more information on the topological degree and some of its applications, we refer the reader to [1–4, 7]

The plan of the paper is as follows: after this introduction, Section 2 is reserved for the mathematical background in three subsections: the first for a brief overview of the Berkovits topological degree used in this paper, the second for various basic properties of Lebesgue and Sobolev exponent spaces and the third for some details related to the Gâteaux differentiability. The last section deals with our problem by proving our main result under some assumptions after giving some technical lemmas.

2. MATHEMATICAL BACKGROUND

In this section we recall the definitions and theorems that will be used in this paper. This will be done in two parts: one is kept to explain the method of the topological degree considered and the other to the functional framework employed.

2.1. BERKOVITS DEGREE

We start by an outline of Berkovits degree theory. For more details we can see [7].

Let X be a real separable reflexive Banach space with dual X^* and with continuous pairing $\langle . \, , \, . \rangle$, Ω be a nonempty subset of X and Y be a real Banach space.

We recall that a mapping $F : \Omega \subset X \to Y$ is *bounded,* denoted by $F \in BD$, if it takes any bounded set into a bounded set. F is said to be *demicontinuous,* denote $F \in DC$, if for any $(u_n) \subset \Omega$, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$. F is said to be *compact* if it is continuous and the image of any bounded set is relatively compact.

A mapping $F : \Omega \subset X \to X^*$ is said to be *of class* (S_+), denote $F \in (S_+)$, if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $\limsup \langle Fu_n, u_n - u \rangle \leq 0$, it follows that $u_n \to u$. F is said to be *quasimonotone,* if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, it follows that $\limsup \langle Fu_n, u_n - u \rangle \geq 0$.

For any operator $F : \Omega \subset X \to X$ and any bounded operator $T : \Omega_1 \subset X \to X^*$ such that $\Omega \subset \Omega_1$, we say that F satisfies *condition* $(S_+)_T$, denoted by $F \in (S_+)_T$, if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $\limsup \langle Fu_n, y_n - y \rangle \leq 0$, we have $u_n \to u$.

Let $\mathcal{O}$ be the collection of all bounded open set in X. For any $\Omega \subset X$, we consider the following classes of operators:

$$\begin{aligned}
\mathcal{F}_1(\Omega) &:= \{F : \Omega \to X^* \mid F \in BD \cap DC \cap (S_+)\},\\
\mathcal{F}_{T,B}(\Omega) &:= \{F : \Omega \to X \mid F \in BD \cap DC \cap (S_+)_T\},\\
\mathcal{F}_T(\Omega) &:= \{F : \Omega \to X \mid F \in DC \cap (S_+)_T\},\\
\mathcal{F}_B(X) &:= \{F \in \mathcal{F}_{T,B}(\overline{G}) \mid G \in \mathcal{O}, \mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{G}})\}.
\end{aligned}$$

Here, $T \in \mathcal{F}_1(\overline{G})$ is called an *essential inner map* to F.

Definition 2.1. Let G be a bounded open subset of a real reflexive Banach space X, $T \in \mathcal{F}_1(\overline{G})$ be continuous and let $F, S \in \mathcal{F}_T(\overline{G})$. The affine homotopy $H : [0,1] \times \overline{G} \to X$ defined by $H(t,u) := (1-t)Fu + tSu$ for $(t,u) \in [0,1] \times \overline{G}$ is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 2.2 ([7]). The above affine homotopy satisfies condition $(S_+)_T$.

We introduce the topological degree for the class $\mathcal{F}_B(X)$ due to Berkovits [7].

Theorem 2.3. *Let*

$$\mathcal{M} = \{(F,G,h) | G \in \mathcal{O}, T \in \mathcal{F}_1(\overline{G}), F \in \mathcal{F}_{T,B}(\overline{G}), h \notin F(\partial G)\}.$$

There exists a unique degree function $d : \mathcal{M} \to \mathbb{Z}$ that satisfies the following properties

- *(Existence) If $d(F,G,h) \neq 0$, then the equation $Fu = h$ has a solution in G.*
- *(Additivity) Let $F \in \mathcal{F}_{T,B}(\overline{G})$. If G_1 and G_2 are two disjoint open subset of G such that $h \notin F(\overline{G} \setminus (G_1 \cup G_2))$, then we have*
$$d(F,G,h) = d(F,G_1,h) + d(F,G_2,h).$$
- *(Homotopy invariance) If $H : [0,1] \times \overline{G} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and $h : [0,1] \to X$ is a continuous path in X such that $h(t) \notin H(t, \partial G)$ for all $t \in [0,1]$, then the value of $d(H(t,.), G, h(t))$ is constant for all $t \in [0,1]$.*
- *(Normalization) For any $h \in G$, we have $d(I,G,h) = 1$.*

Lemma 2.4 ([7, Lemma 2.2 and 2.4]). *Suppose that $T \in \mathcal{F}_1(\overline{G})$ is continuous and $S : D_S \subset X^* \to X$ is demicontinuous such that $T(\overline{G}) \subset D_s$, where G is a bounded open set in a real reflexive Banach space X. Then the following statement are true:*

- *If S is quasimonotone, then $I + S \circ T \in \mathcal{F}_T(\overline{G})$, where I denotes the identity operator.*
- *If S is of class (S_+), then $S \circ T \in \mathcal{F}_T(\overline{G})$.*

2.2. VARIABLE LEBESGUE AND SOBOLEV SPACES

In this subsection, we state some basic properties of the variable exponent Sobolev space and introduce some notations. For more details, we refer the reader to [11, 14].

Let Ω be an open bounded subset of $I\!R^N$, $N \geq 2$, with a Lipschitz boundary denoted by $\partial\Omega$. Denote

$$C_+(\overline{\Omega}) = \left\{ h \in C(\overline{\Omega}) | \inf_{x \in \overline{\Omega}} h(x) > 1 \right\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ := \max \{h(x), x \in \overline{\Omega}\}, \quad h^- := \min\{h(x), x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$ we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to I\!R \text{ measurable }; \int_\Omega |u(x)|^{p(x)} \, \mathrm{d}x < \infty \right\}$$

endowed with the *Luxembourg norm*

$$|u|_{p(x)} = \inf \left\{ \alpha > 0/\rho_{p(x)} \left(\frac{u}{\alpha} \right) \leq 1 \right\}.$$

where

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} \, \mathrm{d}x, \quad \forall u \in L^{p(x)}(\Omega).$$

$(L^{p(x)}(\Omega), | \, . \, |_{p(x)})$ is a Banach space [14, Theorem 2.5], separable and reflexive [14, Corollary 2.7]. Its conjugate space is $L^{p'(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \Omega$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, Hölder inequality holds [14, Theorem 2.1]

$$\left| \int_\Omega uv \, \mathrm{d}x \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(x)} |v|_{p'(x)} \leq 2 |u|_{p(x)} |v|_{p'(x)}. \tag{3}$$

Notice that if (u_n) and $u \in L^{p(x)}(\Omega)$ then the following relations hold true (see [11])

$$|u|_{p(x)} < 1 (= 1; > 1) \quad \Leftrightarrow \quad \rho_{p(x)}(u) < 1 (= 1; > 1),$$

$$|u|_{p(x)} > 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}, \tag{4}$$

$$|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}, \tag{5}$$

From (4) and (5), we can deduce the inequality

$$\rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-} + |u|_{p(x)}^{p^+}. \tag{6}$$

If $p, q \in C_+(\overline{\Omega})$, $p(x) \leq q(x)$ for any $x \in \overline{\Omega}$, then the embedding $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ is continuous.

Define the Sobolev space with variable exponent

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega)/|\nabla u| \in L^{p(x)}(\Omega)\}.$$

Equipped with the norm $\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$ it is a separable, reflexive Banach space. We also define $W_0^{1,p(x)}(\Omega)$ as the subspace of $W^{1,p(x)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,p(x)}$. If the exponent $p(\cdot)$ satisfies the log-Hölder continuity condition, i.e. there is a constant $\beta > 0$ such that for every $x, y \in \Omega, x \neq y$ with $|x-y| \leq \frac{1}{2}$ one has

$$|p(x) - p(y)| \leq \frac{\beta}{-\log|x-y|}, \tag{7}$$

then we have the Poincaré inequality (see [22]), i.e. there exists a constant $C > 0$ depending only on Ω and the function p such that

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)}, \forall u \in W_0^{1,p(x)}(\Omega). \tag{8}$$

In particular, the space $W_0^{1,p(x)}(\Omega)$ has a norm $\|\cdot\|$ given by

$$\|u\| = |\nabla u|_{p(\cdot)} \text{ for all } u \in W_0^{1,p(x)}(\Omega),$$

which is equivalent to $\|\cdot\|_{1,p(x)}$. In addition, we have the compact embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ (see [14]). The space $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ is a Banach space, separable and reflexive (see [11, 14]). The dual space of $W_0^{1,p(x)}(\Omega)$, denoted $W^{-1,p'(x)}(\Omega)$, is equipped with the norm

$$\|v\|_{-1,p'(x)} = \inf\left\{|v_0|_{p'(x)} + \sum_{i=1}^{N} |v_i|_{p'(x)}\right\},$$

where the infimum is taken on all possible decompositions $v = v_0 - \text{div}F$ with $v_0 \in L^{p'(x)}(\Omega)$ and $F = (v_1, ..., v_N) \in (L^{p'(x)}(\Omega))^N$.

When $a \in L^\infty(\Omega)$ with $\text{essinf}_\Omega a > 0$, we define

$$L_{a(x)}^{p(x)}(\Omega) := \left\{u : \Omega \to \mathbb{R} \text{ mesurable } / \int_\Omega a(x)|u(x)|^{p(x)}\mathrm{d}x < +\infty\right\}$$

with the norm

$$|u|_{p(x),a(x)} = \inf\left\{\alpha > 0 : \int_\Omega a(x)|\frac{u(x)}{\alpha}|^{p(x)}\mathrm{d}x \leq 1\right\}.$$

For any $u \in W_0^{1,p(x)}(\Omega)$, define

$$\|u\|_a = \inf\left\{\alpha > 0 : \int_\Omega \left(|\frac{\nabla u(x)}{\alpha}|^{p(x)} + a(x)|\frac{u(x)}{\alpha}|^{p(x)}\right)\mathrm{d}x \leq 1\right\}.$$

Then, it is easy to see that $\|\cdot\|_a$ is a norm on $W_0^{1,p(x)}(\Omega)$ equivalent to $\|\cdot\|_{1,p(x)}$ and $\|\cdot\|$.

Proposition 2.5 ([8, Proposition 2.2]). *Set*

$$\rho(u) = \int_\Omega (|\nabla u|^{p(x)} + a(x)|u|^{p(x)})\mathrm{d}x.$$

For $u \in X = W_0^{1,p(x)}(\Omega)$ *we have*

- $\|u\|_a < (=;>)1 \Leftrightarrow \rho(u) < (=;>)1$,
- $\|u\|_a < 1 \Rightarrow \|u\|_a^{p^+} \leq \rho(u) \leq \|u\|_a^{p^-}$,
- $\|u\|_a > 1 \Rightarrow \|u\|_a^{p^-} \leq \rho(u) \leq \|u\|_a^{p^+}$.

2.3. GÂTEAUX DERIVATIVE

We now try to recall some details about the concept of derivative in the sense of Gâteau, which will be used in the next section.

Let X be a Banach space and $F : X \to \mathbb{R}$ is a functional.

Definition 2.6 (Directional derivative). The directional derivative of F at $x \in X$ along the direction $h \in X$ is the limit

$$\mathrm{DF}(x;h) = \lim_{\epsilon \to 0} \frac{F(x + \epsilon h) - F(x)}{\epsilon}$$

if it exists.

It is not necessary that the operator defined by $h \mapsto \mathrm{DF}(x;h)$ to be a linear operator. If it is then we may talk about the concept of the Gâteaux derivative.

Definition 2.7 (Gâteaux derivative). A functional F is called Gâteaux (or weakly) differentiable at $x \in X$ if $\mathrm{DF}(x;h)$ exists for any direction $h \in X$ and the operator $h \mapsto \mathrm{DF}(x;h)$ is linear and continuous. Then, by the Riesz representation theorem, there exists some $\mathrm{DF}(x) \in X^*$ such that

$$\lim_{\epsilon \to 0} \frac{F(x + \epsilon h) - F(x)}{\epsilon} =: \mathrm{DF}(x;h) = \langle \mathrm{DF}(x), h\rangle, \quad \forall h \in X.$$

The element $\mathrm{DF}(x)$ is called the Gâteaux derivative of F at x.

The Gâteaux derivative if it exists is unique, a result that follows naturally by the uniqueness of the limit. If X is a finite dimensional space, $X = \mathbb{R}^N$, then the Gâteaux derivative coincides with the gradient ∇F.

Furthermore, it is straightforward to extend the above definition so as to define Gâteaux differentiability and the Gâteaux derivative for a functional F on an open subset $A \subset X$; we simply have to ensure that the above definition holds for every $x \in A$.

3. WEAK SOLUTIONS

In this section, we study the Dirichlet boundary value problem (1) based on the degree theory in Subsection 2.1, where $\Omega \subset I\!R^N$, $N \geq 2$, is a bounded domain with a Lipschitz boundary $\partial\Omega$, $p \in C_+(\overline{\Omega})$ satisfying the log-Hölder continuity condition (7) and $2 \leq p^- \leq p(x) \leq p^+ < \infty$.

3.1. BASIC ASSUMPTIONS AND TECHNICAL LEMMAS

Consider the following functional

$$\Phi(u) := \int_\Omega \frac{1}{p(x)}\Big(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}} + a(x)|u|^{p(x)}\Big)\,\mathrm{d}x, \quad \forall u \in X.$$

Using arguments similar to those in [21], we can show that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous (i.e. for any sequence $(u_n) \subset X$ such that $u_n \rightharpoonup u$ in X it holds that $\liminf \Phi(u_n) \geq \Phi(u)$). Its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, given by

$$\langle \Phi' u, v\rangle := \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla v\,\mathrm{d}x + \int_\Omega a(x)|u|^{p(x)-2}uv\,\mathrm{d}x \tag{9}$$

for all $v \in X$. Moreover, as in [21, Proposition 3.1], we have the following lemma.

Lemma 3.1. *The mapping $\Phi' : X \to X^*$ is strictly monotone, bounded homeomorphism and of type (S_+).*

Let $f : \Omega \times I\!R \times I\!R^N \to I\!R$ be a real-valued function such that:

(A_1): f satisfies the Carathéodory condition, i.e. $f(.,s,\xi)$ is measurable on Ω for all $(s,\xi) \in I\!R \times I\!R^N$ and $f(x,.,.)$ is continuous on $I\!R \times I\!R^N$ for a.e. $x \in \Omega$.

(A_2): f verifies the flowing growth condition

$$|f(x,s,\xi)| \leq c(k(x) + |s|^{q(x)-1} + |\xi|^{q(x)-1})$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in I\!R \times I\!R^N$, where c is a positive constant, $k \in L^{p'(x)}(\Omega)$ and $q \in C_+(\overline{\Omega})$ with $q^+ < p^-$.

A typical example of f satisfying (A_1) and (A_2) is

$$|f(x,s,\xi)| = k(x) + |s|^{q(x)-2}s\log(1+|s|) + |\xi|^{q(x)-1}$$

where $k \in L^{p'(x)}(\Omega)$ is positive and $q \in C_+(\overline{\Omega})$ with $q^+ < p^-$.

Lemma 3.2 ([4]). *Under assumptions (A_1) and (A_2), the operator $S : X \to X^*$ defined as $\langle Su, v\rangle = -\int_\Omega \lambda f(x,u,\nabla u)v\,\mathrm{d}x, \quad \forall u,v \in X$ is compact.*

3.2. EXISTENCE RESULT

When we cannot have a strong solution to an ordinary or partial differential equation (a function that is everywhere differentiable, in some sense, satisfying the equation), we think of a weak solution. This is a function that is nevertheless considered to satisfy the equation in a precisely defined sense. There are many different definitions of a weak solution, appropriate for different classes of equations. One of the most important is based on the notion of distributions or critical points. According to equality (9), we can adopt this definition

Definition 3.3. We say that a function $u \in X = W_0^{1,p(x)}(\Omega)$ is a weak solution of problem (1) if

$$\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla v \, \mathrm{d}x + \int_\Omega a(x)|u|^{p(x)-2}uv \, \mathrm{d}x = \int_\Omega \lambda f(x,u,\nabla u)v \, \mathrm{d}x$$

holds for all $v \in X$.

Theorem 3.4. *Under assumptions* (A_1) *and* (A_2)*, problem* (1) *has a weak solution* u *in* $W_0^{1,p(x)}(\Omega)$.

Proof. Let Φ' and $S : X \to X^*$ be as in Subsection 3.1. Then $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (1) if and only if

$$\Phi' u = -Su \tag{10}$$

Thanks to the properties of the operator Φ' seen in Lemma 3.1 and in view of Minty-Browder Theorem (see [25, Theorem 26A]), the inverse operator $T := \Phi'^{-1} : X^* \to X$ is bounded, continuous and satisfies condition (S_+). Moreover, note by Lemma 3.2 that the operator S is bounded, continuous and quasimonotone. Consequently, equation (10) is equivalent to

$$v + S \circ Tv = 0. \tag{11}$$

where $u = Tv$.

To solve equation (11), we will apply the degree theory introduced in Subsection 2.1. To do this, we first claim that the set

$$B := \{v \in X^* / v + tS \circ Tv = 0 \text{ for some } t \in [0,1]\}$$

is bounded. Indeed, let $v \in B$. Set $u := Tv$, then $\|Tv\|_a = \|u\|_a$.

If $\|u\|_a \leq 1$, then $\|Tv\|_a$ is bounded.

If $\|u\|_a > 1$, then we get by the third assertion of Proposition 2.5, the growth condition (A_2), the Hölder inequality (3), the inequality (6) and the Young inequality the estimate

$$\begin{aligned}
\|Tv\|_a^{p^-} &= \|u\|_a^{p-} \leq \rho(u) \leq \langle \Phi' u, u\rangle \\
&= -t\langle S \circ Tv, Tv\rangle = -t\langle Su, u\rangle = t\lambda \int_\Omega f(x,u,\nabla u)u \, \mathrm{d}x \\
&\leq c_1\Big(\int_\Omega |k(x)u(x)| \, \mathrm{d}x + \rho_{q(x)}(u) + \int_\Omega |\nabla u|^{q(x)-1}|u| \, \mathrm{d}x\Big) \\
&\leq c_2\Big(|k|_{p'(x)}|u|_{p(x)} + |u|_{q(x)}^{q^+} + |u|_{q(x)}^{q^-} + \frac{1}{q'^-}\rho_{q(x)}(\nabla u) + \frac{1}{q^-}\rho_{q(x)}(u)\Big) \\
&\leq c_3\big(|u|_{p(x)} + |u|_{q(x)}^{q^+} + |u|_{q(x)}^{q^-} + |\nabla u|_{q(x)}^{q^+} + |\nabla u|_{q(x)}^{q^+}\big).
\end{aligned}$$

From the Poincaré inequality (8) and the continuous embedding $L^{p(x)} \hookrightarrow L^{q(x)}$, we can deduce the estimate

$$\begin{aligned} \|Tv\|_a^{p^-} &\leq c_4(|\nabla u|_{p(x)} + |\nabla u|_{q(x)}^{q^+} + |\nabla u|_{q(x)}^{q^+}) \\ &= c_4(\|u\| + \|u\|^{q^+} + \|u\|^{q^-}). \end{aligned}$$

Since the two norms $\|\cdot\|$ and $\|\cdot\|_a$ are equivalent in X, $\|u\|_a > 1$ and $1 < q^- \leq q^+$, we can conclude that $\|Tv\|_a^{p^-} \leq c_5\|Tv\|_a^{q^+}$, because the previous $c_i, i = 1, \ldots, 5$ are constants. This means that $\{Tv|v \in B\}$ is bounded.

Since the operator S is bounded, it follows from (11) that the set B is bounded in X^*. Consequently, there exists $R > 0$ such that $|v|_{-1,p'(x)} < R$ for all $v \in B$. This says that $v + tS \circ Tv \neq 0$ for all $v \in \partial B_R(0)$ and all $t \in [0, 1]$. From Lemma 2.4 it follows that $I + S \circ T \in \mathcal{F}_T(\overline{B_R(0)})$ and $I = \Phi' \circ T \in \mathcal{F}_T(\overline{B_R(0)})$.

Since the operators I, S and T are bounded, $I + S \circ T$ is also bounded. We conclude that $I + S \circ T \in \mathcal{F}_{T,B}(\overline{B_R(0)})$ and $I \in \mathcal{F}_{T,B}(\overline{B_R(0)})$. Consider the homotopy $H : [0, 1] \times \overline{B_R(0)} \to X^*$ given by $H(t, v) := v + tS \circ Tv$ for $(t, v) \in [0, 1] \times \overline{B_R(0)}$. Applying the homotopy invariance and normalization property of the degree d stated in Theorem 2.3, we get $d(I + S \circ T, B_R(0), 0) = d(I, B_R(0), 0) = 1$, and hence there exists a point $v \in B_R(0)$ such that $v + S \circ Tv = 0$. We conclude that $u = Tv$ is a weak solution of (1). $\square$

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