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CERTAIN COMMUTATIVITY CRITERIA FOR 3-PRIME NEAR-RINGS VIA HOMODERIVATIONS

SLIMANE EL AMRANI and ABDERRAHMANE RAJI

Abstract. In this work, we study the commutativity of 3-prime near-rings admitting homoderivations that satisfy certain conditions. Additionally, we offer a counter example to demonstrate that the assumption of 3-primeness in our theorems is crucial.

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1. INTRODUCTION

Throughout this paper, $\mathcal{N}$ will be a left zero-symmetric near-ring with multiplicative center $\mathcal{Z}(\mathcal{N})$, that is, $\mathcal{Z}(\mathcal{N}) = \{x \in \mathcal{N} \mid xy = yx \text{ for all } y \in \mathcal{N}\}$. By left distributivity, we can see that $x.0 = 0$, and consequently, $\mathcal{N}$ is zero-symmetric if and only if 0 is an element of $\mathcal{Z}(\mathcal{N})$. Unless otherwise specified, in this note, we will use the word "near-ring" to mean zero-symmetric left near-ring.

A near-ring $\mathcal{N}$ is called 3-prime if $\mathcal{N}$ has the property that for all $x, y \in \mathcal{N}$, if $x\mathcal{N}y = \{0\}$ then $x = 0$ or $y = 0$, and is called 2-torsion-free if $(\mathcal{N}, +)$ has no elements of order 2. We will write, for all $x, y \in \mathcal{N}$, as usual, $[x, y] = xy - yx$ and $x \circ y = xy + yx$ for the Lie product and Jordan product, respectively.

A nonempty subset $\mathcal{U}$ of $\mathcal{N}$ is called a semigroup left ideal (resp. semigroup right ideal) if $\mathcal{N}\mathcal{U} \subseteq \mathcal{U}$ (resp. $\mathcal{U}\mathcal{N} \subseteq \mathcal{U}$); and if $\mathcal{U}$ is both a semigroup left ideal and a semigroup right ideal, then $\mathcal{U}$ is said to be a semigroup ideal.

An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$, or equivalently as noted in [14], that $d(xy) = xd(y) + d(x)y$ for any $x, y \in \mathcal{N}$.

In 1957, to assess the relationship between a derivation and the property of commutativity of a ring, E. C. Posner [9] introduced a technique for studying the structure of a prime ring. In the context of near-rings, the study of derivations was initiated by H. E. Bell and G. Mason [4]. Thereby, several authors have investigated the structure of prime and semiprime rings that admit suitably constrained additive mappings, such as automorphisms, derivations,

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skew derivations, and generalized derivations acting on appropriate subsets of the rings. Furthermore, many authors have proved analogous results for prime and semiprime near-rings (see [5–8, 10–12], etc.).

In 2000, El Sofy [13] defined a homoderivation on a prime ring $\mathcal{R}$ to be an additive mapping h from $\mathcal{R}$ into itself such that

$$h(xy) = h(x)h(y) + h(x)y + xh(y) \text{ for all } x, y \in \mathcal{R}.$$

An example of such mapping is to let $h(x) = f(x) - x$ for all $x \in \mathcal{R}$ where f is an endomorphism on $\mathcal{R}$. A homoderivation h is also a derivation if and only if $h(x)h(y) = 0$ for all $x, y \in \mathcal{R}$. Hence, if $\mathcal{R}$ is a prime ring, the only additive map that is both a derivation and a homoderivation is the zero mapping.

Recently, Al Harfie et al. [1] proved the commutativity of rings admitting a homoderivation h satisfies any one of the following conditions:

(i) $xh(y) \pm xy \in \mathcal{Z}(\mathcal{R})$;

(ii) $xh(y) \pm yx \in \mathcal{Z}(\mathcal{R})$;

(iii) $xh(y) \pm [x, y] \in \mathcal{Z}(\mathcal{R})$;

(iv) $[h(x), y] \pm xy \in \mathcal{Z}(\mathcal{R})$;

(v) $[h(x), y] \pm yx \in \mathcal{Z}(\mathcal{R})$ for all $x, y \in \mathcal{I}$, where $\mathcal{I}$ is an ideal of $\mathcal{R}$.

Motivated by these results, the main objective of this paper is to study the action of a homoderivation satisfying certain differential identities on the structure of a 3-prime near-ring.

2. PRELIMINARY LEMMAS

We begin this section by introducing the following lemmas, which are essential for developing the proof of our results.

LEMMA 2.1 ([3, Lemma 1.4 (i) & Lemma 1.3 (i)]). *Let $\mathcal{N}$ be a 3-prime near-ring and $\mathcal{U}$ be a nonzero semigroup ideal of $\mathcal{N}$.*

(i) *If $x, y \in \mathcal{N}$ and $x\,\mathcal{U}\,y = \{0\}$, then $x = 0$ or $y = 0$.*

(ii) *If $x \in \mathcal{N}$ and $x\,\mathcal{U} = \{0\}$ or $\mathcal{U}\,x = \{0\}$, then $x = 0$.*

LEMMA 2.2 ([3, Lemma 1.2 (iii) & Lemma 1.5]). *Let $\mathcal{N}$ be a 3-prime near-ring.*

(i) *If $z \in \mathcal{Z}(\mathcal{N}) \setminus \{0\}$ and $xz \in \mathcal{Z}(\mathcal{N})$, then $x \in \mathcal{Z}(\mathcal{N})$.*

(ii) *If $\mathcal{N} \subseteq \mathcal{Z}(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

3. MAIN RESULT

In [1, Theorem 2.1], the authors proved the commutativity of a prime ring $\mathcal{R}$ admitting a nonzero homoderivation h which is zero-power valued on a nonzero ideal $\mathcal{I}$ of $\mathcal{R}$.

Our purpose in this section is to prove a similar result under weakened assumptions and this is by considering the case in which h is only a nonzero homoderivation on a near-ring $\mathcal{N}$ instead of $\mathcal{I}$ being an ideal of the ring $\mathcal{R}$.

Note also that, to avoid repetition, in our proofs we only show that $\mathcal{N}$ is a commutative ring under the conditions proposed in the various theorems, knowing that these conditions are immediately satisfied if $\mathcal{N}$ is a commutative ring.

Theorem 3.1. *Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero homoderivation h, then the following assertions are equivalent*

(i) $[[h(x), y] + yx, t] \in \mathcal{Z}(\mathcal{N})$ *for all* $x, y, t \in \mathcal{N}$,

(ii) $[[h(x), y] - yx, t] \in \mathcal{Z}(\mathcal{N})$ *for all* $x, y, t \in \mathcal{N}$,

(iii) $\mathcal{N}$ *is a commutative ring.*

Proof. (i) $\Rightarrow$ (iii). We are assuming that

$$[[h(x), y] + xy, t] \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y, t \in \mathcal{N}. \tag{1}$$

Accordingly, $\Big[\big[[h(x), y] + xy, t\big], r\Big] = 0$ for all $x, y, t, r \in \mathcal{N}$. Replacing t by $\big([h(x), y] + xy\big)t$ in the latter expression and invoking (1), we get

$$\begin{aligned} 0 &= \Big[([h(x), y] + xy)[[h(x), y] + xy, t], r\Big] \\ &= \big[[h(x), y] + xy, t\big]\,\big[[h(x), y] + xy, r\big] \text{ for all } x, y, t, r \in \mathcal{N}. \end{aligned}$$

Left multiplying by n, where $n \in \mathcal{N}$, and using (1) we infer that

$$\big[[h(x), y] + xy, t\big]n\big[[h(x), y] + xy, r\big] = 0,$$

which can be written as $\big[[h(x), y] + xy, t\big]\mathcal{N}\big[[h(x), y] + xy, r\big] = \{0\}$ for all $x, y, t, r \in \mathcal{N}$. In view of 3-primeness of $\mathcal{N}$, the preceding relation shows that either $[[h(x), y] + xy, t] = 0$ or $[[h(x), y] + xy, r] = 0$ for all $x, y, t, r \in \mathcal{N}$. But, both conditions yield to

$$[h(x), y] + yx \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y \in \mathcal{N}. \tag{2}$$

Substituting $h(x)y$ for y, where $x \in \mathcal{N}$, in (2), we find that

$$h(x)([h(x), y] + yx) \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y \in \mathcal{N}.$$

Using Lemma 2.2 (i) together with (2), we find that

$$h(x) \in \mathcal{Z}(\mathcal{N}) \text{ or } [h(x), y] + yx = 0 \text{ for all } x, y \in \mathcal{N}. \tag{3}$$

Suppose that

$$[h(x), y] + yx = 0 \text{ for all } x, y \in \mathcal{N}. \tag{4}$$

Taking $y = h(x)$ in (4), we obtain $h(x)x = 0$ for all $x \in \mathcal{N}$. Again, replacing y by xy in (4), we get $-xyh(x) + xyx = 0$ for all $x, y \in \mathcal{N}$ which means that $x\mathcal{N}(-h(x) + x) = 0$ for all $x \in \mathcal{N}$.

By the 3-primeness of $\mathcal{N}$, we find that $h = id_{\mathcal{N}}$. So, (4) reduces to $xy = 0$ for all $x, y \in \mathcal{N}$ which, in virtue of 3-primeness of $\mathcal{N}$, forces that $\mathcal{N} = \{0\}$, a contradiction. Consequently, equation (3) ensures the existence of two elements $x_0, y_0 \in \mathcal{N}$ such that $[h(x_0), y_0] + y_0x_0 \neq 0$ and therefore, we will have $x_0 \neq 0$ and $h(x_0) \in \mathcal{Z}(\mathcal{N})$.

Putting $y = h(x_0)$ in (2), we infer that $h(x_0)x \in \mathcal{Z}(\mathcal{N})$ for all $x \in \mathcal{N}$ and hence by applying of Lemma 2.2 (i), we obtain

$$x \in \mathcal{Z}(\mathcal{N}) \text{ or } h(x_0) = 0 \text{ for all } x \in \mathcal{N}.$$

If $h(x_0) \neq 0$, the previous relation shows that $\mathcal{N} \subseteq \mathcal{Z}(\mathcal{N})$ and therefore $\mathcal{N}$ is a commutative ring by Lemma 2.2 (ii).

Now, suppose that $h(x_0) = 0$; putting x_0 instead of x in (2), we find that $yx_0 \in \mathcal{Z}(\mathcal{N})$ for all $y \in \mathcal{N}$. In virtue of Lemma 2.2 (i) and $x_0 \neq 0$, we obtain $y \in \mathcal{N}$ for all $y \in \mathcal{N}$. Consequently, $\mathcal{N}$ must be a commutative ring by Lemma 2.2 (ii).

For (ii)⇒(iii), using similar techniques as above, we get the required result. □

As an application of the previous theorem, we consider differential identities in which we combining commutators and anti-commutators. We obtain the following results.

Theorem 3.2. *Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero homoderivation h, then the following propositions are equivalent*

(i) $[[h(x), y] + yx, t] \circ r \in \mathcal{Z}(\mathcal{N})$ *for all* $x, y, t, r \in \mathcal{N}$,

(ii) $[[h(x), y] - yx, t] \circ r \in \mathcal{Z}(\mathcal{N})$ *for all* $x, y, t, r \in \mathcal{N}$,

(iii) $\mathcal{N}$ *is a commutative ring.*

Proof. Let's show that (i)⇒(iii). Suppose that

$$[[h(x), y] + yx, t] \circ r \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y, t, r \in \mathcal{N}. \tag{5}$$

Assume that $\mathcal{Z}(\mathcal{N}) = \{0\}$. In this case, (5) gives

$$[[h(x), y] + yx, t]r = r(-[[h(x), y] + yx, t]) \text{ for all } x, y, t, r \in \mathcal{N}. \tag{6}$$

Putting $r = mr$, where $m \in \mathcal{N}$, in (6) and using it again, we find that $m(-[[h(x), y]+yx, t])r = mr(-[[h(x), y]+yx, t])$ for all $x, y, t, r, m \in \mathcal{N}$ which implies that $\mathcal{N}[-[[h(x), y] + yx, t], r] = \{0\}$ for all $x, y, t, r \in \mathcal{N}$.

Applying Lemma 2.1 (ii) together with our assumption that $\mathcal{Z}(\mathcal{N}) = \{0\}$, we obtain $[[h(x), y] + yx, t] = 0$ for all $x, y, t \in \mathcal{N}$. The last equation gives $[h(x), y] + yx = 0$ for all $x, y \in \mathcal{N}$. Since the previous relation is similar to (4), we deduce that $\mathcal{N} = \{0\}$ leading to a contradiction and therefore $\mathcal{Z}(\mathcal{N}) \neq \{0\}$.

Choosing $0 \neq z_0 \in \mathcal{Z}(\mathcal{N})$ and replacing r by z_0 in (5),we get

$$z_0\big([[h(x), y] + yx, t] + [[h(x), y] + yx, t]\big) \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y, t \in \mathcal{N}. \tag{7}$$

Invoking Lemma 2.2 (i) and as $z_0 \neq 0$, (7) shows that $[[h(x), y] + yx, t] + [[h(x), y]+yx, t] \in \mathcal{Z}(\mathcal{N})$ for all $x, y, t \in \mathcal{N}$. Once again, taking $[[h(x), y]+yx, t]$ instead of r in (5), we get

$$[[h(x), y]+yx, t]([[h(x), y]+yx, t]+[[h(x), y]+yx, t]) \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y, t \in \mathcal{N}.$$

Using Lemma 2.2 (i), the former equation yields

$$[[h(x),y]+yx,t]+[[h(x),y]+yx,t]=0 \text{ or } [[h(x),y]+yx,t]\in\mathcal{Z}(\mathcal{N}),$$

for all $x,y,t\in\mathcal{N}$. Let $x,y,t\in\mathcal{N}$ such that $[[h(x),y]+yx,t]+[[h(x),y]+yx,t]\neq 0$ then $[[h(x),y]+yx,t]\in\mathcal{Z}(\mathcal{N})$.

Now, suppose that there are $x_0,y_0,t_0\in\mathcal{N}$ such that $[[h(y_0),x_0]+y_0x_0,t_0]+[[h(y_0),x_0]+y_0x_0,t_0]=0$, then $[[h(y_0),x_0]+y_0x_0,t_0]=-[[h(y_0),x_0]+y_0x_0,t_0]$.

Putting $x=x_0, y=y_0, t=t_0$ and $r=[[h(y_0),x_0]+y_0x_0,t_0]r$ in (5), we find that

$$[[h(y_0),x_0]+y_0x_0,t_0][[h(y_0),x_0]+y_0x_0,t_0]\circ r\in\mathcal{Z}(\mathcal{N}) \text{ for all } r\in\mathcal{N}$$

which, in virtue of Lemma 2.2 (i), means that

$$[[h(y_0),x_0]+y_0x_0,t_0]\circ r=0 \text{ for all } r\in\mathcal{N} \text{ or } [[h(y_0),x_0]+y_0x_0,t_0]\in\mathcal{Z}(\mathcal{N}).$$

If there is an element $r_0\in\mathcal{N}$ such that $[[h(y_0),x_0]+y_0x_0,t_0]\circ r_0\neq 0$, the last relation shows that $[[h(y_0),x_0]+y_0x_0,t_0]\in\mathcal{Z}(\mathcal{N})$. Else, suppose that $[[h(y_0),x_0]+y_0x_0,t_0]\circ r=0$ for all $r\in\mathcal{N}$, thus $[[h(y_0),x_0]+y_0x_0,t_0]r=-r[[h(y_0),x_0]+y_0x_0,t_0]=r(-[[h(y_0),x_0]+y_0x_0,t_0])=r([[h(y_0),x_0]+y_0x_0,t_0])$ for all $r\in\mathcal{N}$. Accordingly, $[[h(y_0),x_0]+y_0x_0,t_0]\in\mathcal{Z}(\mathcal{N})$.

So, in both conditions, we find that $[[h(y_0),x_0]+y_0x_0,t_0]\in\mathcal{Z}(\mathcal{N})$. Consequently, $[[h(x),y]+yx,t]\in\mathcal{Z}(\mathcal{N})$ for all $x,y,t\in\mathcal{N}$ and hence $\mathcal{N}$ is a commutative ring by Theorem 3.1.

For (ii)⇒(iii), we can use a similar approach as above with suitable changes. □

THEOREM 3.3. *Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero homoderivation h, then the following conditions are equivalent*

(i) $([h(x),y]+yx)\circ t\in\mathcal{Z}(\mathcal{N})$ *for all* $x,y,t\in\mathcal{N}$,

(ii) $([h(x),y]-yx)\circ t\in\mathcal{Z}(\mathcal{N})$ *for all* $x,y,t\in\mathcal{N}$,

(iii) $\mathcal{N}$ *is a commutative ring.*

Proof. We prove that (i)⇒ (iii). Let h a homoderivation of $\mathcal{N}$ verifying the following condition

$$([h(x),y]+yx)\circ t\in\mathcal{Z}(\mathcal{N}) \text{ for all } x,y,t\in\mathcal{N}. \tag{8}$$

Our goal is to prove that $\mathcal{Z}(\mathcal{N})\neq\{0\}$. Indeed, suppose that $\mathcal{Z}(\mathcal{N})=\{0\}$. In this case, the relation (8) yields

$$([h(x),y]+yx)t=t(-([h(x),y]+yx)) \text{ for all } x,y,t\in\mathcal{N}. \tag{9}$$

Putting $t=mt$, where $m\in\mathcal{N}$, in (9) and using it again, we find that $m(-([h(x),y]+yx))t=mt(-([h(x),y]+yx))$ for all $x,y,t,m\in\mathcal{N}$ which implies that $\mathcal{N}[-([h(x),y]+yx),t]=\{0\}$ for all $x,y,t\in\mathcal{N}$.

Applying Lemma 2.1 (ii) and using the hypothesis that $\mathcal{Z}(\mathcal{N})=\{0\}$, we obtain $[h(x),y]+yx=0$ for all $x,y\in\mathcal{N}$. Since the latter result is similar to (4), we conclude that $\mathcal{N} = \{0\}$, but this outcome is not possible. Thus, $\mathcal{Z}(\mathcal{N}) \neq \{0\}$.

Choosing $0 \neq z_0 \in \mathcal{Z}(\mathcal{N})$ and taking $t = z_0$ in (8), we get $z_0([h(x), y] + yx + [h(x), y] + yx) \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{N}$. In virtue of Lemma 2.2 (i), we infer that $2\big([h(x), y] + yx\big) \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{N}$. Now, replacing t by $[h(x), y] + yx$ in (8), we obtain $([h(x), y] + yx)([h(x), y] + yx + [h(x), y] + yx) \in \mathcal{Z}(\mathcal{N})$ for all $x, y, \in \mathcal{N}$. By Lemma 2.2 (i), the previous relation shows that

$$2([h(x), y] + yx) = 0 \text{ or } [h(x), y] + yx \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y \in \mathcal{N}. \tag{10}$$

Let x and y be two arbitrary elements of $\mathcal{N}$. If $2([h(x), y] + yx) \neq 0$, then according to the previous relation, we conclude that $[h(x), y] + yx \in \mathcal{Z}(\mathcal{N})$. Otherwise, suppose there exist $x_0, y_0 \in \mathcal{N}$ such that $2([h(y_0), x_0] + y_0x_0) = 0$, then $[h(y_0), x_0] + y_0x_0 = -([h(y_0), x_0] + y_0x_0)$.

Taking $x = x_0, y = y_0$ and $t = ([h(y_0), x_0] + y_0x_0)t$ in (8), we obtain

$$([h(y_0), x_0] + y_0x_0)([h(y_0), x_0] + y_0x_0) \circ t \in \mathcal{Z}(\mathcal{N}) \text{ for all } t \in \mathcal{N}.$$

It follows that

$$([h(y_0), x_0] + y_0x_0) \circ t = 0 \text{ or } [h(y_0), x_0] + y_0x_0 \in \mathcal{Z}(\mathcal{N}) \text{ for all } t \in \mathcal{N}.$$

If there is an element $t_0 \in \mathcal{N}$ such that $([h(y_0), x_0] + y_0x_0) \circ t_0 \neq 0$, the last relation shows that $[h(y_0), x_0] + y_0x_0 \in \mathcal{Z}(\mathcal{N})$.

Now, suppose that $([h(y_0), x_0] + y_0x_0) \circ t = 0$ for all $t \in \mathcal{N}$, then for each $t \in \mathcal{N}$, we have

$$\begin{aligned} ([h(y_0), x_0] + y_0x_0)t &= -t([h(y_0), x_0] + y_0x_0) \\ &= t(-([h(y_0), x_0] + y_0x_0)) \\ &= t([h(y_0), x_0] + y_0x_0). \end{aligned}$$

Accordingly, $[h(y_0), x_0] + y_0x_0 \in \mathcal{Z}(\mathcal{N})$. Consequently, (10) reduces to

$$[h(x), y] + yx \in \mathcal{Z}(\mathcal{N}),$$

for all $x, y \in \mathcal{N}$ which is like to (2) in Theorem 3.1. Then, by using the same reasoning as in Theorem 3.1, we get the required result.

(ii)⇒(iii). Using similar arguments to achieve the desired result. □

In [2], M. Ashraf and A. Boua studied the commutativity of near-rings admitting a semiderivation d that satisfies $d([x, y]) + x \circ y \in \mathcal{Z}(\mathcal{N})$ and $d([x, y]) - x \circ y \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{N}$. In the following result, we prove the analogous commutativity theorem using the concept of homoderivations.

Theorem 3.4. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a nonzero homoderivation h, then the following assertions are equivalent*

(i) $(h([x, y]) + x \circ y) \circ t \in \mathcal{Z}(\mathcal{N})$ *for all* $x, y, t \in \mathcal{N}$,

(ii) $(h([x, y]) - x \circ y) \circ t \in \mathcal{Z}(\mathcal{N})$ *for all* $x, y, t \in \mathcal{N}$,

(iii) $\mathcal{N}$ *is a commutative ring.*

Proof. Let's show that (i)⇒(iii). Assuming that

$$(h([x,y]) + x \circ y) \circ t \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y, t \in \mathcal{N}. \tag{11}$$

Suppose that $\mathcal{Z}(\mathcal{N}) = \{0\}$. In this case, (11) becomes $(h([x,y])+x\circ y)\circ t = 0$ for all $x, y, t \in \mathcal{N}$. Thus $(h([x,y]) + x \circ y)t = t(-(h([x,y]) + x \circ y))$ for all $x, y, t \in \mathcal{N}$.

Putting $t = rt$, where $r \in \mathcal{N}$, in last equation, we find that $r(-(h([x,y])+x\circ y))t = rt(-(h([x,y])+x\circ y))$ for all $x, y, r, t \in \mathcal{N}$ which leads to $\mathcal{N}[-(h([x,y])+ x \circ y), t] = \{0\}$ for all $x, y, t \in \mathcal{N}$.

Applying Lemma 2.1 (ii), we obtain $-(h([x,y]) + x \circ y) \in \mathcal{Z}(\mathcal{N}) = \{0\}$; so that $h([x,y]) + x \circ y = 0$ for all $x, y \in \mathcal{N}$.

Taking $y = x$ and by 2-torsion freeness of $\mathcal{N}$, we get $x^2 = 0$ for all $x \in \mathcal{N}$, and hence we can observe that $x(x + y)^2 = 0$ for all $x, y \in \mathcal{N}$ which yields $xyx = 0$ for all $x, y \in \mathcal{N}$.

By the 3-primeness of $\mathcal{N}$, we conclude that $\mathcal{N} = \{0\}$, leading to a contradiction. Consequently, $\mathcal{Z}(\mathcal{N}) \neq \{0\}$.

Putting z_0 instead of t in (11), where $z_0 \in \mathcal{Z}(\mathcal{N}) \setminus \{0\}$, we find that $z_0(h([x,y]) + x \circ y + h([x,y]) + x \circ y) \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{N}$. Once again, replacing t by $h([x,y]) + x \circ y$ in (11), we obtain $(h([x,y]) + x \circ y)(h([x,y]) + x \circ y + h([x,y]) + x \circ y) \in \mathcal{Z}(\mathcal{N})$ for all $x, y, \in \mathcal{N}$. In view of Lemma 2.2 (i), the previous relation shows that

$$2\big(h([x,y]) + x \circ y\big) = 0 \text{ or } h([x,y]) + x \circ y \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y \in \mathcal{N}. \tag{12}$$

Since $\mathcal{N}$ is 2-torsion free, the first condition yields $h([x,y]) + x \circ y = 0$ and hence (12) shows that

$$h([x,y]) + x \circ y \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y \in \mathcal{N}. \tag{13}$$

Putting $y = x$, in (13), we find that $2x^2 \in \mathcal{Z}(\mathcal{N})$ for all $x \in \mathcal{N}$. In particular, replacing x by $2x^2$ in (13), we get $2x^2(y + y) \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{Z}(\mathcal{N})$.

Using Lemma 2.2 (i) together 2-torsion freeness of $\mathcal{N}$, we obtain

$$x^2 = 0 \text{ or } y + y \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y \in \mathcal{N}$$

But the condition $x^2 = 0$ for all $x \in \mathcal{N}$ cannot be verified, and thus $y + y \in \mathcal{Z}(\mathcal{N})$ for all $y \in \mathcal{N}$. Putting $y = y^2$ in the last result, we get $y(y + y) \in \mathcal{Z}(\mathcal{N})$ for all $y \in \mathcal{N}$. Hence, in view of Lemma 2.2 (i) and the 2-torsion freeness of $\mathcal{N}$ we conclude that $\mathcal{N} \subseteq \mathcal{Z}(\mathcal{N})$. Then $\mathcal{N}$ is a commutative ring by Lemma 2.2 (ii).

(i)⇒(iii). Reasoning as above, we obtain the required result. □

THEOREM 3.5. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a nonzero homoderivation h, then the following assertions are equivalent*

(i) $[h([x,y]) + x \circ y, t] \circ r \in \mathcal{Z}(\mathcal{N})$ *for all* $x, y, t, r \in \mathcal{N}$,

(ii) $[h([x,y]) - x \circ y, t] \circ r \in \mathcal{Z}(\mathcal{N})$ *for all* $x, y, t, r \in \mathcal{N}$,

(iii) $\mathcal{N}$ *is a commutative ring.*

Proof. We show that (i)⇒(iii). By hypothesis given, we have

$$[h([x,y]) + x \circ y, t] \circ r \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y, t, r \in \mathcal{N}. \tag{14}$$

Our aim in this case is to show that $\mathcal{Z}(\mathcal{N}) \neq \{0\}$.

In fact, suppose that $\mathcal{Z}(\mathcal{N}) = \{0\}$ and putting $[h([x,y]) + x \circ y, t]$ instead of r in (14), we get $2([h([x,y]) + x \circ y, t])^2 = 0$ for all $x, y, t \in \mathcal{N}$.

By 2-torsion freeness of $\mathcal{N}$, the latter result yields $([h([x,y]) + x \circ y, t])^2 = 0$ for all $x, y, t \in \mathcal{N}$.

On the other hand, from (14) we have $[h([x,y]) + x \circ y, t]r + r[h([x,y]) + x \circ y, t] = 0$ for all $x, y, t, r \in \mathcal{N}$. Left multiplying the previous relation by $[h([x,y]) + x \circ y, t]$, we find that $[h([x,y]) + x \circ y, t]r[h([x,y]) + x \circ y, t] = 0$ for all $x, y, t, r \in \mathcal{N}$ and hence it follows that $[h([x,y]) + x \circ y, t]\mathcal{N}[h([x,y]) + x \circ y, t] = \{0\}$ for all $x, y, t \in \mathcal{N}$.

In the light of the 3-primeness of $\mathcal{N}$, we obtain $[h([x,y]) + x \circ y, t] = 0$ for all $x, y, t \in \mathcal{N}$, which means that $h([x,y]) + x \circ y = 0$ for all $x, y \in \mathcal{N}$. In particular, for $y = x$ and by 2-torsion freeness of $\mathcal{N}$, we infer that $x^2 = 0$ for all $x \in \mathcal{N}$.

Using the same arguments as used in the proof of the Theorem 3.4, we arrive at $\mathcal{N} = \{0\}$, a contradiction which means that $\mathcal{Z}(\mathcal{N}) \neq \{0\}$.

Now, let $0 \neq z_0 \in \mathcal{Z}(\mathcal{N})$ and replacing r by z_0 in (14), we get

$$z_0([h([x,y]) + x \circ y, t] + [h([x,y]) + x \circ y, t]) \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y, t \in \mathcal{N}. \tag{15}$$

Invoking Lemma 2.2 (i) and using the fact that $z_0 \neq 0$, (15) gives

$$2\big[h([x,y]) + x \circ y, t\big] \in \mathcal{Z}(\mathcal{N})$$

for all $x, y, t \in \mathcal{N}$.

Once again, replacing r by $[h([x,y]) + x \circ y, t]$ in (14), we get

$$\big[h([x,y]) + x \circ y, t\big]\Big([h([x,y]) + x \circ y, t] + [h([x,y]) + x \circ y, t]\Big) \in \mathcal{Z}(\mathcal{N})$$

for all $x, y, t \in \mathcal{N}$. Using Lemma 2.2 (i), the former equation yields

$$2[h([x,y]) + x \circ y, t] = 0 \text{ or } [h([x,y]) + x \circ y, t] \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y, t \in \mathcal{N}.$$

By the 2-torsion freeness of $\mathcal{N}$, both conditions assure that

$$[h([x,y]) + x \circ y, t] \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y, t \in \mathcal{N}. \tag{16}$$

Substituting $\big(h([x,y]) + x \circ y\big)t$ for t in (16) and invoking Lemma 2.2 (i), we conclude that

$$[h([x,y]) + x \circ y, t] = 0 \text{ or } h([x,y]) + x \circ y \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y, t \in \mathcal{N}$$

that is,

$$h([x,y]) + x \circ y \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y \in \mathcal{N}. \tag{17}$$

Since (17) is similar to (13), by using techniques identical to those described after (13) in Theorem 3.4, we achieve the desired result.

(ii)⇒(iii). A proof can be given by using a similar approach to the one used in the first part. □

The following example demonstrates that the "3-primeness of $\mathcal{N}$" used in all theorems cannot be omitted.

EXAMPLE 3.6. Let $(\mathbb{R}, +)$ be the usual additive group of real numbers and $(M_2(\mathbb{Z}), +)$ be the usual additive group of 2×2 matrices with integer coefficients. Let $\mathcal{S} = \mathbb{R} \times M_2(\mathbb{Z})$ and define the additive law in $\mathcal{S}$ by $(s, x) \oplus (t, y) = (s + t, x + y)$, and the multiplication law $*$ on $\mathcal{S}$ by $(s, x) * (t, y) = (0, \det(x)y)$ for all $(s, x), (t, y) \in \mathcal{S}$. We can prove that $(\mathcal{S}, \oplus, *)$ is a 2-torsion free left near-ring that is not 3-prime. Define $\mathcal{N}$ and h as follows:

$$\mathcal{N} = \left\{ \begin{pmatrix} (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) \\ (s,x) & (t,y) & (0,0) \end{pmatrix} \mid (0,0), (s,x), (t,y) \in \mathcal{S} \right\},$$

$$h \begin{pmatrix} (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) \\ (s,x) & (t,y) & (0,0) \end{pmatrix} = \begin{pmatrix} (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) \\ (s,x) & (0,0) & (0,0) \end{pmatrix}.$$

Clearly, $\mathcal{N}$ is a 2-torsion free left near-ring which is not 3-prime, and h is a nonzero homoderivation on $\mathcal{N}$ that satisfies the following conditions:

(i) $[[h(A), B] \pm BA, C] \in \mathcal{Z}(\mathcal{N})$,

(ii) $[[h(A), B] \pm BA, C] \circ D \in \mathcal{Z}(\mathcal{N})$,

(iii) $([h(A), B] \pm BA) \circ C \in \mathcal{Z}(\mathcal{N})$,

(iv) $(h([A, B]) \pm A \circ B) \circ C \in \mathcal{Z}(\mathcal{N})$,

(v) $[h([A, B]) \pm A \circ B, C] \circ D \in \mathcal{Z}(\mathcal{N})$,

for all $A, B, C, D \in \mathcal{N}$. But, since the multiplicative law of $\mathcal{N}$ is not commutative, then $\mathcal{N}$ cannot be a commutative ring.

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Sultan Moulay Slimane University
Faculty of Sciences and Technology
LMACS Laboratory
Beni Mellal, Morocco
E-mail: slimaneelamrani1994@gmail.com

E-mail: rajiabd2@gmail.com
https://orcid.org/0000-0002-4091-7097