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A SHORT NOTE ON LOCALLY CONVEX TOPOLOGIES

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Abstract. Given a normed space $(X, ||\cdot||)$ and a linear (unbounded and not densely defined) operator $A\colon D \subset X \to X$, we show how to define norms $|||\cdot|||$ on X such that $A\colon (D, ||\cdot||) \to (X, |||\cdot|||)$ becomes bounded and D becomes dense with respect to $|||\cdot|||$. We then apply our results to m-accretive operators.

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1. INTRODUCTION

Let X be a set, D a nonempty subset of X, and $\mathcal{T}$ a topology on X such that $\overline{D}^{\mathcal{T}} \neq X$, where $\overline{D}^{\mathcal{T}}$ (or, simply, $\overline{D}$, when there is no danger of confusion) stands for the closure of D in the topological space $(X, \mathcal{T})$. Denote by

$$\mathcal{M} := \{\mathcal{T}' \mid \mathcal{T}' \text{ is a topology on } X,\ \mathcal{T}' \subseteq \mathcal{T},\ \overline{D}^{\mathcal{T}'} = X\}. \tag{1.1}$$

The set $\mathcal{M}$ is nonempty, since it contains the indiscrete topology on X. Other elements in $\mathcal{M}$ are, for instance, the sets $\{\emptyset, O, X\}$, where $O \in \mathcal{T}$ is such that $O \cap D \neq \emptyset$. Zorn's Lemma yields that there is at least one maximal element in $(\mathcal{M}, \subseteq)$, where $\subseteq$ is the inclusion of sets.

The problem of constructing elements in $\mathcal{M}$, respectively, topics related to this problem have a pretty long history. We will briefly sketch a short overview, without claiming to cover all contributions made in this direction. First contributions are due to P. Alexandroff and P. Urysohn ([1] and [2]), followed by A. Tychonoff ([14]), E. Čech ([5]), M. H. Stone ([13]), E. Hewitt ([8]), and M. Katětov ([9] and [10]). Among other topics, these papers contain early systematic studies of extension/coarsening methods. The presented constructions give canonical ways to alter a topology so that a given subset has prescribed closure properties. We further mention [4] and [11]. The monograph [12] contains many illustrative examples showing what can/cannot happen when modifying topologies to alter density or separation properties. Developed in the context of groups, [6] characterises when a subset can be dense in some coarser group topology, offering insight into constructing topologies that force density. Moreover, this paper offers examples how to build coarser/finer topologies forcing density under algebraic constraints. Finally we mention [7]

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as providing techniques for building dense linear subspaces and for controlling density properties.

In this note, we show how one can obtain topologies $\mathcal{T}'$ in $\mathcal{M}$ in the following setting:

- $(X, ||\cdot||)$ is a normed space (thus $\mathcal{T}$ is the topology induced by the norm $||\cdot||$),
- D is a linear subspace of X and the domain of a linear operator $A\colon D \to X$,
- $\mathcal{T}'$ is induced by a norm $|||\cdot|||$.

So, these topologies $\mathcal{T}'$ are all locally convex topologies on X. The main tool in constructing such norms $|||\cdot|||$ on X is the resolvent $R(\lambda, A)$ of A, where λ belongs to the resolvent set $\rho(A)$ of A. Furthermore, the resulting topologies $\mathcal{T}'$ will have the property that $A\colon (D, ||\cdot||) \to (X, |||\cdot|||)$ is bounded.

In this paper, all linear spaces are considered over the field $\mathbb{K}$ of real or complex numbers. If X, Y are linear spaces over $\mathbb{K}$, then, for a linear operator A defined on a subspace of X with values in Y, the domain of A is denoted by $D(A)$ and its range by $R(A)$.

2. THE MAIN RESULT

The first result emphasises how one can define norms on a linear space in order to ensure the boundedness of certain linear operators. For this, we introduce the following notations.

Notation. Let $(X, ||\cdot||)$ be a normed space and let $B\colon X \to X$ be a linear and injective operator. We denote by $||\cdot||_B\colon X \to \mathbb{R}$ the norm defined by

$$||x||_B = ||B(x)||,\ \forall\, x \in X,$$

and by $\mathcal{T}_B$ the topology induced by it.

Remark 2.1. Let $(X, ||\cdot||)$ be a normed space, and denote by $\mathcal{T}$ the topology induced by the norm $||\cdot||$. If B, $C\colon X \to X$ are linear and injective operators, and $D \subseteq X$, then the following (straightforward) equivalences hold.

$$\mathcal{T}_B \subseteq \mathcal{T} \;\Leftrightarrow\; B\colon (X, ||\cdot||) \to (X, ||\cdot||) \text{ is bounded.} \tag{2.1}$$

$$\overline{D}^{\mathcal{T}_B} = X \;\Leftrightarrow\; R(B) \subseteq \overline{B(D)}^{\mathcal{T}}. \tag{2.2}$$

$$\mathcal{T}_B \subseteq \mathcal{T}_C \;\Leftrightarrow\; \exists a > 0 \text{ such that } ||Bx|| \leq a||Cx||,\ \forall x \in X. \tag{2.3}$$

Lemma 2.2. *Let $(X, ||\cdot||)$ be a normed space, $A\colon D(A) \subset X \to X$ a linear operator, and $B\colon X \to X$ a linear and injective operator such that $BA\colon (D(A), ||\cdot||) \to (X, ||\cdot||)$ is bounded. Then the operator $A\colon (D(A), ||\cdot||) \to (X, ||\cdot||_B)$ is bounded.*

Proof. Let $c > 0$ be so that $||BAx|| \leq c||x||$, for every $x \in D(A)$. The relations $||A(x)||_B = ||B(A(x))|| \leq c||x||$, for every $x \in D(A)$, imply the conclusion. □

In what follows, $(X, ||\cdot||)$ is a normed space, $\mathcal{T}$ the topology induced by $||\cdot||$, D a linear subspace of X such that $\overline{D}^{\mathcal{T}} \neq X$, and $A\colon D \to X$ a linear operator. Let I be the identity operator on X. Recall that the resolvent set of A is defined by

$$\rho(A) := \{\lambda \in \mathbb{K} \,|\lambda I - A\colon D \to X \text{ is bijective and } (\lambda I - A)^{-1} : X \to X \text{ is bounded}\}.$$

For $\lambda \in \rho(A)$, denote the resolvent $(\lambda I - A)^{-1}$ of A at λ by R_λ. Then $R(R_\lambda) = D$ and the following equalities hold

$$\lambda R_\lambda x - x = AR_\lambda x, \forall\, x \in X, \tag{2.4}$$

$$\lambda R_\lambda x - x = R_\lambda Ax, \forall\, x \in D. \tag{2.5}$$

Note that (2.4) yields in particular that

$$AR_\lambda x \in D, \forall\, x \in D. \tag{2.6}$$

Fix arbitrary λ, $\mu \in \rho(A)$. Then

$$(\mu - \lambda)I = (\mu I - A) - (\lambda I - A).$$

Taking on both sides of the above equality the composition (on the left) with R_μ, we get

$$(\mu - \lambda)R_\mu = I - R_\mu(\lambda I - A).$$

Taking now on both sides of the above equality the composition (on the right) with R_λ, we obtain the so-called resolvent identity

$$R_\lambda - R_\mu = (\mu - \lambda)R_\mu R_\lambda,\ \forall \lambda,\, \mu \in \rho(A). \tag{2.7}$$

In particular, this identity yields

$$R_\lambda R_\mu = R_\mu R_\lambda,\ \forall \lambda,\, \mu \in \rho(A). \tag{2.8}$$

For $\lambda \in \rho(A)$, we set the notations

$$||\cdot||_\lambda := ||\cdot||_{R_\lambda},\ \mathcal{T}_\lambda := \mathcal{T}_{R_\lambda}. \tag{2.9}$$

Lemma 2.3. *Let $(X, ||\cdot||)$ be a normed space, $A\colon D \to X$ a linear operator, and $\lambda \in \rho(A)$. Then the operator $A\colon (D, ||\cdot||) \to (X, ||\cdot||_\lambda)$ is bounded.*

Proof. Relation (2.5) implies that the operator $R_\lambda A\colon (D, ||\cdot||) \to (X, ||\cdot||)$ is bounded. Thus the assertion follows from Lemma 2.2. □

Lemma 2.4. *Let $(X, ||\cdot||)$ be a normed space, $A\colon D \to X$ a linear operator, and λ, $\mu \in \rho(A)$. Then the following assertions hold.*

1° $R_\lambda(D) = R_\mu(D)$.

2° $\mathcal{T}_\lambda = \mathcal{T}_\mu$.

Proof. 1° Equality (2.8) yields that $R_\lambda(R(R_\mu)) = R_\mu(R(R_\lambda))$, thus $R_\lambda(D) = R_\mu(D)$.

2° Pick $x \in X$. Using (2.7) and (2.8), we get

$$||R_\lambda x|| \leq ||R_\mu x|| + |\mu - \lambda| \cdot ||R_\mu R_\lambda x|| \leq ||R_\mu x|| + |\mu - \lambda| \cdot ||R_\lambda|| \cdot ||R_\mu x||.$$

Hence

$$||R_\lambda x|| \leq ||R_\mu x|| \cdot (1 + |\mu - \lambda| \cdot ||R_\lambda||).$$

According to (2.3), we get that $\mathcal{T}_\lambda \subseteq \mathcal{T}_\mu$. The converse inclusion is obtained similarly. This proves the claim. □

LEMMA 2.5. *Let $(X, ||\cdot||)$ be a normed space, $A\colon D \to X$ a linear operator, $\lambda \in \rho(A)$, and $(\lambda_n)_{n\in\mathbb{N}}$ a sequence in $\rho(A)$ such that $\lim\limits_{n\to\infty} ||R_{\lambda_n}|| = 0$. Then the following assertions hold for every $x \in D$.*

1° $\lim\limits_{n\to\infty} ||\lambda_n R_{\lambda_n} x - x|| = 0.$

2° $\lim\limits_{n\to\infty} ||\lambda_n A R_{\lambda_n} x - Ax||_\lambda = 0.$

Proof. Pick $x \in D$.

1° According to (2.5), the following inequality holds

$$||\lambda_n R_{\lambda_n} x - x|| \leq ||R_{\lambda_n}|| \cdot ||Ax||, \forall\, n \in \mathbb{N}.$$

Since $\lim\limits_{n\to\infty} ||R_{\lambda_n}|| = 0$, the conclusion follows.

Assertion 2° folows from 1° and from Lemma 2.3. □

THEOREM 2.6. *Let $(X, ||\cdot||)$ be a normed space and $A\colon D \to X$ a linear operator with the property that* $\inf\{||R_\mu|| \mid \mu \in \rho(A)\} = 0$. *Then the following assertions hold for every $\lambda \in \rho(A)$.*

1° *The operator $A\colon (D, ||\cdot||) \to (X, ||\cdot||_\lambda)$ is bounded.*

2° *D is dense in X with respect to $||\cdot||_\lambda$.*

Proof. Let $\lambda \in \rho(A)$.

Assertion 1° follows from Lemma 2.3.

2° By the hypothesis, there is a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\rho(A)$ such that $\lim\limits_{n\to\infty} ||R_{\lambda_n}|| = 0$. Pick an arbitrary $x \in D$. In view of assertion 1° of Lemma 2.4,

$$\lambda_n R_{\lambda_n} x \in R_\lambda(D),\ \forall n \in \mathbb{N}.$$

Assertion 1° of Lemma 2.5 implies then that

$$x \in \overline{R_\lambda(D)}^{\mathcal{T}}.$$

Since $x \in D$ was arbitrarily chosen, the following inclusion follows

$$D = R(R_\lambda) \subseteq \overline{R_\lambda(D)}^{\mathcal{T}}.$$

The conclusion follows now from (2.2). □

REMARK 2.7. Under the assumptions of Theorem 2.6, and also taking into account (2.1), note that the topology $\mathcal{T}_\lambda$ belongs to the set $\mathcal{M}$ defined in (1.1).

3. AN APPLICATION TO CERTAIN ACCRETIVE OPERATORS

In this section, $(X, ||\cdot||)$ is assumed to be a Banach space, and $A\colon D(A) \subset X \to X$ an m-accretive operator (where $D(A)$ is a linear subspace of X), i.e., A is linear and satisfies the properties

(i) $||x + \lambda Ax|| \geq ||x||$, for every $x \in D(A)$ and every $\lambda > 0$,
(ii) $R(I + A) = X$.

REMARK 3.1. By [3, Proposition 3.3], the equality $R(I + \lambda A) = X$ holds for every $\lambda > 0$. It follows that $-\frac{1}{\lambda} \in \rho(A)$ and that $\left|\left|R_{-\frac{1}{\lambda}}\right|\right| \leq \lambda$, for every $\lambda > 0$.

For $\lambda > 0$, denote by $A_\lambda\colon X \to X$ the operator

$$A_\lambda := -\frac{1}{\lambda}\left(-\frac{1}{\lambda}R_{-\frac{1}{\lambda}} - I\right),$$

and by

$$|||\cdot|||_\lambda := ||\cdot||_{-\frac{1}{\lambda}},$$

where $||\cdot||_{-\frac{1}{\lambda}}$ is defined according to (2.9). The operators $(A_\lambda)_{\lambda>0}$ are called the *Yosida approximants of* A. In view of (2.4),

$$A_\lambda = -\frac{1}{\lambda}AR_{-\frac{1}{\lambda}}, \text{ for every } \lambda > 0. \tag{3.1}$$

THEOREM 3.2. *Let $(X, ||\cdot||)$ be a Banach space, $A\colon D(A) \subset X \to X$ an m-accretive operator, and $(A_\lambda)_{\lambda>0}$ the family of Yosida approximants of A. Then the following assertions hold for every $\mu > 0$.*

1° *The operator $A\colon (D(A), ||\cdot||) \to (X, |||\cdot|||_\mu)$ is bounded.*
2° $\lim\limits_{\lambda \searrow 0} |||A_\lambda x - Ax|||_\mu = 0, \forall\, x \in D(A)$.
3° *$D(A)$ is dense in X with respect to $|||\cdot|||_\mu$.*

Proof. Pick $\mu > 0$.
Assertion 1° follows from Lemma 2.3.
Assertion 2° follows from the equality

$$\lim_{\lambda \searrow 0} \left|\left|R_{-\frac{1}{\lambda}}\right|\right| = 0,$$

from (3.1), and from assertion 2° of Lemma 2.5.
Assertion 3° is a consequence of assertion 2° of Theorem 2.6. □

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