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JOIN-ESSENTIAL ELEMENT GRAPH OF A LATTICE

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Abstract. Let L be a bounded distributive lattice. The join-essential element graph $\mathbb{JE}(L)$ of L is a graph whose vertices are all nontrivial elements (i.e. different from 1 and 0) of L and two distinct elements x and y are adjacent if and only if $x \vee y$ is an essential element of L. The basic properties and possible structures of the graph $\mathbb{JE}(L)$ and its subgraph $\mathbb{PE}(L)$ induced by vertices which are not essential as elements of L are investigated.

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1. INTRODUCTION

All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words they are bounded. There are many studies on various graphs associated to algebraic structures. Beck [4], Anderson and Naseer [3], and Anderson and Livingston [2] et. al. have studied graphs on commutative rings.

In this paper, we extend several concepts of graphs of modules to lattice theory. With a careful generalization, we can cover some basic corresponding results in the former setting. The main difficulty is figuring out what additional hypotheses the lattice must satisfy to get similar results. Nevertheless, growing interest in developing the algebraic theory of lattices can be found in several papers and books (see for example [5–8, 10]).

The aim of this paper is to study a graph associated to a lattice L called the join-essential element graph of L. This will result in characterization of lattices in terms of some specific properties of those graphs. For a given bounded lattice L, the join-essential element graph of L is a simple graph $\mathbb{JE}(L)$ whose vertices are nontrivial elements (i.e. different from 1 and 0) and two distinct vertices are adjacent if and only if the join of the corresponding elements is an essential element of L.

We also study the proper join-essential element graph of $\mathbb{PE}(L)$ which is a subgraph of $\mathbb{JE}(L)$ generated by vertices which, as elements of L are not essential. The graph $\mathbb{JE}(L)$ was introduced and studied by Nimbhorkar and Deshmukh in [10] in a special case when L is a finite lattice.

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The concept of the essential ideal graph of a commutative ring R was introduced and studied by Amjadi [1].

The essential ideal graph of R, denoted by $\Gamma_R(R)$, is a graph whose vertex set is the set of all nonzero proper ideals of R and two vertices I and J are adjacent whenever $I + J$ is an essential ideal.

The sum-essential graph $\Gamma_M(R)$ of a left R-module M is a graph whose vertices are all nontrivial submodules of M and two distinct submodules are adjacent if and only if their sum is an essential submodule of M was introduced and investigated by Matczuk and Majidinya [9].

Among many results in this paper, Section 2 consists of some preliminary definitions and results concerning the join-essential element graph of a lattice needed later on. It is shown in Example 2.4 that the join-essential element graph of a lattice may not be connected.

We also show that if L is a complete distributive lattice, then every element of L has a complement in L. Moreover, if s is a complement of $x \neq 0$, then $x \vee s$ is essential in L.

Proposition 2.10 shows that if the graph $\mathbb{JE}(L)$ is complete, then one of the following conditions hold:

(i) L is a uniform lattice;

(ii) There exist atoms a_1 and a_2 of L such that $a_1 \wedge a_2 = 0$.

In Section 3, a necessary condition for $\mathbb{JE}(L)$ graph to be complete is proved (see Theorem 3.1). The semisimple lattice and the semisimple element of a lattice are defined (Definition 3.5). Henceforth we will assume that all considered lattices are complete distributive lattices with 0 and 1.

By using Definition 3.5, we observe in Theorem 3.9 that if s is a nonzero element of a lattice L, then s is semisimple if and only if $s = \bigvee_{i\in\Lambda} a_i$, where $\{a_i\}_{i\in\Lambda}$ is the set of all atoms a_i of L with $a_i \leq s$.

We also show in Proposition 3.10 that the lattice L is semisimple if and only if L contains no proper essential element. We show in Theorem 3.13 that $\mathbb{JE}(L)$ is always connected with $\mathrm{diam}(\mathbb{JE}(L)) \leq 3$ and $\mathrm{gr}(\mathbb{JE}(L)) \in \{3, \infty\}$ (Proposition 3.20). We classify all lattices L whose join-essential element graph is triangle-free graph (Theorem 3.17), k-regular graph (Theorem 3.22) and n-partite graph (Theorem 3.25). We also prove in Theorem 3.24 that every vertex of $\mathbb{JE}(L)$ is of finite degree if and only if the graph has only finitely many vertices.

Section 4 is mainly devoted to investigation lattices L such that the subgraph $\mathbb{PE}(L)$ contains a vertex of degree one. Theorem 4.10 offers necessary and sufficient condition for an element u of L to be of degree one as a vertex in $\mathbb{PE}(L)$. We also prove in Theorem 4.13 that every vertex of $\mathbb{PE}(L)$ is of finite degree if and only if the graph is finite. We classify all lattices whose proper join-essential element graph is tree (Theorem 4.16). The diameter and girth $\Gamma_P(L)$ are described (Theorem 4.17).

2. PRELIMINARIES

Let G be a simple graph. The vertex set of G is denoted by $\mathcal{V}(G)$, $\deg_G(v)$ stands for the degree of $v \in \mathcal{V}(G)$, i.e. the cardinality of the set of all vertices which are adjacent to v.

A graph G is said to be connected if there exists a path between any two distinct vertices, G is a complete graph if every pair of distinct vertices of G are adjacent and K_n will stand for a complete graph with n vertices.

The graph G is k-regular, if $\deg_G(v) = k < \infty$ for every $v \in \mathcal{V}(G)$. Let $u, v \in \mathcal{V}(G)$. We say that u is a universal vertex of G if u is adjacent to all other vertices of G and we write $u \backsim v$ if u and v are adjacent. The distance $d(u, v)$ is the length of the shortest path from u to v if such path exists, otherwise, $d(a, b) = \infty$.

The diameter of G is $\operatorname{diam}(G) = \sup\{d(a, b) : a, b \in \mathcal{V}(G)\}$. The girth of a graph G, denoted by $\operatorname{gr}(G)$, is the length of a shortest cycle in G. If G has no cycles, then $\operatorname{gr}(G) = \infty$. A subset $S \subseteq \mathcal{V}(G)$ is independent if no two vertices of S are adjacent.

For a positive integer k, a k-partite graph is a graph whose vertices can be partitioned into k nonempty independent sets. For terminology and notation not defined here, the reader is referred to [11].

By a lattice we mean a poset $(L, \leq)$ in which every couple elements x, y has a g.l.b. (called the meet of x and y, and written $x \wedge y$) and a l.u.b. (called the join of x and y, and written $x \vee y$).

A lattice L is complete when each of its subsets X has a l.u.b. and a g.l.b. in L. Setting $X = L$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that L is a lattice with 0 and 1).

A lattice L is called a distributive lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in L (equivalently, L is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in L). A lattice L is called modular if $(c \wedge b) \vee a = (c \vee a) \wedge b$ for all $a, b, c \in L$ with $a \leq b$.

Let for $a, b \in L$, $[a, b] = \{x \in L : a \leq x \leq b\}$. Obviously, $[a, b]$ is a sublattice of L and $L = [0, 1]$.

An element x of a lattice L is called essential (written $x \trianglelefteq L$), if there is no nonzero $y \in L$ such that $x \wedge y = 0$. An element x of L is called proper if $x \neq 1$.

For terminology and notation not defined here, the reader is referred to [5, 6].

Remark 2.1. If $1 \neq x \trianglelefteq L$, then x is a universal vertex in $\mathbb{JE}(L)$.

Proof. Let y be any element of L. We claim that $x \vee y \trianglelefteq L$. If $(x \vee y) \wedge z = 0$ for some $z \in L$, then $x \wedge z \leq (x \wedge y) \vee z = 0$; so $x \wedge z = 0$ implies that $z = 0$, as $x \trianglelefteq L$. Hence x is a universal vertex. □

Lemma 2.2. *Let $a, b \in L$. Then the following hold:*

(i) *If $a \leq b$ and $a \trianglelefteq L$, then $b \trianglelefteq L$;*

(ii) *If $a \trianglelefteq L$ and $b \trianglelefteq L$, then $a \wedge b, a \vee b \trianglelefteq L$;*

(iii) *If $a \trianglelefteq [0, b]$ and $b \trianglelefteq L$, then $a \trianglelefteq L$.*

Proof. (i) Let $b \wedge c = 0$, for some $c \in L$. Then from $a \wedge c \leq b \wedge c = 0$, we obtain $c = 0$, since a is essential in L. So $b \trianglelefteq L$.

(ii) Since $a \leq a \vee b$, $a \vee b \trianglelefteq L$ by (1). Let $(a \wedge b) \wedge c = 0$ for some $c \in L$. Then $b \trianglelefteq L$ yields $a \wedge c = 0$; hence $c = 0$, as $a \trianglelefteq L$, as required.

(iii) If $0 \neq c \in L$, then $c \wedge b \neq 0$, which in turn implies $b \wedge c \wedge a = c \wedge a \neq 0$, as needed. □

In a lattice L with 0, an element c is called a complement of b in L if it is maximal relative to the property $b \wedge c = 0$. For an element $x \in L$, we set $(0 :_L x) = \{y \in L : x \wedge y = 0\}$.

Lemma 2.3. *If L is a complete distributive lattice, then every element of L has a complement in L. Moreover, if s is a complement of $x \neq 0$, then $x \vee s \trianglelefteq L$.*

Proof. Let $x \neq 0$ and set $s = \bigvee_{z \in (0:_L x)} z$. Clearly, $x \wedge s = 0$. If $x \wedge y = 0$, then $y \in (0 :_L x)$ and so $y \leq s$. Thus s is a complement of x in L. Let $(x \vee \bigvee_{z \in (0:_L x)} z) \wedge b = 0$ for some $b \in L$. Then $x \wedge b \leq (x \vee \bigvee_{z \in (1:_L x)} z) \wedge b = 0$, from where we have $b \wedge x = 0$ (so $b \in (0 :_L x)$). Similarly, $b \wedge \bigvee_{z \in (1:_L x)} z = 0$. It follows that $b = b \wedge b \leq b \wedge \bigvee_{z \in (1:_L x)} z = 0$; hence $b = 0$, as needed. □

The following example shows that the join-essential element graph of a lattice may not be connected [10, Example 3.4].

Example 2.4. Let $L = \{0, a, b, c, d, e, 1\}$ be a lattice with the relations $0 < d < a < 1$, $0 < d < b < 1$, $0 < e < b < 1$, $0 < e < c < 1$, $a \wedge c = d \wedge c = e \wedge d = a \wedge e = 0$, $b \wedge a = d$ and $b \vee c = b \vee a = e \vee d = 1$. An inspection will show that the set of all essential elements of L is $\{0\}$. Since two vertices b and a of the graph $\mathbb{JE}(L)$ are not connected by a path, we observe that the join-essential element graph of a lattice is not a connected graph.

A lattice L with 0 is called L-domain if $a \wedge b = 0$, then either $a = 0$ or $b = 0$.

Remark 2.5. For a lattice L, the following conditions hold:

(i) If every proper element of L is essential in L, then $\mathbb{JE}(L)$ is a complete graph;

(ii) If L is a L-domain, then $\mathbb{JE}(L)$ is a complete graph;

(iii) If L has a proper essential element and $\mathbb{JE}(L)$ is a k-regular graph, then $\mathbb{JE}(L)$ is a complete graph and $|\mathcal{V}\ (\mathbb{JE}(L))| = k + 1$.

Proof. (i) Let x and y be two distinct nontrivial elements of L. If $x \vee y = 1 \trianglelefteq L$, then $x \backsim y$ is an edge in $\mathbb{JE}(L)$. If $x \vee y \neq 1$, then by assumption, $x \vee y \trianglelefteq L$ and so $x \backsim y$ is an edge in $\mathbb{JE}(L)$.

(ii) Assume that x and y are two distinct nontrivial elements of L and let $(x \vee y) \wedge b = 0$ for some $b \in L$. Since $x \wedge b \leq (x \vee y) \wedge b = 0$, we get $x \wedge b = 0$ which implies that $b = 0$, as L is a L-domain. Hence, $x \vee y \trianglelefteq L$ and so x and y are adjacent in $\mathbb{JE}(L)$.

(iii) Let $\mathbb{JE}(L)$ be a k-regular graph. Suppose L contains a proper essential element x. Then x is adjacent to all other vertices of $\mathbb{JE}(L)$ by Remark 2.1 which implies that $k + 1$ is the number of vertices of $\mathbb{JE}(L)$. Since $\mathbb{JE}(L)$ is k-regular, we deduce that $\mathbb{JE}(L)$ is a complete graph. □

The following example shows that the condition "L has a proper essential element" in Remark 2.5 (iii) can not be omitted

EXAMPLE 2.6. Let $L = \{0, a, b, c, 1\}$ be a lattice with the relations $b < a$, $a \wedge c = 0$ and $b \vee c = 1$. An inspection will show that the set of all essential elements of L is $\{0\}$. Its join-essential element graph is the regular graph $K_{1,2}$ with $\{c\}$ and $\{a, b\}$ as the two partite sets, but it is not a complete graph.

We say that an element x in a lattice L is an atom (resp. coatom) if there is no $y \in L$ such that $0 < y < x$ (resp. $x < y < 1$). We need the following theorem proved in [10, Theorem 3.2].

THEOREM 2.7. *$\mathbb{JE}(L)$ is a complete graph if and only if every nonzero non-essential element in L is an atom.*

An element $u \in L$ is called uniform if for every $x, y \in L$ the following implication holds: if $0 < x \leq u$ and $0 < y \leq u$, then $x \wedge y \neq 0$ (i.e. all nonzero elements from $[0, u]$ are essential in $[0, u]$). A lattice L is called uniform if 1 is uniform in L [5].

REMARK 2.8. (i) If u is uniform in L and $x \leq u$, then x is uniform in L (for if $0 < b \leq x$ and $0 < c \leq x$, then u uniform shows that $b \wedge c \neq 0$).

(ii) It is clear that a lattice L is uniform if and only if every non-zero element of L is essential in L.

LEMMA 2.9. *Let c be a universal vertex of $\mathbb{JE}(L)$. If c is not essential in L, then c is an atom in L.*

Proof. Assume to the contrary, that c is a nonessential element of L with $c \neq 1$ that is not atom. Let b be an element of L such that $0 < b < c$. Then c is a universal vertex, hence $c \vee b = c$ is an essential element which is impossible. Thus c is atom. □

Proposition 2.10. *If the graph* $\mathbb{JE}(L)$ *is complete, then one of the following conditions hold:*

(i) L *is a uniform lattice;*

(ii) *There exist atoms* a_1 *and* a_2 *of* L *such that* $a_1 \wedge a_2 = 0$.

Proof. Assume that $\mathbb{JE}(L)$ is a complete graph and assume L is not uniform. So there exists a nontrivial element a_1 of L such that it is not essential in L which implies that a_1 is an atom of L by Lemma 2.9. Now a_1 not essential in L implies there is a nontrivial nonessential element a_2 such that $a_1 \wedge a_2 = 0$ (so a_2 is an atom by Lemma 2.9), as required. □

3. BASIC PROPERTIES OF $\mathbb{JE}(L)$

Let us begin this section with the following theorem:

Theorem 3.1. *Let* L *be a modular lattice. Then the graph* $\mathbb{JE}(L)$ *is complete if one of the following conditions hold:*

(i) L *is a uniform lattice;*

(ii) *There exist different coatoms* c_1 *and* c_2 *of* L *such that* $c_1 \wedge c_2 = 0$.

Proof. Assume that L is a uniform lattice and let a, b be two distinct vertices of the graph $\mathbb{JE}(L)$ (so $a, b \trianglelefteq L$). Hence $a \vee b$ is essential in L by Lemma 2.2. Suppose that the conditions of (2) holds. Let x be a nontrivial element of L. It is easy to see that either $x \vee c_i = 1$ or $x \leq c_i$ $(i = 1, 2)$. If $x \leq c_1$ and $x \leq c_2$, then $x = 0$ which is impossible. Without loss of generality, let $x \leq c_1$ and $x \vee c_2 = 1$. Then $x = x \vee (c_1 \wedge c_2) = c_1 \wedge (c_2 \vee x) = c_1$. Thus every nontrivial element of L is a coatom. Let x, y be two distinct vertices of the graph $\mathbb{JE}(L)$. If $x \vee y \neq 1$, then $x = y$, a contradiction. Hence $x \vee y = 1 \trianglelefteq L$. Therefore $\mathbb{JE}(L)$ is a complete graph. □

In the next example we show that the condition "L is a modular lattice" is not superficial in Theorem 3.1.

Example 3.2. Let $L = \{0, a, b, c, 1\}$ be a lattice with the relations $b < a$, $a \wedge c = 0$ and $b \vee c = 1$. Clearly, the set of all essential elements of L is $\{0\}$ (so L is not uniform). By assumption, $(c \wedge a) \vee b = b \neq (c \vee b) \wedge a = a$ implies that L is not modular. Moreover, $a \wedge c = 0$, with a, c coatoms in L. But $a \vee b = a$ demonstrates that L is not complete.

Proposition 3.3. *Assume that* L *is a modular lattice and let* c *be an element of* $\mathcal{V}(\mathbb{JE}(L))$ *with degree* 1. *If* $c \trianglelefteq L$ *which is not an atom, then there exists an atom* $a < c$ *such that* $\mathcal{V}(\mathbb{JE}(L)) = \{c, a\}$.

Proof. Let a be an element of L such that $0 < a < c$. Since $c \vee a = c$, c and a are adjacent in $\mathbb{JE}(L)$. If b is an element of L such that $0 < b < a$, then $c \vee b = c$ implies $\deg_{\mathbb{JE}(L)}(c) \geq 2$, which is a contradiction. Thus a is an atom and $\mathcal{V}(\mathbb{JE}(L)) = \{c, a\}$. □

Remark 3.4. Let a be an element of a distributive lattice L.

(i) If $a \nleq c_1, c_1$, where c_1, c_2 are coatoms of L, then $a \vee (c_1 \wedge c_2) = 1$;

(ii) If $a \wedge b \leq c$, where c is a coatom, then either $a \leq c$ or $b \leq c$.

Proof. (i) By assumption, $a \vee (c_1 \wedge c_2) = (a \vee c_1) \wedge (a \vee c_2) = 1 \wedge 1 = 1$.

(ii) Assume to the contrary, that $a \nleq c$ and $b \nleq c$; so $c = c \vee (a \wedge b) = 1$, a contradiction. □

Definition 3.5. (i) A nonzero element x of a lattice L is called *semisimple*, if for each element y of L with $y < x$, there exists an element z of L such that $x = y \vee z$ and $y \wedge z = 0$. In this case, we say that y is a direct join of x, and we write $x = y \oplus z$.

(ii) A lattice L is called semisimple if 1 is semisimple in L

Lemma 3.6. *Let c be a coatom of a modular lattice L. If $1 = c \oplus a$ for some element $a \in L$, then a is an atom.*

Proof. Let $0 < a' \leq a$. The $a' \wedge c = 0$ results in $a' \nleq c$; so $a' \vee c = 1$. Hence, $a = a \wedge (c \vee a') = a' \vee (c \wedge a) = a'$. Thus a is atom. □

Lemma 3.7. *Let L be a modular lattice. If s is semisimple and $0 \neq u \leq s$, then u is semisimple.*

Proof. Let w be an element of L such that $w < u$. By assumption, $s = w \vee v$ and $w \wedge v = 0$ for some element v of L. Then $u = u \wedge s = u \wedge (w \vee v) = w \vee (v \wedge u)$ and $w \wedge (v \wedge u) = 0$, as needed. □

Remark 3.8. If b is semisimple, $a < b$ and $a \trianglelefteq [0, b]$, then $a = 0$.

Proof. Assume to the contrary, that $a \neq 0$. By assumption, $b = a \vee a'$ and $a \wedge a' = 0$ for some $a' \in L$. Then a is essential in $[0, b]$, hence $a' = 0$. Therefore $a = b$, which is a contradiction. □

A subset $\{a_i\}_{i \in \Lambda}$ of non-zero elements of a complete lattice L (with 0) is called independent if for every $i \in \Lambda$ the equality $a_i \wedge (\bigvee_{i \neq j, j \in \Lambda} a_j) = 0$ holds [5]. Henceforth we will assume that all considered lattices are complete distributive lattices with 0 and 1.

Theorem 3.9. *If s is a nonzero element of a lattice L, then s is semisimple if and only if $s = \bigvee_{i \in \Lambda} a_i$, where $\{a_i\}_{i \in \Lambda}$ is the set of all atoms a_i of L with $a_i \leq s$.*

Proof. Suppose that $s = \bigvee_{i \in \Lambda} a_i$, where $\{a_i\}_{i \in \Lambda}$ is the set of all atoms a_i of L with $a_i \leq s$ and let $b < s$. Set $\Omega = \{I \subseteq \Lambda : b \wedge (\bigvee_{i \in I} a_i) = 0\}$ and $\{a_i\}_{i \in I}$ is independent. If for each $i \in \Lambda$, $a_i \leq b$, then $a = b$, a contradiction. Thus there exists $i \in \Lambda$ such that $a_i \nleq b$; so $b \wedge a_i = 0$. Hence $\Omega \neq \emptyset$. Therefore by Zorn's lemma Ω has a maximal element H (so $b \wedge (\bigvee_{i \in H} a_i) = 0\}$).

Put $z = b \vee h$, where $h = \bigvee_{i \in H} a_i$. It is enough to show that $s = z$.

We claim that for each $j \in \Lambda$, $z \wedge a_j \neq 0$. Assume to the contrary, that there exists $j \in \Lambda$ such that $z \wedge a_j = 0$. If $j \in H$, then $a_j \leq \bigvee_{i\in H} a_i \leq z$, which is a contradiction. So $j \notin H$. If $H' = H \cup \{j\}$, then it can be easily seen that $\{a_i\}_{i\in H'}$ is independent and also $b \wedge \bigvee_{i\in H'} a_i = b \wedge (a_j \vee h) = b \wedge a_j \leq z \wedge a_j = 0$, which is a contradiction by maximality of H. Hence for each $j \in \Lambda$, $z \wedge a_j \neq 0$ (so $a_j \leq z$); hence $s \leq z$ and so $s = z$. Thus s is semisimple.

Conversely, assume that s is semisimple and let $e = \bigvee_{i\in\Lambda} a_i$, where $\{a_i\}_{i\in\Lambda}$ is the set of all atoms a_i of L with $a_i \leq s$. It is enough to show that $s = e$. Suppose to the contrary, that $s \neq e$ (so $e < s$). Then we have $\sum = \{e \leq z : z < s\} \neq \emptyset$ since $e \in \sum$. Hence $\sum$ has a maximal element by Zorn's Lemma, say g (so $e \leq g$ and $g < s$). By assumption, there is an element g' of L such that $s = g \vee g'$ and $g \wedge g' = 0$. If g' is an atom, then $g' \leq e \leq g$ implies $s = g$, a contradiction. So there exists an element g_1 such that $0 < g_1 < g'$. Again $s = g_1 \vee k$ and $g_1 \wedge k = 0$ for some element k of L. Moreover, $(g \vee g_1) \wedge (g \vee k) = g \vee (g_1 \wedge k) = g$. If $g \vee k = g$, then $k \leq g$ entails $g = g \wedge s = g \wedge (g_1 \vee k) = k$. Thus $g' = g' \wedge (g_1 \vee k) = g_1$, a contradiction. Hence $g \neq g \vee k$. If $g \vee g_1 = g$, the $g_1 \leq g$ results in $g_1 \leq g \wedge g' = 0$, which is impossible. Thus $g \vee g_1 \neq g$. It follows that $g \vee g_1, g \vee k \notin \sum$ by maximality of g, but $g \vee g_1, g \vee k < s$ and $e \leq g \vee g_1, g \vee k$. Thus $s = e$. □

Proposition 3.10. *For the lattice L, the following conditions hold.*

(i) *An element $s \in L$ is semisimple if and only if for every $x \in L$ the following implication holds: if $x < s$, then x is not essential in $[0, s]$.*

(ii) *The lattice L is semisimple if and only if L contains no proper essential element.*

Proof. (i) Let s be semisimple. By Theorem 3.9, $s = \bigvee_{i\in I} c_i$, where $\{c_i\}_{i\in I}$ is a set of independent atoms of L with $c_i \leq s$ for every $i \in I$. Suppose to the contrary, that $u < s$ with $u \trianglelefteq [0, s]$. Then for each $i \in I$, $c_i \leq u$; hence $u = s$, a contradiction. Conversely, let c be any element of L such that $c < s$. If c' is a complement of c in $[0, s]$, then $c \oplus c' \trianglelefteq [0, s]$ by Lemma 2.3; so $c \vee c' = s$, as needed.

(ii) By (i), 1 is semisimple if and only if for every $x \in L$ the following implication holds: if $x < 1$, then x is not essential in $[0, 1]$. This completes the proof. □

Proposition 3.11. *Let c be an element of $\mathcal{V}(\mathbb{JE}(L))$ with degree 1. If c is not an essential element in L, then it is an atom.*

Proof. Suppose that s is an element of L such that $0 < s \leq c$ and let c' be a complement of c in L; so c and c' are adjacent in $\mathbb{JE}(L)$ by Lemma 2.3. If $s \vee c' \neq 1$, then $c \vee (s \vee c') = c \vee c'$ is essential in L, which implies that c is adjacent to c' and $s \vee c'$. This yields $s = 0$ which is impossible. Thus $s \vee c' = 1 = c \vee c'$. By assumption, $c = c \wedge (s \vee c') = s$. This shows that c is an atom. □

Theorem 3.12. *Assume that L is a semisimple lattice and let $c \in L$. Then* $\deg_{\mathbb{JE}(L)}(c) = 1$ *if and only if c is an atom and has a unique nonzero complement in L.*

Proof. Suppose $\deg_{\mathbb{JE}(L)}(c) = 1$. Then by Proposition 3.11, c is an atom. By Lemma 2.3, it is clear that the complement to c in L is unique. For the reverse implication, assume that $c' \neq 0$ is the unique complement of c in L and let b and c be adjacent in $\mathbb{JE}(L)$ with $c' \neq b$; so $b \vee c = 1 = c' \vee c$ since L is semisimple. If $c \wedge b = c$, then $c \leq b$, hence $b = 1$, which is impossible. So we may assume that $b \wedge c = 0$, as c is an atom. Thus $b \leq c'$, hence $c' = c' \wedge (c \vee b) = b$, which leads to a contradiction. Thus $\deg_{\mathbb{JE}(L)}(c) = 1$. □

Theorem 3.13. $\mathbb{JE}(L)$ *is a connected graph with* $\operatorname{diam}(\mathbb{JE}(L)) \leq 3$.

Proof. Suppose there are nontrivial elements $f \neq k$ such that f and k are not adjacent. Let h, h' denote complements to $f \vee k$ and $f \wedge k$ in L, respectively. If $f \wedge k = 0$, then $h \wedge (f \vee k) = 0$, thus $h \wedge f = 0 = h \wedge k$ and we have the path $f \backsim k \oplus h \backsim f \oplus h \backsim k$, as $h \vee (f \vee k) \trianglelefteq L$. If $f \wedge k \neq 0$, then we have the path $f \backsim h' \backsim k$, as $h' \vee (f \wedge k) \trianglelefteq L$, as required. □

In the next example we show that the condition that “L is a distributive lattice” is not superficial in Theorem 3.13.

Example 3.14. Let L be the lattice as described in Example 2.4. Then $(b \vee d) \wedge e = d \wedge e = 0 \neq (e \wedge b) \vee (e \wedge d) = b \vee 0 = b$ shows that L is not distributive. Matczuk and Majidinya [9, Theorem 1.5 and Corollary 3.8] have proved that for a module over a commutative ring, the diameter of the sum-essential graphs of modules is at most 3 and its girth is 3 or ∞. This example shows that these bounds do not hold for the join-essential element graph of a lattice (also, see Theorems 3.1 and 3.2 in [1]). So the condition that “L is a distributive lattice” in Theorem 3.13 can not be omitted

The set of all coatom (resp. atom) elements of L is denoted by $\mathcal{CA}(L)$ (resp. $\mathcal{A}(L)$). The radical of L is the meet of all coatom elements of L, and is denoted as $\operatorname{rad}(L)$ (i.e. $\operatorname{rad}(L) = \bigwedge_{c \in \mathcal{CA}(L)} c$).

Theorem 3.15. *If* $\operatorname{rad}(L) = 0$ *and* $\mathcal{CA}(L) = \{c_1, c_2, \ldots, c_n\}$, *then L is semisimple.*

Proof. By Remark 3.4, $c_1 \vee (\bigwedge_{i=2}^n c_i) = 1$, $e = \bigwedge_{i=2}^n c_i \neq 0$ and $e \wedge c_1 = 0$; so e is an atom by Lemma 3.6. By a similar way, $\bigwedge_{i=1, i\neq j}^n c_i \neq 0$ is an atom element for each $1 \leq j \leq n$. An easy inspection will show that $\bigvee_{j=1}^n (\bigwedge_{i=1, i\neq j}^n c_i) = 1$, which implies that 1 is semisimple by Theorem 3.9, as needed. □

Lemma 3.16. *Let $a_1, a_2 \in \mathcal{A}(L)$. If $a_1 \oplus a_2 = 1$, then $\mathcal{V}(\mathbb{JE}(L)) = \{a_1, a_2\}$. In particular, $\mathbb{JE}(L)$ is a complete graph.*

Proof. Let $b \in \mathcal{V}(\mathbb{JE}(L))$. We claim that either $b = a_1$ or $b = a_2$. It is easy to see that either $b \wedge a_i = 0$ or $a_i \leq b$ $(i = 1, 2)$. If $a_1 \leq b$ and $a_2 \leq b$, then $b = 1$, which is impossible. If $a_1 \wedge b = 0$ and $a_2 \wedge b = 0$, then $b = b \wedge (a_1 \vee a_2) = 0$, a contradiction. Without loss of generality, let $a_1 \leq b$ and $b \wedge a_2 = 0$. Then $b = b \wedge (a_1 \vee a_2) = a_1 \vee (a_2 \wedge b) = a_1$. Thus every vertex of $\mathbb{JE}(L)$ is an atom.

Finally, let x, y be two distinct vertices of the graph $\mathbb{JE}(L)$. If $x \vee y \neq 1$, then $x = y$, a contradiction. Hence $x \vee y = 1 \trianglelefteq L$. Therefore $\mathbb{JE}(L)$ is a complete graph. □

Compare the next theorem with Theorem 2.8 in [1].

THEOREM 3.17. *Let $\mathcal{CA}(L) = \{c_1, c_2\}$. If $\mathbb{JE}(L)$ is triangle-free, then the following conditions hold:*

(i) *c_1 and c_2 are nonessential elements of L;*

(ii) *L is a semisimple lattice;*

(iii) *$\mathbb{JE}(L) \cong K_2$ the complete graph with two vertices.*

Proof. (i) Suppose, to the contrary, that c_i is essential for some i, say $i = 1$. If c_2 is essential, then $c_1 \vee c_2, c_1 \wedge c_2 \trianglelefteq L$ by Lemma 2.2 (ii) which implies that c_1, c_2 and $c_1 \wedge c_2$ would form a triangle, a contradiction. Henceforth, we may assume that c_2 is not essential. Let s be a complement to c_2 in L (so $c_2 \vee s \trianglelefteq L$ by Lemma 2.3). Then $c_2 \wedge s = 0$, from which we have $s \not\leq c_2$ (so $s \vee c_2 = 1$). We claim that $s \leq c_1$. Assume to the contrary, that $s \not\leq c_1$ (so $s \vee c_1 = 1$). Then $c_1 \vee c_2 = 1$, so $1 = (s \vee c_1) \wedge (c_2 \vee c_1) = c_1 \vee (s \wedge c_2) = c_1$, a contradiction. Thus $s \leq c_1$. If $s = c_1$, then $c_1 \wedge c_2 = 0$ which is impossible since c_1 is essential; hence $s < c_1$. It follows that c_1, c_2 and s would form a triangle, a contradiction. Thus, c_1 is not essential. Similarly, c_2 is not essential.

(ii) It suffices to show that $\mathrm{rad}(L) = 0$ by Theorem 3.15. Assume to the contrary, that $\mathrm{rad}(L) \neq 0$. As $c_1 \wedge c_2 \leq c_1$, we get that $\mathrm{rad}(L)$ is not essential in L by Lemma 2.2. Now $\mathbb{JE}(L)$ is a connected triangle-free graph, which shows that $\mathrm{rad}(L) \vee b$ is essential for some element b of L. We claim that either $b \leq c_1$ or $b \leq c_2$. Otherwise, $b \not\leq c_1$ and $b \not\leq c_2$. Then $(c_1 \wedge c_2) \vee b \leq c_1 \vee b, c_2 \vee b$ implies that c_1, c_2 and b would form a triangle, which is a contradiction. So we may assume that $b \leq c_1$. It follows that c_1 is an essential element of L which again is a contradiction. Thus, $\mathrm{rad}(L) = 0$.

(iii) Clearly, $1 = c_1 \oplus c_2$. By Lemma 3.16, it is enough to show that c_1 and c_2 are atoms. If c_1 is not atom, then $0 < c < c_1$, for some $c \in L$ with $c \not\leq c_2$ (so $c \vee c_2 = 1$). Hence, $c_1 = c_1 \wedge (c \vee c_2) = c_1 \wedge c = c$, a contradiction. Thus c_1 is atom. Similarly, c_2 is an atom, as needed. □

In the next example we show that the condition that "L is a distributive lattice" is not superficial in Theorem 3.17.

Example 3.18. Let L be the lattice as described in the Example 3.14. This lattice is triangle-free and has two coatoms d and e. However, $d \wedge e \neq 0$ and $\mathbb{JE}(L) \neq K_2$. This shows that Theorem 3.17 does not hold in the lattice context.

Lemma 3.19. *If a is a proper element of the lattice L, then there exists a coatom c such that $a \leq c$.*

Proof. Set $\Omega = \{g : g \text{ is an element of L with } a \leq g, g \neq 1\}$. Then $\Omega \neq \emptyset$ since $a \in \Omega$. Moreover, $(\Omega, \leq)$ is a partial order. Clearly, Ω is closed under taking joins of chains and so the result follows by Zorn's Lemma. □

Proposition 3.20. *For the lattice L, the girth of the graph $\mathbb{JE}(L)$ is either 3 or ∞.*

Proof. At first, we note that if c_1 and c_2 are distinct coatoms of L, then $c_1 \vee c_2 = 1 \trianglelefteq L$; so c_1 and c_2 are adjacent in $\mathbb{JE}(L)$. If $\mathbb{JE}(L)$ contains a triangle, then $\mathrm{gr}(\mathbb{JE}(L)) = 3$. If $\mathbb{JE}(L)$ is triangle-free, then $|\mathcal{CA}(L)| \leq 2$. If $\mathcal{CA}(L) = \{c\}$, then c is adjacent to all vertices of $\mathbb{JE}(L)$ by Lemma 3.19. As $\mathbb{JE}(L)$ is triangle-free, $\mathbb{JE}(L)$ is a star graph and so $\mathrm{gr}(\mathbb{JE}(L)) = \infty$. We may assume that $\mathcal{CA}(L) = \{c_1, c_2\}$. By Theorem 3.17, $\mathbb{JE}(L) \cong K_2$; hence $\mathrm{gr}(\mathbb{JE}(L)) = \infty$. □

Lemma 3.21. *If $\mathbb{JE}(L)$ is a k-regular graph, then every atom is a coatom.*

Proof. Let a be an atom of L. If c is a coatom of L with $a \nleq c$, then $c \vee a = 1$ implies that a and c are adjacent in $\mathbb{JE}(L)$. As $\deg_{\mathbb{JE}(L)}(a) = k$, k is the number of coatoms c_i of L such that $a \nleq c_i$. Let $c_1, c_2, \ldots, c_k$ be such coatom elements. If $a \leq b$, then $a \vee c_1 \leq c_1 \vee b$, which leads to c_1 and b being adjacent in $\mathbb{JE}(L)$. It follows that c_1 is adjacent to $c_2, c_3, \ldots, c_k$ and to an arbitrary proper element b of L with $a \leq b$ which is a contradiction since $\mathbb{JE}(L)$ is k-regular; hence there is no $b \in L$ such that $a < b < 1$. Thus a is a coatom. □

Compare the next theorem with Theorem 2.4 in [1] and Theorem 3.6 in [9].

Theorem 3.22. *$\mathbb{JE}(L)$ is a k-regular graph if and only if $\mathbb{JE}(L)$ is a complete graph and $|\mathcal{V}\ (\mathbb{JE}(L))| = k + 1$.*

Proof. One side is clear. To prove the other side, let $\mathbb{JE}(L)$ be a k-regular graph. Suppose L contains a proper essential element. Then $\mathbb{JE}(L)$ is a finite complete graph by Remark 2.5 (iii). Suppose now that L does not have proper essential element, i.e. L is semisimple by Proposition 3.10. We want to show that $\mathbb{JE}(L) \cong K_2$, the complete graph with two vertices. Let a be an atom of L. Then a is a coatom by Lemma 3.21. By assumption, $1 = a \oplus a'$ for some $a' \in L$. By Lemma 3.6, a' is an atom (so it is a coatom). An easy inspection will show that if c is an atom (resp. a coatom) of L, then $c = a$ or $c = a'$; hence $\mathcal{A}(L) = \{a, a'\}$. Now the assertion follows from Lemma 3.16. □

$a = \oplus_{i\in I} a_i$ is said to be a direct decomposition of a into the join of the elements $\{a_i : i \in I\}$ if $a = \bigvee_{i\in I} a_i$ and $\{a_i : i \in I\}$ is independent.

LEMMA 3.23. *Let there exist distinct atoms $a_1, \ldots, a_n$ of L such that $1 = \oplus_{i=1}^n a_i$. Then $\mathcal{A}(L) = \{a_1, \ldots, a_n\}$ and $\mathcal{V}$ $(\mathbb{JE}(L))$ is a finite set.*

Proof. Suppose that a is an atom of L; we show that $a \in \{a_1, \ldots, a_n\}$. Assume to the contrary, that $a \neq a_i$ $(1 \leq i \leq n)$; so $a \wedge a_i = 0$. Then $a = a \wedge (\bigvee_{i=1}^n a_i) = \bigvee_{i=1}^n (a \wedge a_i) = 0$, a contradiction. Thus $\mathcal{A}(L) = \{a_1, \ldots, a_n\}$. Finally, since L is semisimple, every vertex of $\mathbb{JE}(L)$ is finite join of atoms, as needed. □

Compare the next theorem with Theorem 2.1 in [9].

THEOREM 3.24. *For the lattice L, the following conditions are equivalent:*

(i) *L has only finitely many elements;*

(ii) *Every vertex of $\mathbb{JE}(L)$ is of finite degree;*

(iii) *Either*

 (a) *L contains a proper essential element c of finite degree in $\mathbb{JE}(L)$ or*

 (b) *There exist atoms $a_1, \ldots, a_n$ of L such that $1 = \oplus_{i=1}^n a_i$ and $\mathcal{V}$ $(\mathbb{JE}(L))$ is a finite set.*

Proof. (i)⇒(ii) is clear.

(ii)⇒(iii) Suppose L does not contain a proper essential element, i.e. L is semisimple by Proposition 3.10. Let $1 = \oplus_{i\in I} a_i$, where $\{a_i\}_{i\in I}$ is the set of all atoms of L. Pick $k \in I$. For any proper subset J of $I \setminus \{k\}$, set $a = \bigvee_{k\neq i\in I} a_i$ and $b = \bigvee_{i\in J} a_i$. Since $a \vee (a_k \vee b) = 1 \trianglelefteq L$, a is adjacent to $a_k \vee b$, which implies that the set $I \setminus \{k\}$ has finitely many subsets, so I is finite, say $I = \{1, \ldots, n\}$ which yields $1 = \oplus_{i=1}^n a_i$. Now the result is a consequence of Lemma 3.23.

(iii)⇒(i). If L is as in (iii) (a), then Remark 2.1 proves that a is a universal vertex; hence L has only finitely many elements. Suppose (iii) (b) holds. Now (i) is a consequence of (iii) (b). □

The following theorem is a lattice counterpart of Proposition 3.14 in [9] describing the structure of n-partite graphs.

THEOREM 3.25. *Let the lattice L have exactly $2 \leq n < \infty$ coatom elements. Then $\mathbb{JE}(L)$ is a n-partite graph if and only if L is a semisimple lattice.*

Proof. Suppose that $c_1, c_2, \ldots, c_n$ are the coatom elements of L and let $\mathbb{JE}(L)$ be a n-partite graph.

It is enough to show that $\mathrm{rad}(L) = 0$ by Theorem 3.15.

Assume to the contrary, that $\mathrm{rad}(L) \neq 0$. Suppose that c is the complement of $\mathrm{rad}(L)$ in L (so $\mathrm{rad}(L) \vee c \trianglelefteq L$). As $\mathrm{rad}(L) \vee c \subseteq c \vee c_i$, we obtain that $c \vee c_i \trianglelefteq L$ for each i, $1 \leq i \leq n$.

Then c, $c_1, \ldots, c_n$ would generate a complete subgraph of $\mathbb{JE}(L)$ which is impossible, since $\mathbb{JE}(L)$ is a n-partite graph; so $\mathrm{rad}(L) = 0$.

Conversely, assume that L is semisimple. For each $1 \leq k \leq n$, we put

$$V_k = \{a \in \mathcal{V}(\mathbb{JE}(L)) : a \leq c_k \text{ and } a \nleq c_i \text{ for } i < k\}.$$

Let $x, y \in V_i$ (so $x \vee y \leq c_i$). If x adjacent to y, then c_i is essential in L which is impossible since L is semisimple. Let $a \in V_i$ and $b \in V_j$ ($1 \leq i < j \leq n$). Then $a \leq c_i$, $a \nleq c_j$ and $b \leq c_j$. We claim that $c_i \wedge c_j = 0$. Suppose the result is false. Then there exists k such that $k \neq i$ and $k \neq j$ with $c_i \wedge c_j \leq c_k$ which is impossible by Remark 3.4. Thus $c_i \oplus c_j = 1$, from which we have that c_i and c_j are atoms of L, by Lemma 3.6. Hence $a \vee b = 1 \trianglelefteq L$. Therefore $V_1, \ldots, V_n$ form n-partitioning subsets of $\mathcal{V}$ ($\mathbb{JE}(L)$). □

4. BASIC PROPERTIES OF $\mathbb{PE}(L)$

In this section, we will investigate the proper join-essential element graph of L, denoted by $\mathbb{PE}(L)$, which is a subgraph of $\mathbb{JE}(L)$ induced by vertices which are not essential elements of L. As a direct application of Proposition 3.10 we obtain the following proposition.

Proposition 4.1. *For the lattice L the following hold:*

(i) *L is semisimple if and only if $\mathbb{PE}(L) = \mathbb{JE}(L)$;*

(ii) *L is semisimple if and only if there exists a vertex c of $\mathbb{PE}(L)$ such that $\deg_{\mathbb{PE}(L)}(c) = \deg_{\mathbb{JE}(L)}(c)$.*

Proposition 4.2. *$\mathbb{PE}(L)$ is a complete graph if and only if every non-zero vertex of $\mathbb{PE}(L)$ is an atom.*

Proof. Let us notice that every universal vertex $\mathbb{PE}(L)$ is also universal in $\mathbb{JE}(L)$ and hence a nonempty graph $\mathbb{PE}(L)$ is complete if and only if $\mathbb{JE}(L)$ is such. Now the assertion follows from Theorem 2.7. □

Theorem 4.3. *For a semisimple lattice L, the following conditions are equivalent:*

(i) *$\mathbb{PE}(L)$ is a complete graph;*

(ii) *$\mathbb{PE}(L)$ contains a universal vertex;*

(iii) *$1 = a_1 \oplus a_2$ for some atoms a_1 and a_2.*

Proof. (i)⇒(ii) By (i), every vertex is universal, and so (ii) holds.

(ii)⇒(iii) Let a_1 be a universal vertex of $\mathbb{PE}(L)$ (so it is not essential in L). Then Lemma 2.9 shows that a_1 is an atom. By assumption, there is an element a_2 of L such that $1 = a_1 \oplus a_2$. It suffices to show that a_2 is an atom. Let $0 < c < a_2$ for some $c \in L$. By Lemma 3.7, $a_2 = c \oplus c'$ for some $c' \in L$ which implies that $1 = (a_1 \vee c) \vee c'$ with $(a_1 \vee c) \wedge c' = 0$ which is impossible since $a_1 \vee c \trianglelefteq L$. Thus a_2 is an atom.

The implication (iii)⇒(i) follows from Lemma 3.16. □

The join of all the atoms of L, denoted $s(L)$, is called the socle of the lattice L (i.e. $s(L) = \bigvee_{a \in \mathcal{A}(L)} a$). If $b \in L$, then the join of all the atoms a of L with $a \leq b$, denoted $s(b)$, is called the socle of the element b (i.e. $s(b) = \bigvee_{a \in \mathcal{A}(L), a \leq b} a$). An inspection will show that if b is an element of L, then $s(b)$ is the largest semisimple element of L such that $s(b) \leq b$. Moreover, $b = s(b)$ if and only if b is semisimple by Theorem 3.9. We need the following lemma proved in [8, Lemma 2 and Lemma 3].

Lemma 4.4. (i) *Let $a < b < c < d$ be elements of L. If b is essential in $[a, c]$ and c is essential in $[a, d]$, then b is essential in $[a, d]$.*

(ii) *Let $a, b, c, d \in L$ and $b \wedge d = 0$. If a and c are essential in $[0, b]$ and $[0, d]$ respectively, then $a \vee c$ is essential in $[0, b \vee d]$.*

Proposition 4.5. *If $a \in \mathcal{V}(\mathbb{JE}(L))$ with degree 1, then the following hold:*

(i) *a is uniform in L;*

(ii) $\deg_{\mathbb{PE}(L)}(x) = 1$ *for every vertex $x \leq a$;*

(iii) *The complement of a in L is a semisimple element.*

(iv) *If b is an element of L with $a \wedge b = 0$ and $a \vee b \trianglelefteq L$, then $s(L) \leq b$.*

Proof. (i) Assume to contrary, that there exist $0 < b \leq a$ and $0 < c \leq a$ with $b \wedge c = 0$. By Lemma 2.3, there is an element $y \in L$ such that $a \vee y \trianglelefteq L$ and $a \wedge y = 0$ (so y is not essential in L). If $y \vee b = y$, then $b \leq y$ and so $b = a \wedge b \leq a \wedge y = 0$, which is impossible. Thus $y \vee b \neq y$. Now $\deg_{\mathbb{PE}(L)}(a) = 1$, $y \neq b \vee y$ and $a \vee (y \vee b) = a \vee y \trianglelefteq L$, which leads to $y \vee b \trianglelefteq L$. Similarly, $c \vee y \trianglelefteq L$. It follows from Lemma 2.2 (ii) that $(y \vee b) \wedge (y \vee c) = y \trianglelefteq L$, a contradiction. Thus a is uniform in L.

(ii) By Lemma 2.3, there is an element $y \in L$ such that $a \vee y \trianglelefteq L$ and $a \wedge y = 0$. By an argument like that in (i), we have $x \vee y \trianglelefteq L$ and so x and y are adjacent in $\mathbb{PE}(L)$. If z and x are adjacent in $\mathbb{PE}(L)$ with $z \neq y$, then $x \vee z \leq a \vee z$ and $x \vee z \trianglelefteq L$, we have that a and z are adjacent in $\mathbb{PE}(L)$, which is a contradiction. Thus $\deg_{\mathbb{PE}(L)}(x) = 1$.

(iii) Suppose that s is the complement of a in L; so $a \vee s \trianglelefteq L = [0, 1]$. If $b < s$ is an element of L with $b \trianglelefteq [0, s]$, then $a \trianglelefteq [0, a]$ and $a \wedge s = 0$ shows that $a \vee b$ is essential in $[0, a \vee s]$, by Lemma 4.4 (ii), which implies that $a \vee b \trianglelefteq L$, by Lemma 4.4 (i), as $0 < a \vee b < a \vee s < 1$. Thus, as $\deg_{\mathbb{PE}(L)}(a) = 1$, if $x < s$, then x is not essential in $[0, s]$, so it is semisimple and (iii) holds by Proposition 3.10.

(iv) By (i), a is uniform. Let b be an element of L satisfying the assumptions of (iv) and c be an atom of L. If $c \leq a$, then c is essential in $[0, a]$, which implies that $c \wedge (a \wedge b) \neq 0$ and so $c = c \wedge (a \wedge b) \leq b$. Now suppose that $c \wedge a = 0$ (so $c \leq s$, where s is a complement of a in L). If $c \leq b$, then we are done. So suppose that $c \wedge b = 0$ (so $b \neq s$). Then b would not be essential in L and a would be adjacent to two different vertices, b and s. Hence $c \leq b$ and so $s(L) \leq b$. □

Corollary 4.6. *If the graph $\mathbb{PE}(L)$ has only finitely many vertices of degree 1, then for every such vertex x, there exists an atom a such that $a \leq x$.*

Proof. Let x be an element of $\mathbb{PE}(L)$ with degree 1. Assume the result is false. Then there exist infinitely many elements with degree 1 by Proposition 4.5 (ii), which is a contradiction. □

Proposition 4.7. *Suppose that a is a proper uniform element of L and let $c = a \wedge s(L)$. Assume that for any element b of L with $a \wedge b = 0$, $b \leq s(L)$. Then the following hold:*

(i) *$a \vee s(L)$ is essential in L;*

(ii) *If $c = 0$, then $s = s(L)$ is the unique complement to a in L;*

(iii) *If $c \neq 0$, then c is an atom and an element s is a complement to a in L if and only if s is the complement to c in $[0, s(L)]$. In particular, if s is a complement to a in L, then $s \oplus c = s(L)$.*

Proof. (i) If s is a complement to a in L, then $a \wedge s = 0$ and $a \vee s \trianglelefteq L$ by Lemma 2.3. Since by assumption $s \leq s(L)$, $a \vee s \leq a \vee s(L)$, which implies that $a \vee s(L) \trianglelefteq L$.

(ii) If s is a complement to a in L, then $s \leq s(L)$, from which we get that $s(L)$ is maximal with respect to $a \wedge s(L) = 0$; hence $s = s(L)$.

(iii) Suppose now that $c \neq 0$. By Lemma 3.7, $c \leq s(L)$ implies that c is semisimple. If $c' < c$, then c' is essential in $[0, c]$, as a is uniform. This yields $c' = 0$ by Remark 3.8; hence c is an atom.

Let s be a complement to a in L (so $s \leq s(L)$) which implies that $s \wedge (a \wedge s(L)) = s \wedge c = 0$. Let $g \leq s(L)$ be any element of L such that $g \wedge (a \wedge s(L) = a \wedge g = 0$. It follows that $g \leq s$ and so s is maximal with respect to $s \wedge c = 0$. Thus s is a complement to c in $[0, s(L)]$. The reverse implication is similar. Finally, if s is a complement to a in L, then $s \vee c \trianglelefteq [0, s(L)]$; so $s \oplus c = s(L)$, as $s(L)$ is semisimple. This completes the proof. □

Corollary 4.8. *Suppose $a \in \mathcal{V}(\mathbb{JE}(L))$ is a vertex of degree one. Let s be the unique complement to a in L and $c = a \wedge s(L)$. Then the following hold:*

(i) *If $b \wedge a = 0$ for some element b of L, then $b \leq s(L)$;*

(ii) *$c = 0$ if and only if $s = s(L)$;*

(iii) *$c \neq 0$ if and only if $s \neq s(L)$ and in this case, s is the unique complement of c in $[0, s(L)]$.*

Proof. (i) Since s is semisimple by Proposition 4.5 (iii) and $b \wedge a = 0$, $b \leq s \leq s(L)$.

(ii) If $c = 0$, then $s(L) \leq s$; hence $s = s(L)$. The reverse implication is similar.

(iii) If $c \neq 0$, then (iii) is just Proposition 4.7 (iii). □

Proposition 4.9. *Suppose u is a nontrivial uniform element of L with a nonzero complement $s \leq s(L)$ and let there exist an atom element a of L such that $a \nleq u$. Then* $\deg_{\mathbb{PE}(L)}(u) > 1$.

Proof. If u has two different complements in L, then we are done. So we may assume that s is the unique complement of u in L (so $u \vee s \trianglelefteq L$ and $u \wedge s = 0$). Let $0 \neq f \leq u$. Since u is uniform, $f \trianglelefteq [0, u]$, which implies that $f \vee s \trianglelefteq [0, u \vee s]$ by Lemma 4.4 (ii), as $s \trianglelefteq [0, s]$ and $u \wedge s = 0$. Now $0 < f \vee s < u \vee s < 1$ shows that $f \vee s \trianglelefteq [0, 1] = L$, by Lemma 4.4 (i). Since $a \nleq u$, $a \wedge u = 0$ (so $f \wedge a = 0$), which implies that $a \leq s$. Then s semisimple yields $s = a \vee w$ and $a \wedge w = 0$, for some $w \in L$. Moreover, $a \vee (f \vee w) = f \vee s$ is essential in L. Now from $(f \vee w) \wedge a = 0$, we get that $f \vee w$ is not essential in L and the essentiality of $a \vee (f \vee w)$ in L shows that a is adjacent with two distinct vertices s and $f \vee w$. Hence $\deg_{\Gamma_P(L)}(F) > 1$. □

We next give three other characterizations of vertices of degree one in $\mathbb{PE}(L)$. Compare the next theorem with Theorem 2.13 in [9].

Theorem 4.10. *If u is an element of $\mathcal{V}(\mathbb{JE}(L))$, then the following conditions are equivalent:*

(i) $\deg_{\mathbb{PE}(L)}(u) = 1$*;*

(ii) (a) *u is uniform and the complement to u in L is unique and semisimple;*

(b) *For every atom element a of L, $a \leq u$;*

(iii) (a) *u is uniform and the complement to u in L is unique and semisimple;*

(b) *If $e \wedge u \neq 0$ and $u \vee e \trianglelefteq L$, for an element e of L, then $s(L) \leq e$;*

(iv) *For any element e of L the following conditions hold:*

(a) *If $u \wedge e = 0$, then $e \leq s(L)$;*

(b) *If $u \wedge e \neq 0$ and $u \vee e \trianglelefteq L$, then $s(L) \leq e$;*

(c) *u is uniform and $u \wedge s(L)$ has a unique complement in $s(L)$.*

Proof. (i)⇒(ii) Let $\deg_{\Gamma_P(L)}(F) = 1$. Then part (a) holds by Proposition 4.5 (i) and Proposition 4.5 (iii). Part (b) follows from Proposition 4.9.

(ii)⇒(iii) Let a be an atom of L. Then condition (ii) (b) implies that $a \leq u$. Let e be an element of L such that $e \wedge u \neq 0$ and $u \vee e \trianglelefteq L$. As u is uniform, $a \wedge (e \wedge u) = a \wedge e \neq 0$; hence $a \leq e$ and (b) follows.

(iii)⇒(i) Let s be a complement of u in L. By (iii) (a), s is unique and semisimple. Let e be an element of L such that $e \vee u \trianglelefteq L$. If $e \leq s$, then $e \wedge u \neq 0$, and so $s \leq s(L) \leq e$ by (iii) (b); hence $e = s$. Thus, to prove the implication, it is enough to show that if $e \nleq s$, then $e \trianglelefteq L$. To this end, suppose $e \nleq s$. Then $e \wedge u \neq 0$, as the complement s is unique. Hence, $s(L) \leq e$ by (iii) (b). Notice also that the condition (iii) (a) guarantees that u satisfies assumptions of Corollary 4.6. Set $c = u \wedge s(L)$.

If $c = 0$, then $s = s(L)$ (so $u \vee s(L) \trianglelefteq L$). Since u is uniform, $e \wedge u$ is essential in $[0, u]$, which implies that $(e \wedge u) \vee s(L) \trianglelefteq [0, u \vee s(L)]$ by Corollary 4.8, as $s(L) \trianglelefteq [0, s(L)]$. Now, by Corollary 4.8, $0 < (e \wedge u) \vee s(L) < u \vee s(L) < 1$ establishes that $(e \wedge u) \vee s(L) \trianglelefteq [0, 1] = L$, which implies that e is essential in L, as $(e \wedge u) \vee s(L) \leq e$.

If $c \neq 0$, then Proposition 4.7 (iii) implies that c is an atom, s is the unique complement to c in $[0, s(L)]$ and $s(L) = s \vee c$. Since $c \trianglelefteq [0, u]$ and $s \trianglelefteq [0, s]$, $c \vee s$ is essential in $[0, u \vee s]$; hence $s(L) = s \vee c$ is essential in $[0, 1] = L$ by Corollary 4.8. So $e \trianglelefteq L$ follows, as $s(L) \leq e$. This completes the proof of (i). Note that the condition (iii) (a) guarantees that u satisfies assumptions of Corollary 4.6. Then the equivalence (iii)⇔(iv) follows from this lemma. □

Lemma 4.11. *Let L contain no infinite direct join of nonzero elements. Then the following hold:*

(i) *For every nonzero element a, $[0, a]$ contains a uniform element;*

(ii) *There exists a direct join of uniform elements which is essential in L, so that L has a rank, and in fact, the rank is then finite.*

Proof. (i) If a is not itself uniform, then $[0, a]$ contains a direct join $a_1 \oplus a'_1$. Now either a'_1 is uniform or $[0, a'_1]$ contains a direct join $a_2 \oplus a'_2$ and continuing in this way we get in L the direct join $a_1 \oplus a_2 \oplus \ldots$, which is impossible. Thus $[0, a]$ contains a uniform element.

(ii) Let $\bigvee_{i=1}^{n} a_i$ be a direct join of uniform elements in L. Either it is essential in L or there is an nonzero element g such that $g \wedge \bigvee_{i=1}^{n} = 0$. By (i), $[0, g]$ contains a uniform element a_{n+1} and $\bigvee_{i=1}^{n+1} a_i$ is a direct join of $n+1$ terms. By continuing this way, we obtain in L a direct join $a_1 \oplus a_2 \oplus \ldots$, which is a contradiction. This shows that L has a rank and that this rank is finite. □

If L has finite rank (or finite uniform dimension), say n, then we write $\mathrm{udim}(L) = n$. If L does not satisfy this conditions, we put $\mathrm{udim}(L) = \infty$. It is easy to see that if $\mathrm{udim}(L) = 2$, then every nonessential nonzero element a is uniform.

Proposition 4.12. *If every vertex of $\mathbb{PE}(L)$ is of finite degree, then $s(L)$ is essential in L and $[0, s(L)]$ contains only finitely many elements.*

Proof. Assume that $\mathbb{PE}(L)$ is not an empty graph and let there be nonzero elements a_i such that $\bigoplus_{i=1}^{\infty} a_i$ is essential in L. Then $b = \bigoplus_{i=2}^{\infty} a_i$ would be adjacent to infinitely many vertices $a_1 \oplus a_m$ for positive integer m which is impossible since the degree of b is finite; hence $\mathrm{udim}(L)$ is finite, say n, by Lemma 4.11. Then there exist uniform elements $u_1, \ldots, u_n$ of L such that $\bigoplus_{i=1}^{n} u_i \trianglelefteq L$.

Let c_i be a complement of u_i in L (so $c_i \vee u_i \trianglelefteq L$). If k_i is any nonzero element of $[0, u_i]$ (so $k_i \trianglelefteq [0, u_i]$), then $k_i \vee c_i \trianglelefteq [0, u_i \vee c_i]$, by Lemma 4.4, which implies that $k_i \vee c_i \trianglelefteq L$ by Lemma 2.2 (iii). This shows that $[0, u_i]$ contains only finitely many elements, and in particular, it contains an atom $(1 \leq i \leq n)$.

Let a_i be an atom with $a_i \leq u_i$ $(1 \leq i \leq n)$. Since $\bigvee_{i=1}^n a_i \subseteq s(L)$ and $\bigvee_{i=1}^n a_i \trianglelefteq [0, \bigvee_{i=1}^n u_i]$, $\bigvee_{i=1}^n a_i \trianglelefteq L$; hence $s(L)$ is essential in L. It follows from the assumption and Proposition 4.1 that every vertex of the graph $\mathbb{PE}([0, s(L)]) = \mathbb{JE}([0, s(L)])$ is of finite degree; hence $[0, s(L)]$ contains only finitely many elements by Theorem 3.24. □

Compare the next theorem with Theorem 2.3 in [9].

THEOREM 4.13. *For the lattice L the following conditions are equivalent:*

(i) *Every vertex of $\mathbb{PE}(L)$ is of finite degree;*

(ii) *The graph $\mathbb{PE}(L)$ is finite.*

Proof. (i)⇒(ii) By Proposition 4.12, $x \wedge s(L) \neq 0$ for every nontrivial element x of L and $[0, s(L)]$ contains only finitely many elements. Assume that k is any nontrivial element of $[0, s(L)]$ and let $A_k = \{c \in L : c \wedge s(L) = k\}$.

At first we show that $\mathcal{V}(\mathbb{PE}(L)) = \bigcup_{s(L) \neq k \in [0, s(L]} A_k$. Since the inclusion $\mathcal{V}(\mathbb{PE}(L)) \subseteq \bigcup_{s(L) \neq k \in [0, s(L)]} A_k$ is clear, we will prove the reverse inclusion.

Suppose that $e \in \bigcup_{s(L) \neq k \in [0, s(L)]} A_k$. Then there is a $s(L) \neq c \in [0, s(L)]$ such that $e \wedge s(L) = c$. Then c is not essential, hence e is not essential; therefore $e \in \mathcal{V}(\mathbb{PE}(L))$ and so we have equality. Now it is enough to show that A_k is a finite set for every nontrivial element k of $[0, s(L)]$. Let c_k be a complement of k in L. Clearly, $c_k \neq 0$. Let $u \in A_k$. Then $c_k \vee k = c_k \vee (u \wedge s(L)) \leq c_k \vee u$ and $c_k \vee k \trianglelefteq L$, which implies that $c_k \vee u \trianglelefteq L$; hence c_k is adjacent to any $u \in A_k$. As c_k is of finite degree, we obtain that A_k is finite. This completes the proof.

The implication (ii)⇒(i) is clear. □

DEFINITION 4.14. We say that nonzero elements a and b of L are strongly distinct if $a \wedge b = 0$.

It is easy to see that if a and b are strongly distinct, then $a \neq b$.

Compare the next theorem with Theorem 3.11 in [9].

THEOREM 4.15. *If $\mathbb{PE}(L)$ is not an empty graph, then the following conditions are equivalent:*

(i) *$\mathbb{PE}(L)$ is a triangle-free graph;*

(ii) (a) $\mathrm{udim}(L) = 2$;

(b) *If a and b are elements of $\mathcal{V}$ $(\mathbb{PE}(L))$ with $a \vee b \trianglelefteq L$, then a and b are strongly distinct.*

(iii) *If a and b are elements of $\mathcal{V}$ $(\mathbb{PE}(L))$ with $a \vee b \trianglelefteq L$, then a and b are strongly distinct.*

Proof. (i)⇒(ii) Assume that $\mathbb{PE}(L)$ is triangle-free and that there are nontrivial elements a, b and c such that $a \oplus b \oplus c \trianglelefteq L$. Then $a \vee b$, $a \vee c$, $b \vee c$ would form a triangle, which is a contradiction. Thus $\mathrm{udim}(L) \leq 2$. If $\mathrm{udim}(L) = 1$, then any two vertexes are adjacent, which is impossible. Thus $\mathrm{udim}(L) = 2$; so (ii) (a) holds. Let a, b be elements of $\mathcal{V}$ $(\mathbb{PE}(L))$ such that $a \vee b \trianglelefteq L$. We show that a and b are strongly distinct. Let c' be the complement of $c = a \wedge b$ in L. If $c \neq 0$, then c' is a proper nonessential element of L, $a \vee c' \trianglelefteq L$ and $b \vee c' \trianglelefteq L$ (as $c \vee c' \leq a \wedge c', b \wedge c'$), which implies that b, a, c' would form a triangle. This is impossible, so $c = 0$.

The implication (ii)⇒(iii) is clear.

(iii)⇒(i) Let a, b, c be elements of $\mathcal{V}$ $(\mathbb{PE}(L))$ such that $a \vee c \trianglelefteq L$ and $b \vee c \trianglelefteq L$. Then $a \wedge c = 0$ and $b \wedge c = 0$ by (iii). It is enough to show that $a \vee b$ is not essential in L. Assume to the contrary, that $a \vee b \trianglelefteq L$, so $c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b) \neq 0$, which is impossible. This completes the proof. □

Theorem 4.16. *If $\mathbb{PE}(L)$ is not an empty graph, then the following conditions are equivalent:*

(i) *$\mathbb{PE}(L)$ is a tree;*

(ii) *If a and b are elements of $\mathcal{V}$ $(\mathbb{PE}(L))$ with $a \vee b \trianglelefteq L$, then a and b are strongly distinct and one of a and b is an atom element;*

(iii) *$\mathbb{PE}(L)$ is a star graph with the center c, for an atom element c of L.*

Proof. (i)⇒(ii) Since every tree is triangle-free, Theorem 4.15 shows that the first part of the statement (ii) holds. Let $0 < a' < a$ and $0 < b' < b$. Since every tree is a bipartite graph, we have a cycle $a' \backsim b \backsim a \backsim b' \backsim a'$ in a tree, which is impossible. Thus one of a and b is an atom element and so (ii) holds.

(ii)⇒(iii) By (ii), L contains an atom element. Moreover, $\mathbb{PE}(L)$ is triangle-free and $\mathrm{udim}(L) = 2$ by Theorem 4.15; so L has at most two atom elements a, a' with $a \neq a'$.

If $s(L) = a$, then $\mathbb{PE}(L)$ is a star graph with center a by (ii). If $s(L)$ is not simple, then $\mathrm{udim}(L) = 2$ implies that $a \vee a' = s(L)$ is essential in L.

Let x be any element of $\mathcal{V}$ $(\mathbb{PE}(L))$. Then $x \wedge (a \vee a') = (x \wedge a) \vee (x \wedge a') \neq 0$, which implies $[0, x]$ contains one of a or a'. Let $a \trianglelefteq [0, u] \neq L$ and $a' \trianglelefteq [0, u'] \neq L$. Then $u \vee u' \trianglelefteq L$ implies u, $u \vee u'$, u' would form a triangle, which is impossible. So one of the atoms, say a, does not have proper essential extensions in L. Hence, if x is an element of $\mathcal{V}$ $(\mathbb{PE}(L))$ with $x \neq a$, then $a' \leq x$ and $\mathbb{PE}(L)$ is a star graph with center a.

The implication (iii)⇒(i) is clear. □

Theorem 4.17. *If $\mathbb{PE}(L)$ is not an empty graph, then the following hold.*

(i) *$\mathbb{PE}(L)$ is a connected graph with* $\mathrm{diam}(\mathbb{PE}(L)) \leq 3$.

(ii) $\mathrm{gr}(\Gamma_P(L)) \in \{3, 4, \infty\}$.

Proof. (i) Suppose there are nontrivial elements $f \neq k$ such that f and k are not adjacent. Let h, h' denote complements to $f \vee k$ and $f \wedge k$ in L, respectively. If $f \wedge k = 0$, then $h \wedge (f \vee k) = 0$ implies $h \wedge f = 0 = h \wedge k$ and so $(h \vee f) \wedge k = 0 = (h \vee k) \wedge f$; hence $h \vee f$ and $h \vee k$ are not essential in L. Then $h \vee (f \vee k) \trianglelefteq L$, which establishes that $f \backsim k \oplus h \backsim f \oplus h \backsim k$ is a path of length 3 in $\mathbb{PE}(L)$. If $f \wedge k \neq 0$, then $f \backsim h' \backsim k$ is a path of length 2 in $\mathbb{PE}(L)$, as $h' \vee (f \wedge k) \trianglelefteq L$ and f, h' and k are not essential in L, as needed.

(ii) If $\mathbb{PE}(L)$ contains a triangle, then $\mathrm{gr}(\mathbb{PE}(L)) = 3$. So we can assume that $\mathbb{PE}(L)$ is triangle-free. If $\mathbb{PE}(L)$ is tree, then $\mathrm{gr}(\mathbb{PE}(L)) = \infty$. If $\mathbb{PE}(L)$ is not tree, then there exist two strongly distinct uniform elements a and b in $\mathcal{V}$ $(\mathbb{PE}(L))$ such that $a \vee b$ is essential in L and neither a nor b is an atom by Theorems 4.15 and 4.16. Let a', b' be nontrivial elements of $[0, a]$ and $[0, b]$, respectively. Since $a' \trianglelefteq [0, a]$ and $b' \trianglelefteq [0, b]$, we have a cycle $a' \backsim b' \backsim a \backsim b \backsim a'$, which implies that $\mathrm{gr}(\Gamma_M(L)) = 4$. □

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