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ON SOME TCHEBYSHEV-TYPE INEQUALITIES INVOLVING THE k-WEIGHTED FRACTIONAL OPERATOR

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Abstract. In this paper, we use the k-weighted fractional integral of functions with respect to another function to generalize Tchebyshev-type fractional integral inequalities. Some inequalities involving k-weighted fractional integrals are also be proved.

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Key words. Tchebyshev inequality, Hilfer operator, Hölder inequality.

1. INTRODUCTION

In [3], for the Tchebyshev functional (1)

$$T(f,\, g)(x) = \frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\,\frac{1}{b-a}\int_a^b g(x)\,\mathrm{d}x, \tag{1}$$

the Tchebyshev inequality is given as follows:

$$T(f,\, g)(x) \geq 0, \tag{2}$$

where f, and g are two integrable functions, synchronous on $[a,b]$.

Belarbi and Dahmani [2] developed the following result about Tchebyshev's inequality using Riemann-Liouville fractional integral operators. Let f and g be two synchronous functions on $[0,+\infty[$. Then for all $x>0$, $\alpha>0$,

$$\mathrm{I}^{\alpha}(f\,g)(x) \geq \frac{\Gamma(\alpha+1)}{x^{\alpha}}\mathrm{I}^{\alpha}f(x)\,\mathrm{I}^{\alpha}g(x). \tag{3}$$

Over the previous decade, several authors established and verified different new integral inequalities of type (3) for the Tchebyshev functional (1) using the various fractional integral operators, see [1,4,5,7,10–12].

On the other hand, the weighted fractional integrals are defined, for an integrable function f on the interval $[a,b]$ and for a differentiable function μ such that $\mu'(t)\neq 0$ for all $t\in[a,b]$, as follows,

$${}_{a^+}I_w^{\beta}f(x) = \frac{1}{w(x)\Gamma(\beta)}\int_a^x \mu'(s)(\mu(x)-\mu(s))^{\beta-1}w(s)f(s)\mathrm{d}s, \quad x>a,$$

where w is a weighted function (positive measurable function) [6].

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2. k-WEIGHTED FRACTIONAL OPERATOR

In this section, we present a definition of the k-weighted fractional integrals of a function f with respect to the function ψ and we prove that they are bounded in a specified space. Let $[a,b]\subseteq[0,+\infty)$, where $a<b$.

Definition 2.1. The k-weighted fractional integral operators are defined as follows: Let $\alpha>0$, $k>0$ and ψ be a positive, strictly increasing differentiable function such that $\psi'(s)\neq 0$ for all $s\in[a,b]$. The left and right sided k-weighted fractional integral of a function f with respect to the function ψ on $[a,b]$ are defined respectively as follows:

$$ {}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}f(x)=\frac{1}{w(x)k\,\Gamma_k(\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\frac{\alpha}{k}-1}w(t)f(t)\mathrm{d}t \tag{4}$$

for $a<x\leq b$, and

$$ {}_{b^-}\mathrm{J}^{\alpha,\psi}_{k,w}f(x)=\frac{1}{w(x)k\,\Gamma_k(\alpha)}\int_x^b \psi'(t)(\psi(t)-\psi(x))^{\frac{\alpha}{k}-1}w(t)f(t)\mathrm{d}t, \tag{5}$$

for $a\leq x<b$, where w is a weighted non-decreasing function and the k-gamma function defined by

$$\Gamma_k(\alpha)=\int_0^\infty t^{\alpha-1}e^{-\frac{t^k}{k}}\mathrm{d}t.$$

The space $L_p^W[a,b]$ of all real-valued Lebesgue measurable functions f on $[a,b]$ with norm conditions:

$$\parallel f\parallel_p^W=\left(\int_a^b\mid f(x)\mid^p W(x)\mathrm{d}x\right)^{\frac{1}{p}}<\infty,\ 1\leq p<+\infty.$$

is known as weighted Lebesgue space, where W be a weight function (measurable and positive).

(i) Putting $W\equiv 1$, the space $L_p^W[a,b]$ reduces to the classical space $L_p[a,b]$.

(ii) Choosing $W(x)=w^p(x)\,\psi'(x)$, we get

$$ L_{X_w^p}[a,b]=\left\{f:\parallel f\parallel_{X_w^p}=\left(\int_a^b\mid w(x)f(x)\mid^p\psi'(x)\mathrm{d}x\right)^{\frac{1}{p}}<\infty\right\}. \tag{6}$$

In the next theorem, we show that the the k-weighted fractional operators are bounded.

Theorem 2.2. *The fractional integrals* (4), (5) *are defined for functions* $f\in L_{X^p_{\psi'}}[a,b]$, *existing almost everywhere and*

$$ {}_{a^+}\mathrm{I}^{\psi}_{w}f(x)\in L_{X_w^p}[a,b],\qquad {}_{b^-}\mathrm{I}^{\psi}_{w}f(x)\in L_{X_w^p}[a,b]. \tag{7}$$

Moreover

$$\left\|{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w} f(x)\right\|_{X^p_w} \le C\ \|f(x)\|_{X^p_w}, \qquad \left\|{}_{b^-}\mathrm{J}^{\alpha,\psi}_{k,w} f(x)\right\|_{X^p_w} \le C\ \|f(x)\|_{X^p_w}, \tag{8}$$

where

$$C = \frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}.$$

Proof. Let $f \in L_{X^p_w}[a,b]$ and

$$\frac{1}{p}+\frac{1}{p'}=1,$$

we have

$$\left\|{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w} f\right\|^p_{X^p_w} = \int_a^b \mid w(x)\,{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w} f(x) \mid^p \psi'(x)\,\mathrm{d}x$$

$$= \frac{1}{k^p\,\Gamma^p_k(\alpha)} \int_a^b \left| \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\frac{\alpha}{k}-1} w(t) f(t)\,\mathrm{d}t \right|^p \psi'(x)\,\mathrm{d}x$$

$$= \frac{1}{k^p\,\Gamma^p_k(\alpha)} \int_a^b \left| \int_a^x w(t)\,f(t) \left(\psi'(t)(\psi(x)-\psi(t))^{\frac{\alpha}{k}-1}\right)^{\frac{1}{p}+\frac{1}{p'}} \mathrm{d}t \right|^p \psi'(x)\,\mathrm{d}x.$$

Using Hölder inequality for $p \ge 1$, we get

$$\left\|a^+\mathrm{J}^{\alpha,\psi}_{k,w} f\right\|^p_{X^p_{\psi'}},$$

$$\le \frac{1}{k^p\,\Gamma^p_k(\alpha)} \int_a^b \left| \int_a^x w^p(t) f^p(t)\,\psi'(t)(\psi(x)-\psi(t))^{\frac{\alpha}{k}-1}\,\mathrm{d}t \right|$$
$$\times \left| \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\frac{\alpha}{k}-1}\,\mathrm{d}t \right|^{p-1} \psi'(x)\,\mathrm{d}x$$

$$\le \frac{1}{k^p\,\Gamma^p_k(\alpha)} \int_a^b \left(\int_a^x |w(t)\,f(t)|^p\ \psi'(t)(\psi(x)-\psi(t))^{\frac{\alpha}{k}-1}\,\mathrm{d}t \right)$$
$$\times \left(\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\frac{\alpha}{k}-1}\,\mathrm{d}t \right)^{p-1} \psi'(x)\,\mathrm{d}x$$

$$= \frac{k^{p-1}}{\alpha^{p-1}}\ \frac{1}{k^p\,\Gamma^p_k(\alpha)} \int_a^b \int_a^x |w^p(t)\,f^p(t)|\ \psi'(t)(\psi(x)-\psi(t))^{\frac{\alpha}{k}-1}$$
$$\times (\psi(x)-\psi(a))^{\frac{\alpha(p-1)}{k}}\ \psi'(x)\,\mathrm{d}t\,\mathrm{d}x$$

$$\le \frac{\alpha\ (\psi(b)-\psi(a))^{\frac{\alpha(p-1)}{k}}}{k\,\Gamma^p_k(\alpha+k)}$$
$$\times \int_a^b \int_a^x |w^p(t)\,f^p(t)|\ \psi'(t)(\psi(x)-\psi(t))^{\frac{\alpha}{k}-1}\,\psi'(x)\,\mathrm{d}t\,\mathrm{d}x.$$

Applying Fubini's theorem, we deduce

$$
\begin{aligned}
&\left\| {}_{a^+}\mathrm{J}\,^{\alpha,\psi}_{k,w} f \right\|^p_{X^p_{\psi'}} \\
&\leq \frac{\alpha\,(\psi(b)-\psi(a))^{\frac{\alpha(p-1)}{k}}}{k\,\Gamma_k^p(\alpha+k)} \\
&\int_a^b |w^p(t)\,f^p(t)|\;\psi'(t)\left(\int_t^b (\psi(x)-\psi(t))^{\frac{\alpha}{k}-1}\,\psi'(x)\,\mathrm{d}x\right)\mathrm{d}t \\
&= \frac{(\psi(b)-\psi(a))^{\frac{\alpha(p-1)}{k}}}{\Gamma_k^p(\alpha+k)}\int_a^b |w^p(t)\,f^p(t)|\;\psi'(t)\,(\psi(b)-\psi(t))^{\frac{\alpha}{k}}\;\mathrm{d}t \\
&\leq \frac{(\psi(b)-\psi(a))^{\frac{\alpha(p-1)}{k}}}{\Gamma_k^p(\alpha+k)}\;(\psi(b)-\psi(a))^{\frac{\alpha}{k}}\int_a^b |w^p(t)\,f^p(t)|\;\psi'(t)\,\mathrm{d}t \\
&= \frac{(\psi(b)-\psi(a))^{\frac{\alpha p}{k}}}{\Gamma_k^p(\alpha+k)}\;\|f\|^p_{X^p_w}\,.
\end{aligned}
$$

Similarly

$$
\int_a^b |\,w(x)\,{}_{b^-}\mathrm{J}\,^{\alpha,\psi}_{k,w} f(x)\,|^p\;\psi'(x)\mathrm{d}x \leq \frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\;\|f\|_{X^p_w}\,.
$$

This gives us our desired formulas (8) and (7). $\square$

One important feature of the k-weighted fractional operators is that they depend on the choice of the functions w, and ψ, and they give rise to certain types of k-weighted fractional integrals.

(i) Taking $w(\tau)=1$, the k-weighted fractional reduces to the k-Hilfer operator of order $\alpha>0$ (or generalized k-fractional integrals [8])

$$
\begin{aligned}
{}_{a^+}\mathcal{J}_k^{\alpha} f(x) &= \frac{1}{k\,\Gamma_k(\alpha)}\int_a^x (\psi(x)-\psi(s))^{\frac{\alpha}{k}-1}\psi'(s)f(s)\mathrm{d}s, \quad x>a, \\
{}_{b^-}\mathcal{J}_k^{\alpha} f(x) &= \frac{1}{k\,\Gamma_k(\alpha)}\int_x^b (\psi(s)-\psi(x))^{\frac{\alpha}{k}-1}\psi'(s)f(s)\mathrm{d}s, \quad x<b.
\end{aligned}
\tag{9}
$$

(ii) Taking $\psi(\tau)=\tau$, the k-weighted fractional is simplified to the k-weighted Riemann-Liouville fractional operator of order $\alpha>0$

$$
\begin{aligned}
{}_{a^+}\mathcal{RL}_{k,w}^{\alpha} f(x) &= \frac{1}{w(x)k\,\Gamma_k(\alpha)}\int_a^x (x-s)^{\frac{\alpha}{k}-1}w(s)f(s)\mathrm{d}s, \quad x>a, \\
{}_{b^-}\mathcal{RL}_{k,w}^{\alpha} f(x) &= \frac{1}{w(x)k\,\Gamma_k(\alpha)}\int_x^b (s-x)^{\frac{\alpha}{k}-1}w(s)f(s)\mathrm{d}s, \quad x<b.
\end{aligned}
\tag{10}
$$

(iii) Using $\psi(\tau) = \ln\tau$, the k-weighted fractional reduces to the k-weighted Hadamard fractional operator of order $\alpha > 0$

$$(11)\quad \begin{aligned} {}_{a^+}\mathcal{H}^{\alpha}_{k,w} f(x) &= \frac{1}{w(x)k\,\Gamma_k(\alpha)} \int_a^x \left(\ln\frac{x}{s}\right)^{\frac{\alpha}{k}-1} w(s)f(s)\frac{\mathrm{d}s}{s}, \quad x > a > 1,\\ {}_{b^-}\mathcal{H}^{\alpha}_{k,w} f(x) &= \frac{1}{w(x)k\,\Gamma_k(\alpha)} \int_x^b \left(\ln\frac{s}{x}\right)^{\frac{\alpha}{k}-1} w(s)f(s)\frac{\mathrm{d}s}{s}, \quad 1 < x < b. \end{aligned}$$

(iv) Putting

$$\psi(\tau) = \frac{\tau^{\rho+1}}{\rho+1}$$

where $\rho > 0$, the k-weighted fractional makes it similar to the k-weighted Katugompola fractional operators of order $\alpha > 0$, (or (k,s)-weighted fractional) [9].

$$(12)\quad \begin{aligned} {}_{a^+}\mathcal{K}^{\alpha}_{k,w} f(x) &= \frac{(\rho+1)^{1-\frac{\alpha}{k}}}{w(x)k\,\Gamma_k(\alpha)} \int_a^x \left(x^{\rho+1} - s^{\rho+1}\right)^{\frac{\alpha}{k}-1} w(s)f(s)s^{\rho}\mathrm{d}s, \ x > a,\\ {}_{b^-}\mathcal{K}^{\alpha}_{k,w} f(x) &= \frac{(\rho+1)^{1-\frac{\alpha}{k}}}{w(x)k\,\Gamma_k(\alpha)} \int_x^b \left(s^{\rho+1} - x^{\rho+1}\right)^{\frac{\alpha}{k}-1} w(s)f(s)s^{\rho}\mathrm{d}s, \ x < b. \end{aligned}$$

(v) Setting $\psi(\tau) = \frac{(\tau-a)^\theta}{\theta}$ $(\psi(\tau) = -\frac{(b-\tau)^\theta}{\theta})$ respectively where $\theta > 0$, the left sided (right sided) k-weighted fractional respectively is reduced to the k-weighted fractional conformable operator of order $\alpha > 0$ [13].

$$(13)\quad \begin{aligned} &{}_{a^+}\mathcal{C}^{\alpha}_{k,w} f(x)\\ &= \frac{\theta^{1-\frac{\alpha}{k}}}{w(x)k\,\Gamma_k(\alpha)} \int_a^x \Big((x-a)^\theta - (s-a)^\theta\Big)^{\frac{\alpha}{k}-1} \frac{w(s)\,f(s)}{(s-a)^{1-\theta}}\mathrm{d}s, \ x > a,\\ &{}_{b^-}\mathcal{C}^{\alpha}_{k,w} f(x)\\ &= \frac{\theta^{1-\frac{\alpha}{k}}}{w(x)k\,\Gamma_k(\alpha)} \int_x^b \Big((b-x)^\theta - (b-s)^\theta\Big)^{\frac{\alpha}{k}-1} \frac{w(s)\,f(s)}{(b-s)^{1-\theta}}\mathrm{d}s, \ x < b. \end{aligned}$$

3. TCHEBYSHEV-TYPE INEQUALITIES VIA k-WEIGHTED FRACTIONAL OPERATORS

We present some basic notations that we utilize in this study as well as a significant remark.

- A non-decreasing function on $[a,b]$ is an increasing or constant function on $[a,b]$.
- Let $a, b \in R$ where $b > a$. Two functions f and g are said to be synchronous on $[a,b]$ if

$$(f(x) - f(y))(g(x) - g(y)) \geq 0, \ \text{for all } x, y \in [a,b].$$

Remark 3.1. Since w is non-decreasing on $[a,b]$, applying the notion of k-weighted fractional integral (4), we get

$$\begin{aligned} {}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}(1)(x) &= \frac{1}{w(x)k\,\Gamma_k(\alpha)}\int_a^x \psi'(s)(\psi(x)-\psi(s))^{\frac{\alpha}{k}-1}w(s)\,\mathrm{d}s \\ &\leq \frac{1}{w(a)k\,\Gamma_k(\alpha)}\int_a^x \psi'(s)(\psi(x)-\psi(s))^{\frac{\alpha}{k}-1}w(b)\,\mathrm{d}s \\ &= \frac{w(b)}{w(a)\,k\,\Gamma_k(\alpha)}\int_a^x \psi'(s)(\psi(x)-\psi(s))^{\frac{\alpha}{k}-1}\,\mathrm{d}s, \end{aligned}$$

therefore

$$\frac{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}{w(a)\,\Gamma_k(\alpha+k)} \geq {}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}(1)(x). \tag{14}$$

Theorem 3.2. *Let f and g be two synchronous functions on $[a,b]$ and ψ be an increasing and positive function on $[a,b]$, having a continuous derivative ψ' on $[a,b]$ and also $x>a$, $\alpha,\beta,\ k>0$, then the following inequalities hold:*

$$\begin{aligned} &\frac{w(b)\,(\psi(x)-\psi(a))^{\frac{\beta}{k}}}{w(a)\,\Gamma_k(\alpha+k)}\ {}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}(f\,g)(x) \\ &\qquad + \frac{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}{w(a)\,\Gamma_k(\alpha+k)}\ {}_{a^+}\mathrm{J}^{\beta,\psi}_{k,w}(f\,g)(x) \\ &\geq {}_{a^+}\mathrm{J}^{\alpha,\psi}_{k}f(x)\,{}_{a^+}\mathrm{J}^{\beta,\psi}_{k,w}g(x) + {}_{a^+}\mathrm{J}^{\beta,\psi}_{k}f(x)\,{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}g(x). \end{aligned} \tag{15}$$

and

$${}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}(f\,g)(x) \geq \frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\ {}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}f(x)\ {}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}g(x). \tag{16}$$

Proof. Given that f and g are synchronous functions on $[a,b]$, then for all $t,s\in[a,b]$, we have

$$(f(t)-f(s))(g(t)-g(s)) \geq 0,$$

then

$$f(t)g(t)+f(s)g(s) \geq f(t)g(s)+f(s)g(t), \tag{17}$$

Multiplying the inequality (17) by

$$\frac{\psi'(t)(\psi(x)-\psi(t))^{\frac{\alpha}{k}-1}w(t)}{w(x)\,k\,\Gamma_k(\alpha)}$$

and integrating with respect to t over (a,x), we get

$$\begin{aligned} &{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}f(x)g(x) + f(s)g(s)\,{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}(1)(x) \\ &\geq g(s)\,{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}f(x) + f(s)\,{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}g(x). \end{aligned} \tag{18}$$

Now, multiplying the above inequality (18) by

$$\frac{\psi'(s)(\psi(x)-\psi(s))^{\frac{\beta}{k}-1}w(s)}{w(x)k\,\Gamma_k(\beta)}$$

and integrating with respect to s over (a,x), we get

$$\begin{aligned}(19)\qquad & {}_{a^+}\mathrm{J}^{\beta,\psi}_{k,w}(1)(x)\,{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k}(f\,g)(x)+{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}(1)(x)\,{}_{a^+}\mathrm{J}^{\beta,\psi}_{k}(f\,g)(x)\\ &\geq {}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}f(x)\,{}_{a^+}\mathrm{J}^{\beta,\psi}_{k}g(x)+{}_{a^+}\mathrm{J}^{\beta,\psi}_{k,w}f(x)\,{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k}g(x).\end{aligned}$$

Combining inequality (19) and inequality (14) yields inequality (15).

We get the acquired inequality (16) by putting $\beta=\alpha$ through inequality (15). $\square$

THEOREM 3.3. *Let $i=1,2,\ldots,m$ and $f,\ f_i\in L_{X^p_\psi}{}'[a,b]$. If the functions f and f_i are positive increasing functions on $[a,b]$, then for all integer $m\geq 1$ we have*

$$(20)\qquad {}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}\prod_{i=1}^{m}f_i(x)\geq\left(\frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\right)^{m-1}\left(\prod_{i=1}^{m}{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}f_i(x)\right).$$

and

$$(21)\qquad {}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}f^m(x)\geq\left(\frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\right)^{m-1}\left({}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}f(x)\right)^m.$$

Proof. For $m=1$ equality holds. Let $m\neq 1$. By using inequality (16) such that $g=\prod_{i=2}^m f_i$, we get

$$\begin{aligned}&{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}\left\{\prod_{i=1}^{m}f_i(x)\right\}\\
&\geq\frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}f_1(x)\,{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}\left\{\prod_{i=2}^{m}f_i(x)\right\}\\
&\geq\frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k}f_1(x)\\
&\qquad\left(\frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}f_2(x)\,{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}\left\{\prod_{i=3}^{m}f_i(x)\right\}\right)\\
&=\left(\frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\right)^2\left(\prod_{i=1}^{2}{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}f_i(x)\right){}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}\left\{\prod_{i=3}^{m}f_i(x)\right\}\\
&\geq\left(\frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\right)^3\left(\prod_{i=1}^{3}{}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}f_i(x)\right){}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}\left\{\prod_{i=4}^{m}f_i(x)\right\}\end{aligned}$$

$\vdots$

$$\geq \left(\frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\right)^{m-1}\left(\prod_{i=1}^{m-1} {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f_i(x)\right) {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f_m(x)$$

$$= \left(\frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\right)^{m-1}\left(\prod_{i=1}^{m} {}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} f_i(x)\right),$$

this gives us the desired inequality (20).

We get inequality (21) through the use of $f_i = f$ in inequality (20). $\square$

THEOREM 3.4. *Let f and g be two functions defined on $L_{X_w^p}[a,b]$, such that f is monotone and g is differentiable on $]a,b[$ and there exist real numbers $m := \inf_{x\in[a,b]} g'(x)$ and $M := \sup_{x\in[a,b]} g'(x)$.*

- *If f is an increasing function, then*

$$\begin{aligned} {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(f\,g)(x) \geq{}& \frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\ {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x)\ {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} g(x) \\ &- m\,\frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\ {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x)\ {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} I_d(x) \\ &+ m\ {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(I_d f)(x)\,. \end{aligned} \tag{22}$$

- *If f is a decreasing function, then*

$$\begin{aligned} {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(f\,g)(x) \geq{}& \frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\ {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x)\ {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} g(x) \\ &- M\,\frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\ {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x)\ {}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} I_d(x) \\ &+ M\ {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(I_d f)(x)\,, \end{aligned} \tag{23}$$

where $I_d(x)$ is the identity function.

Proof. Taking $G(x) = g(x) - m\,x$, G is differentiable and increasing on $]a,b[$. By using inequality (16), we get

$$\begin{aligned} &{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(f\,G)(x) \\ &\geq \frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\ {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x)\ {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(g - m\,x)(x) \\ &= \frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\ {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x)\ {}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} g(x) \\ &\quad - m\,\frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\ {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x)\ {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} I_d(x). \end{aligned} \tag{24}$$

Additionally, we have

$$(25)\qquad {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(g-mI_d)(x) = {}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}(fg)(x) - m\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}(I_d f)(x).$$

Using inequalities (24)-(25), we get the desired inequality (22).

Consider the situation of decreasing functions f and G assuming that

$$G(x) = g(x) - M\,x.$$

The inequality's (23) proof is identical to the proof of inequality (22). $\square$

4. TCHEBYSHEV INEQUALITIES WITH CERTAIN OPERATORS

Tchebyshev-type inequalities involving some particular operators are now given.

4.1. TCHEBYSHEV-TYPE INEQUALITIES VIA k-HILFER OPERATORS

Setting $w(\tau) = 1$, the left side k-weighted fractional ${}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}$ reduces to the left side k-Hilfer operator (9) of order $\alpha > 0$ and the following results hold.

COROLLARY 4.1. *Let f and g be two synchronous functions on $[a,b]$ and $x > a$, α, β, $k > 0$, then*

$$\frac{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}{w(a)\,\Gamma_k(\alpha+k)}\, {}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}(f\,g)(x) + \frac{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}{w(a)\,\Gamma_k(\alpha+k)}\, {}_{a^+}\mathrm{J}_{k}^{\beta,\psi}(f\,g)(x)$$

$$\geq {}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} f(x)\, {}_{a^+}\mathrm{J}_{k}^{\beta,\psi} g(x) + {}_{a^+}\mathrm{J}_{k}^{\beta,\psi} f(x)\, {}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} g(x).$$

and

$${}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}(f\,g)(x) \geq \frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\, {}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} f(x)\, {}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} g(x).$$

COROLLARY 4.2. *Let $i = 1, 2, ..., m$ and f, $f_i \in L_{X_w^p}[a,b]$. If the functions f and f_i are positive increasing functions on $[a,b]$, then for all integer $m \geq 1$ we have*

$${}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} \prod_{i=1}^{m} f_i(x) \geq \left(\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\right)^{m-1} \left(\prod_{i=1}^{m} {}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} f_i(x)\right).$$

and

$${}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} f^m(x) \geq \left(\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\right)^{m-1} \left({}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} f(x)\right)^{m}.$$

Corollary 4.3. *Let f and g be two functions defined on $L_{X^p_w}[a,b]$, such that f is monotone and g is differentiable on $]a,b[$ and there exist real numbers $m := \inf_{x\in[a,b]} g'(x)$ and $M := \sup_{x\in[a,b]} g'(x)$.*

- *If f is an increasing function, then*

$$\begin{aligned}{}_{a^+}\mathrm{J}_k^{\alpha,\psi}(f\,g)(x) \geq& \frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}f(x)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}g(x)\\ &- m\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}f(x)\,{}_{a^+}\mathrm{J}_w^{\alpha,\psi}I_d(x) + m\,{}_{a^+}\mathrm{J}_w^{\alpha,\psi}(I_d f)(x).\end{aligned}$$

- *If f is a decreasing function, then*

$$\begin{aligned}{}_{a^+}\mathrm{J}_k^{\alpha,\psi}(f\,g)(x) \geq& \frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}f(x)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}g(x)\\ &- M\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}f(x)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}I_d(x) + M\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}(I_d f)(x),\end{aligned}$$

where $I_d(x)$ is the identity function.

Remark 4.4. The above corollaries are new results related to Tchebyshev's inequality according to k-Hilfer operators. If we put $a = 0$, we obtain Theorem 4.5, Theorem 4.1, Theorem 4.9 and Theorem 4.13 in [8].

Setting $k = 1$, $b = +\infty$ and $a = 0$, we obtain Theorem 7, Theorem 6, Theorem 8, Corollary 2 and Theorem 10 in [12].

4.2. TCHEBYSHEV-TYPE INEQUALITIES VIA k-WEIGHTED RIEMANN-LIOUVILLE OPERATORS

Setting $\psi(\tau) = \tau$, the left side k-weighted fractional ${}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}$ reduces to the left side k-weighted Riemann-Liouville operator (10) of order $\alpha > 0$ and the following results hold.

Corollary 4.5. *Let f and g be two synchronous functions on $[a,b]$ and $x > a$, α, β, $k > 0$, then*

$$\begin{aligned}&\frac{w(b)\,(x-a)^{\frac{\alpha}{k}}}{w(a)\,\Gamma_k(\alpha+k)}\,{}_{a^+}\mathrm{RL}_{k,w}^{\alpha}(f\,g)(x) + \frac{w(b)\,(x-a)^{\frac{\alpha}{k}}}{w(a)\,\Gamma_k(\alpha+k)}\,{}_{a^+}\mathrm{RL}_{k,w}^{\beta}(f\,g)(x)\\ &\geq {}_{a^+}\mathrm{RL}_k^{\alpha}f(x)\,{}_{a^+}\mathrm{RL}_{k,w}^{\beta}g(x) + {}_{a^+}\mathrm{RL}_k^{\beta}f(x)\,{}_{a^+}\mathrm{RL}_{k,w}^{\alpha}g(x).\end{aligned}$$

and

$${}_{a^+}\mathrm{RL}_{k,w}^{\alpha}(f\,g)(x) \geq \frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(x-a)^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{RL}_{k,w}^{\alpha}f(x)\,{}_{a^+}\mathrm{RL}_{k,w}^{\alpha}g(x).$$

Corollary 4.6. *Let $i = 1, 2, ..., m$ and $f,\ f_i \in L_{X_w^p}[a, b]$. If the functions f and f_i are positive increasing functions on $[a, b]$, then for all integer $m \geq 1$ we have*

$$ {}_{a^+}\mathrm{RL}_{k,w}^{\alpha} \prod_{i=1}^{m} f_i(x) \geq \left(\frac{w(a)\,\Gamma_k(\alpha + k)}{w(b)\,(x-a)^{\frac{\alpha}{k}}} \right)^{m-1} \left(\prod_{i=1}^{m} {}_{a^+}\mathrm{RL}_{k,w}^{\alpha} f_i(x) \right). $$

and

$$ {}_{a^+}\mathrm{RL}_{k,w}^{\alpha} f^m(x) \geq \left(\frac{w(a)\,\Gamma_k(\alpha + k)}{w(b)\,(x-a)^{\frac{\alpha}{k}}} \right)^{m-1} \left({}_{a^+}\mathrm{RL}_{k,w}^{\alpha} f(x) \right)^m. $$

Corollary 4.7. *Let f and g be two functions defined on $L_{X_w^p}[a, b]$, such that f is monotone and g is differentiable on $]a, b[$ and there exist real numbers $m := \inf_{x\in[a,b]} g'(x)$ and $M := \sup_{x\in[a,b]} g'(x)$.*

- *If f is an increasing function, then*

$$ \begin{aligned} {}_{a^+}\mathrm{RL}_{k,w}^{\alpha}(f\,g)(x) &\geq \frac{w(a)\,\Gamma_k(\alpha + k)}{w(b)\,(x-a)^{\frac{\alpha}{k}}}\ {}_{a^+}\mathrm{RL}_{k,w}^{\alpha} f(x)\ {}_{a^+}\mathrm{RL}_{k,w}^{\alpha} g(x) \\ &- m\frac{w(a)\,\Gamma_k(\alpha + k)}{w(b)\,(x-a)^{\frac{\alpha}{k}}}\ {}_{a^+}\mathrm{RL}_{k,w}^{\alpha} f(x)\ {}_{a^+}\mathrm{RL}_{k,w}^{\alpha} I_d(x) +\ m\ {}_{a^+}\mathrm{RL}_{k,w}^{\alpha} (I_d f)(x). \end{aligned} $$

- *If f is a decreasing function, then*

$$ \begin{aligned} {}_{a^+}\mathrm{RL}_{k,w}^{\alpha}(f\,g)(x) &\geq \frac{w(a)\,\Gamma_k(\alpha + k)}{w(b)\,(x-a)^{\frac{\alpha}{k}}}\ {}_{a^+}\mathrm{RL}_{k,w}^{\alpha} f(x)\ {}_{a^+}\mathrm{RL}_{k,w}^{\alpha} g(x) \\ &- M\frac{w(a)\,\Gamma_k(\alpha + k)}{w(b)\,(x-a)^{\frac{\alpha}{k}}}\ {}_{a^+}\mathrm{RL}_{k,w}^{\alpha} f(x)\ {}_{a^+}\mathrm{RL}_{k,w}^{\alpha} I_d(x) +\ M\ {}_{a^+}\mathrm{RL}_{k,w}^{\alpha} (I_d f)(x), \end{aligned} $$

where $I_d(x)$ is the identity function.

Remark 4.8. The above corollaries are new results related to Tchebyshev's inequality according to k-weighted Riemann-Liouville operators. If we take $w(x) = 1$ and $a = 0$, we obtain Corollary 4.8, Corollary 4.3, Corollary 4.11 and Corollary 4.15 in [8].

4.3. TCHEBYSHEV-TYPE INEQUALITIES VIA k-WEIGHTED HADAMARD OPERATORS

Setting $\psi(\tau) = \ln \tau$, the left side k-weighted fractional ${}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}$ reduces to the left side k-weighted Hadamard operator (11) with order $\alpha > 0$ and the following results hold.

Corollary 4.9. *Let f and g be two synchronous functions on $[a, b]$ and $x > a,\ \alpha, \beta,\ k > 0$, then*

$$ \begin{aligned} &\frac{w(b)\,(\ln(\frac{x}{a}))^{\frac{\alpha}{k}}}{w(a)\,\Gamma_k(\alpha + k)}\ {}_{a^+}\mathrm{H}_{k,w}^{\alpha}(f\,g)(x) + \frac{w(b)\,(\ln(\frac{x}{a}))^{\frac{\alpha}{k}}}{w(a)\,\Gamma_k(\alpha + k)}\ {}_{a^+}\mathrm{H}_{k,w}^{\beta,\psi}(f\,g)(x) \\ &\geq\ {}_{a^+}\mathrm{H}_{k}^{\alpha} f(x)\ {}_{a^+}\mathrm{H}_{k,w}^{\beta} g(x) +\ {}_{a^+}\mathrm{H}_{k}^{\beta} f(x)\ {}_{a^+}\mathrm{H}_{k,w}^{\alpha} g(x). \end{aligned} $$

and

$$ {}_{a^+}\mathrm{H}^{\alpha}_{k,w}(f\,g)(x) \geq \frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\ln(\frac{x}{a}))^{\frac{\alpha}{k}}}\; {}_{a^+}\mathrm{H}^{\alpha}_{k,w}f(x)\; {}_{a^+}\mathrm{H}^{\alpha}_{k,w}g(x). $$

COROLLARY 4.10. *Let $i = 1, 2, ..., m$ and $f,\ f_i \in L_{X^p_w}[a,b]$. If the functions f and f_i are positive increasing functions on $[a,b]$, then for all integer $m \geq 1$ we have*

$$ {}_{a^+}\mathrm{H}^{\alpha}_{k,w}\prod_{i=1}^{m} f_i(x) \geq \left(\frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\ln(\frac{x}{a}))^{\frac{\alpha}{k}}}\right)^{m-1} \left(\prod_{i=1}^{m} {}_{a^+}\mathrm{H}^{\alpha}_{k,w}f_i(x)\right). $$

and

$$ {}_{a^+}\mathrm{H}^{\alpha}_{k,w}f^m(x) \geq \left(\frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\ln(\frac{x}{a}))^{\frac{\alpha}{k}}}\right)^{m-1} \left({}_{a^+}\mathrm{H}^{\alpha}_{k,w}f(x)\right)^m. $$

COROLLARY 4.11. *Let f and g be two functions defined on $L_{X^p_w}[a,b]$, such that f is monotone and g is differentiable on $]a,b[$ and there exist real numbers $m := \inf_{x\in[a,b]} g'(x)$ and $M := \sup_{x\in[a,b]} g'(x)$.*

- *If f is an increasing function, then*

$$ \begin{aligned} {}_{a^+}\mathrm{H}^{\alpha}_{k,w}(f\,g)(x) \geq{}& \frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\ln(\frac{x}{a}))^{\frac{\alpha}{k}}}\; {}_{a^+}\mathrm{H}^{\alpha}_{k,w}f(x)\, {}_{a^+}\mathrm{H}^{\alpha}_{k,w}g(x) \\ &- m\frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\ln(\frac{x}{a}))^{\frac{\alpha}{k}}}\; {}_{a^+}\mathrm{H}^{\alpha}_{k,w}f(x)\, {}_{a^+}\mathrm{H}^{\alpha}_{k,w}I_d(x) + m\, {}_{a^+}\mathrm{H}^{\alpha}_{k,w}(I_d f)(x). \end{aligned} $$

- *If f is a decreasing function, then*

$$ \begin{aligned} {}_{a^+}\mathrm{H}^{\alpha}_{k,w}(f\,g)(x) \geq{}& \frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\ln(\frac{x}{a}))^{\frac{\alpha}{k}}}\; {}_{a^+}\mathrm{H}^{\alpha}_{k,w}f(x)\, {}_{a^+}\mathrm{H}^{\alpha}_{k,w}g(x) \\ &- M\frac{w(a)\,\Gamma_k(\alpha+k)}{w(b)\,(\ln(\frac{x}{a}))^{\frac{\alpha}{k}}}\; {}_{a^+}\mathrm{H}^{\alpha}_{k,w}f(x)\, {}_{a^+}\mathrm{H}^{\alpha}_{k,w}I_d(x) + M\, {}_{a^+}\mathrm{H}^{\alpha}_{k,w}(I_d f)(x), \end{aligned} $$

where $I_d(x)$ is the identity function.

REMARK 4.12. The above corollaries are new results related to Tchebyshev's inequality according to k-weighted Hadamard operators. If we choose $w(x) = 1$ and $\psi(x) = \ln x$ in Corollary 4.9, we obtain Theorem 3 in [5].

4.4. TCHEBYSHEV-TYPE INEQUALITIES VIA k-WEIGHTED KATUGOMPOLA OPERATORS

Putting

$$ \psi(\tau) = \frac{\tau^{\rho+1}}{\rho+1}, $$

where $\rho > 0$, the left side k-weighted fractional ${}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}$ reduces to the left side k-weighted Katugompola fractional operator (12) of order $\alpha > 0$ and the following results hold.

Corollary 4.13. *Let f and g be two synchronous functions on $[a,b]$ and $x>a$, α,β, $k>0$, then*

$$\begin{aligned}&\frac{w(b)\,(x^{\rho+1}-a^{\rho+1})^{\frac{\beta}{k}}}{w(a)\,(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}\,{}_{a^+}\mathcal{K}^{\alpha}_{k,w}(f\,g)(x)\\&\quad+\frac{w(b)\,(x^{\rho+1}-a^{\rho+1})^{\frac{\beta}{k}}}{w(a)\,(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}\,{}_{a^+}\mathcal{K}^{\alpha}_{k,w}(f\,g)(x)\\&\geq {}_{a^+}\mathcal{K}^{\alpha}_{k,w}f(x)\,{}_{a^+}\mathrm{K}^{\beta,\psi}_{k,w}g(x)+{}_{a^+}\mathcal{K}^{\alpha}_{k,w}f(x)\,{}_{a^+}\mathrm{K}^{\alpha,\psi}_{k,w}g(x).\end{aligned}$$

and

$$ {}_{a^+}\mathcal{K}^{\alpha}_{k,w}(f\,g)(x)\geq\frac{w(a)\,(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{w(b)\,(x^{\rho+1}-a^{\rho+1})^{\frac{\beta}{k}}}\,{}_{a^+}\mathcal{K}^{\alpha}_{k,w}f(x),{}_{a^+}\mathcal{K}^{\alpha}_{k}g(x).$$

Corollary 4.14. *Let $i=1,2,...,m$ and f, $f_i\in L_{X^p_w}[a,b]$. If the functions f and f_i are positive increasing functions on $[a,b]$, then for all integer $m\geq 1$ we have*

$$ {}_{a^+}\mathcal{K}^{\alpha}_{k,w}\prod_{i=1}^{m}f_i(x)\geq\left(\frac{w(a)\,(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{w(b)\,(x^{\rho+1}-a^{\rho+1})^{\frac{\beta}{k}}}\right)^{m-1}\left(\prod_{i=1}^{m}{}_{a^+}\mathcal{K}^{\alpha}_{k,w}f_i(x)\right).$$

and

$$ {}_{a^+}\mathcal{K}^{\alpha}_{k,w}f^m(x)\geq\left(\frac{w(a)\,(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{w(b)\,(x^{\rho+1}-a^{\rho+1})^{\frac{\beta}{k}}}\right)^{m-1}\left({}_{a^+}\mathcal{K}^{\alpha}_{k,w}f(x)\right)^m.$$

Corollary 4.15. *Let f and g be two functions defined on $L_{X^p_w}[a,b]$, such that f is monotone and g is differentiable on $]a,b[$ and there exist real numbers $m:=\inf_{x\in[a,b]}g'(x)$ and $M:=\sup_{x\in[a,b]}g'(x)$.*

- *If f is an increasing function, then*

$$\begin{aligned}{}_{a^+}\mathcal{K}^{\alpha}_{k,w}(f\,g)(x)&\geq\frac{w(a)\,(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{w(b)\,(x^{\rho+1}-a^{\rho+1})^{\frac{\beta}{k}}}\,{}_{a^+}\mathcal{K}^{\alpha}_{k,w}f(x)\,{}_{a^+}\mathrm{K}^{\alpha,\psi}_{k,w}g(x)\\&-m\frac{w(a)\,(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{w(b)\,(x^{\rho+1}-a^{\rho+1})^{\frac{\beta}{k}}}\,{}_{a^+}\mathcal{K}^{\alpha}_{k,w}f(x)\,{}_{a^+}\mathcal{K}^{\alpha}_{k,w}I_d(x)+m\,{}_{a^+}\mathcal{K}^{\alpha}_{k,w}(I_df)(x).\end{aligned}$$

- *If f is a decreasing function, then*

$$\begin{aligned}{}_{a^+}\mathcal{K}^{\alpha}_{k,w}(f\,g)(x)&\geq\frac{w(a)\,(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{w(b)\,(x^{\rho+1}-a^{\rho+1})^{\frac{\beta}{k}}}\,{}_{a^+}\mathcal{K}^{\alpha}_{k,w}f(x)\,{}_{a^+}\mathcal{K}^{\alpha}_{k,w}g(x)\\&-M\frac{w(a)\,(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{w(b)\,(x^{\rho+1}-a^{\rho+1})^{\frac{\beta}{k}}}\,{}_{a^+}\mathcal{K}^{\alpha}_{k,w}f(x)\,{}_{a^+}\mathcal{K}^{\alpha}_{k,w}I_d(x)+M\,{}_{a^+}\mathcal{K}^{\alpha}_{k,w}(I_df)(x),\end{aligned}$$

where $I_d(x)$ is the identity function.

Remark 4.16. The above corollaries are new results related to Tchebyshev's inequality according to k-weighted Katugompola operators. If we take $w(x) = 1$, $k = 1$, $a = 0$ and

$$\psi(\tau) = \frac{(\tau)^{\xi+\eta}}{\xi+\eta},$$

we get Theorem 2.2, Theorem 2.1, Theorem 2.3 and Theorem 2.4 in [7].

4.5. TCHEBYSHEV-TYPE INEQUALITIES VIA k-WEIGHTED FRACTIONAL CONFORMABLE OPERATORS

Setting $\psi(\tau) = \frac{(\tau-a)^{\theta}}{\theta}$, where $\theta > 0$, the left side k-weighted fractional ${}_{a^+}\mathrm{J}^{\alpha,\psi}_{k,w}$ reduces to the left side k-weighted fractional conformable operator (13) of order $\alpha > 0$ and we get the corollaries.

Corollary 4.17. *Let f and g be two synchronous functions on $[a,b]$ and $x > a$, α, β, $k > 0$, then*

$$\frac{w(b)\,(x-a)^{\frac{\theta\,\beta}{k}}}{w(a)\,\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}\,{}_{a^+}\mathcal{C}^{\alpha}_{k,w}(f\,g)(x) + \frac{w(b)\,(x-a)^{\frac{\theta\,\beta}{k}}}{w(a)\,\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}\,{}_{a^+}\mathcal{C}^{\alpha}_{k,w}(f\,g)(x)$$
$$\geq {}_{a^+}\mathcal{C}^{\alpha}_{k,w}f(x)\,{}_{a^+}\mathrm{C}^{\beta,\psi}_{k,w}g(x) + {}_{a^+}\mathcal{C}^{\alpha}_{k,w}f(x)\,{}_{a^+}\mathrm{C}^{\alpha,\psi}_{k,w}g(x).$$

and

$${}_{a^+}\mathcal{C}^{\alpha}_{k,w}(f\,g)(x) \geq \frac{w(a)\,\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{w(b)\,(x-a)^{\frac{\theta\,\beta}{k}}}\,{}_{a^+}\mathcal{C}^{\alpha}_{k,w}f(x)\,{}_{a^+}\mathcal{C}^{\alpha}_{k}g(x).$$

Corollary 4.18. *Let $i = 1, 2, ..., m$ and f, $f_i \in L_{X^p_w}[a,b]$. If the functions f and f_i are positive increasing functions on $[a,b]$, then for all integer $m \geq 1$ we have*

$${}_{a^+}\mathcal{C}^{\alpha}_{k,w}\prod_{i=1}^{m} f_i(x) \geq \left(\frac{w(a)\,\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{w(b)\,(x-a)^{\frac{\theta\,\beta}{k}}}\right)^{m-1}\left(\prod_{i=1}^{m}{}_{a^+}\mathcal{C}^{\alpha}_{k,w}f_i(x)\right).$$

and

$${}_{a^+}\mathcal{C}^{\alpha}_{k,w}f^m(x) \geq \left(\frac{w(a)\,\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x-a)^{w(b)\,\frac{\theta\,\beta}{k}}}\right)^{m-1}\left({}_{a^+}\mathcal{C}^{\alpha}_{k,w}f(x)\right)^m.$$

Corollary 4.19. *Let f and g be two functions defined on $L_{X^p_w}[a,b]$, such that f is monotone and g is differentiable on $]a,b[$ and there exist real numbers $m := \inf_{x\in[a,b]} g'(x)$ and $M := \sup_{x\in[a,b]} g'(x)$.*

- *If f is an increasing function, then*

$$
\begin{aligned}
{}_{a^+}\mathcal{C}^{\alpha}_{k,w}(f\,g)(x) \geq & \frac{w(a)\,\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{w(b)\,(x-a)^{\frac{\theta\,\beta}{k}}}\,{}_{a^+}\mathcal{C}^{\alpha}_{k,w}f(x)\,{}_{a^+}\mathrm{C}^{\,\alpha,\psi}_{\,k,w}g(x) \\
& - m\frac{w(a)\,\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{w(b)\,(x-a)^{\frac{\theta\,\beta}{k}}}\,{}_{a^+}\mathcal{C}^{\alpha}_{k,w}f(x)\;{}_{a^+}\mathcal{C}^{\alpha}_{k,w}I_d(x) + m\;{}_{a^+}\mathcal{C}^{\alpha}_{k,w}(I_d f)(x).
\end{aligned}
$$

- *If f is a decreasing function, then*

$$
\begin{aligned}
{}_{a^+}\mathcal{C}^{\alpha}_{k,w}(f\,g)(x) \geq & \frac{w(a)\,\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{w(b)\,(x-a)^{\frac{\theta\,\beta}{k}}}\,{}_{a^+}\mathcal{C}^{\alpha}_{k,w}f(x)\,{}_{a^+}\mathrm{C}^{\,\alpha,\psi}_{\,k,w}g(x) \\
& - M\frac{w(a)\,\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{w(ba\,(x-a)^{\frac{\theta\,\beta}{k}}}\,{}_{a^+}\mathcal{C}^{\alpha}_{k,w}f(x)\;{}_{a^+}\mathcal{C}^{\alpha}_{k,w}I_d(x) + M\;{}_{a^+}\mathcal{C}^{\alpha}_{k,w}(I_d f)(x),
\end{aligned}
$$

where $I_d(x)$ is the identity function.

REMARK 4.20. The above corollaries are new results related to Tchebyshev's inequality according to k-weighted fractional conformable operators. If we set $w(x) = 1$, $k = 1$, $a = 0$ and $\psi(\tau) = \frac{(\tau)^{\theta}}{\theta}$, we deduce Theorem 6, Theorem 5, Theorem 7 and Theorem 8 in [11].

5. CONCLUSION

This paper presents a new result for Tchebyshev-type inequalities utilizing k-weighted fractional operators, as well as other related weight inequalities using particular operators that depend on the functions w and ψ. Certain works can be extended in the future by employing these new weight operators.

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