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CHARACTERIZATION OF VECTOR-VALUED WEAK AFFINE BI-FRAMES ON POSITIVE HALF LINE

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Abstract. In this paper, we introduce the notion of vector valued weak affine bi-frames in reducing subspaces of $L^2(\mathbb{R}_+, \mathbb{C}^L)$ and obtain a characterization of these frames by using Walsh-Fourier transform.

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1. INTRODUCTION

The concept of vector valued frames was introduced by Han and Larson in [18]. Having applications in signal processing, data compression and image analysis, vector valued frames solve the problems of multiplexing in networking, which consists of sending multiple signals or streams of information on a carrier at the same time in the form of a single, complex signal and then recovering the separate signals at the receiving end. During recent times, the research on vector valued frames is mainly focused on Gabor and wavelet frames in $L^2(\mathbb{R}, \mathbb{C}^L)$ [17, 20, 24]. For vector valued wavelet frames, Han and Larson [18] showed that there is no MRA in $L^2(\mathbb{R}, \mathbb{C}^L)$ with $L > 1$ according to usual dilation and translation and obtained characterization of vector valued wavelets in the Fourier domain.

The theory of affine bi-frames has wide variety of applications in image compression, signal denoising, numerical treatment of operator equations. In constructing these frames, extension principles based on refinable functions play a vital role [14, 26, 32]. These principles are based on the fact the that the affine systems are Bessel sequences. In reducing subspaces settings of $L^2(\mathbb{R}^d)$, Jia and Li [19] characterized weak affine bi-frames and starting from a pair of general refinable functions without smoothness restrictions, obtained a construction of weak affine biframes and derived a fast algorithm associated with them.

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In [21], Li and Jia introduced the notions of reducing subspace of a Sobolev space and weak nonhomogeneous wavelet bi-frame in a general pair of dual reducing subspaces of $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$, obtained a refinable function-based construction of these frames, and derived the corresponding fast wavelet algorithm. Recently, Li and Tian [22] provides the characterization of weak affine super bi-frames for reducing subspaces of $L^2(\mathbb{R}, \mathbb{C}^L)$. The author in the series of papers [1–12] established various results related to frames, Gabor frames, framelets on real line and positive half line.

During last two decades there is a substantial body of work that has been concerned with the wavelet and Gabor frames on positive half line. Farkov [15] indicated several differences between the constructed wavelets in Walsh analysis and the classical wavelets, and characterized all compactly supported refinable functions on the Vilenkin group G_p with $p \geq 2$. Albeverio et al. [13] presented a complete characterization of scaling functions generating an p-MRA, suggested a method for constructing sets of wavelet functions, and proved that any set of wavelet functions generates a p-adic wavelet frame.

More Recently, Zhang [28] characterizes the shift-invariant Bessel sequences, frame sequences and Riesz sequences in $L^2(\mathbb{R}^+)$ and gives a characterization of dual wavelet frames using Walsh-Fourier transform.

Motivated and inspired by the above work, we in this paper introduce the notion of vector valued weak affine bi-frames in reducing subspaces of $L^2(\mathbb{R}_+, \mathbb{C}^L)$ and obtain a characterization of these frames by using Walsh-Fourier transform. For more results, reader is referred to [23, 29–31].

The paper is structured as follows. In Section 2, we explain certain results about Walsh -Fourier analysis and some basic definitions that will be used in the paper. Section 3 is devoted to the characterization of vector valued weak affine biframes by using Walsh-Fourier transform.

2. PRELIMINARIES

As usual, let $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and $\mathbb{N} = \mathbb{Z}^+ \setminus \{0\}$. Denote by $[x]$ the integer part of x. Let p be a fixed natural number greater than 1. For $x \in \mathbb{R}_+$ and any positive integer j, we set

$$x_j = [p^j x](\operatorname{mod} p), \qquad x_{-j} = [p^{1-j} x](\operatorname{mod} p), \tag{1}$$

where $x_j, x_{-j} \in \{0, 1, \dots, p-1\}$. Clearly, x_j and x_{-j} are the digits in the p-expansion of x:

$$x = \sum_{j<0} x_{-j} p^{j-1} + \sum_{j>0} x_j p^{-j}.$$

Moreover, the first sum on the right is always finite. Besides,

$$[x] = \sum_{j<0} x_{-j} p^{-j-1}, \qquad \{x\} = \sum_{j>0} x_j p^{-j},$$

where $[x]$ and $\{x\}$ are, respectively, the integer and fractional parts of x.

Consider on $\mathbb{R}_+$ the addition defined as follows:

$$x \oplus y = \sum_{j<0} \zeta_j p^{-j-1} + \sum_{j>0} \zeta_j p^{-j},$$

with $\zeta_j = x_j + y_j (\operatorname{mod} p)$, $j \in \mathbb{Z} \setminus \{0\}$, where $\zeta_j \in \{0, 1, \dots, p-1\}$ and x_j, y_j are calculated by (1). Clearly, $[x \oplus y] = [x] \oplus [y]$ and $\{x \oplus y\} = \{x\} \oplus \{y\}$. As usual, we write $z = x \ominus y$ if $z \oplus y = x$, where $\ominus$ denotes subtraction modulo p in $\mathbb{R}_+$.

Let $\varepsilon_p = \exp(2\pi i/p)$, we define a function $r_0(x)$ on $[0,1)$ by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/p) \\ \varepsilon_p^{\ell}, & \text{if } x \in \left[\ell p^{-1}, (\ell+1)p^{-1}\right), \quad \ell = 1, 2, \dots, p-1. \end{cases}$$

The extension of the function r_0 to $\mathbb{R}_+$ is given by the equality $r_0(x+1) = r_0(x), \forall\, x \in \mathbb{R}_+$. Then, the system of *generalized Walsh functions*

$$\left\{w_m(x) : m \in \mathbb{Z}^+\right\}$$

on $[0,1)$ is defined by

$$w_0(x) \equiv 1 \quad \text{and} \quad w_m(x) = \prod_{j=0}^{k} \left(r_0(p^j x)\right)^{\mu_j}$$

where $m = \sum_{j=0}^{k} \mu_j p^j$, $\mu_j \in \{0, 1, \dots, p-1\}$, $\mu_k \neq 0$. They have many properties similar to those of the Haar functions and trigonometric series, and form a complete orthogonal system.

Further, by a Walsh polynomial we shall mean a finite linear combination of Walsh functions. For $x, y \in \mathbb{R}_+$, let

$$\chi(x,y) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j y_{-j} + x_{-j} y_j)\right), \tag{2}$$

where x_j, y_j are given by equation (1).

The *Walsh-Fourier transform* of a function $f \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}_+} f(x)\, \overline{\chi(x,\xi)}\, \mathrm{d}x,$$

where $\chi(x,\xi)$ is given by (2). The Walsh-Fourier operator $\mathcal{F} : L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$, $\mathcal{F}f = \hat{f}$, extends uniquely to the whole space $L^2(\mathbb{R}_+)$. The properties of the Walsh-Fourier transform are quite similar to those of the classic Fourier transform (see [16, 25]). In particular, if $f \in L^2(\mathbb{R}_+)$, then $\hat{f} \in L^2(\mathbb{R}_+)$ and

$$\left\|\hat{f}\right\|_{L^2(\mathbb{R}_+)} = \left\|f\right\|_{L^2(\mathbb{R}_+)}. \tag{3}$$

Moreover, if $f \in L^2[0,1]$, then we can define the Walsh-Fourier coefficients of f as

$$\hat{f}(n) = \int_0^1 f(x)\overline{w_n(x)}\,\mathrm{d}x.$$

The series $\sum_{n\in\mathbb{Z}_+} \hat{f}(n) w_n(x)$ is called the *Walsh-Fourier series* of f. Therefore, from the standard L^2-theory, we conclude that the Walsh-Fourier series of f converges to f in $L^2[0,1]$ and Parseval's identity holds:

$$\|f\|_2^2 = \int_0^1 \left|f(x)\right|^2 \mathrm{d}x = \sum_{n\in\mathbb{Z}_+} \left|\hat{f}(n)\right|^2.$$

By p-adic interval $I \subset \mathbb{R}_+$ of range n, we mean intervals of the form

$$I = I_n^k = \left[kp^{-n}, (k+1)p^{-n}\right),\ k \in \mathbb{Z}^+.$$

The p-adic topology is generated by the collection of p-adic intervals and each p-adic interval is both open and closed under the p-adic topology (see [16]). The family $\left\{[0, p^{-j}) : j \in \mathbb{Z}\right\}$ forms a fundamental system of the p-adic topology on $\mathbb{R}_+$. Therefore, the generalized Walsh functions $w_j(x), 0 \le j \le p^n - 1$, assume constant values on each p-adic interval I_n^k and hence are continuous on these intervals. Thus, $w_j(x) = 1$ for $x \in I_n^0$.

Let $E_n(\mathbb{R}_+)$ be the space of p-adic entire functions of order n, that is, the set of all functions which are constant on all p-adic intervals of range n. Thus, for every $f \in E_n(\mathbb{R}_+)$, we have

$$f(x) = \sum_{k\in\mathbb{Z}^+} f(p^{-n}k)\chi_{I_n^k}(x), \quad x \in \mathbb{R}_+.$$

Clearly each Walsh function of order up to p^{n-1} belongs to $E_n(\mathbb{R}_+)$. The set $E(\mathbb{R}_+)$ of p-adic entire functions on $\mathbb{R}_+$ is the union of all the spaces $E_n(\mathbb{R}_+)$. It is clear that $E(\mathbb{R}_+)$ is dense in $L^p(\mathbb{R}_+), 1 \le p < \infty$ and each function in $E(\mathbb{R}_+)$ is of compact support. Thus, we consider the following set of functions

$$E^0(\mathbb{R}_+) = \left\{ f \in E(\mathbb{R}_+) :\ \hat{f} \in L^\infty(\mathbb{R}_+) \text{ and supp } f \subset \mathbb{R}_+ \setminus \{0\} \right\}.$$

Given a positive integer L and separable Hilbert spaces $H_1, H_2, ..., H_L$, we denote by $\bigoplus_{\ell=l}^L H_\ell$ their direct sum space endowed with the inner product

$$\langle \mathbf{f}, \tilde{\mathbf{f}} \rangle = \sum_{\ell=1}^L \langle f_l, \tilde{f}_l \rangle$$

for $\mathbf{f} = (f_1, f_2, ..., f_L),\ \tilde{\mathbf{f}} = (\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_L) \in \bigoplus_{l=l}^L H_l$. We also say that $\bigoplus_{\ell=l}^L H_\ell$ is a super space. In particular, $\bigoplus_{\ell=l}^L L^2(\mathbb{R}_+)$ is exactly the vector-valued space $L^2(\mathbb{R}_+, \mathbb{C}^L)$. A Bessel sequence, frame, tight frame, Parseval frame and a pair of dual frames in $\bigoplus_{\ell=l}^L H_\ell$ are called a vector valued Bessel sequence, frame, tight frame, Parseval frame and a pair of super bi-frames, respectively.

For $f \in L^2(\mathbb{R}_+, \mathbb{C}^L)$, we always denote its ℓ-th component by f_ℓ. Given $p > 0$ and $k \in \mathbb{Z}_+$, define the dilation operator D_p and translation operator T_k on $L^2(\mathbb{R}_+, \mathbb{C}^L)$ by

$$D_p f(x) = \left(\sqrt{p} f_1(p^{-1}x), \sqrt{p} f_2(p^{-1}x), ..., \sqrt{p} f_L(p^{-1}x)\right),$$

$$T_k f(x) = (f_1(x \ominus k), f_2(x \ominus k), ..., f_L(x \ominus k)),$$

for $f \in L^2(\mathbb{R}_+, \mathbb{C}^L)$, respectively.

The vector-valued Fourier transform is defined by $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_L)$ for $f \in L^2(\mathbb{R}_+, \mathbb{C}^L)$. Given a subset E of $\mathbb{R}_+$ with nonzero measure, we define closed subspaces $L^2(E, \mathbb{C}^L)$ and $FL^2(E, \mathbb{C}^L)$ of $L^2(\mathbb{R}_+, \mathbb{C}^L)$ by

$$L^2(E, \mathbb{C}^L) = \left\{f \in L^2(\mathbb{R}_+, \mathbb{C}^L) : \operatorname{supp}(f) \subset E\right\}$$

and

$$FL^2(E, \mathbb{C}^L) = \left\{f \in L^2(\mathbb{R}_+, \mathbb{C}^L) : \operatorname{supp}(\tilde{f}) \subset E\right\}$$

where $\operatorname{supp}(h) = \{\xi \in \mathbb{R}_+ : h(\xi) \neq 0\}$ for a vector-valued measurable function h, which is well-defined up to a set of measure zero. For simplicity, we write $L^2(E)$ and $FL^2(E)$ for $L^2(E, C)$ and $FL^2(E, C)$ respectively.

Let $L \in \mathbb{N}$, and $p \in \mathbb{R}_+$. For $f \in L^2(\mathbb{R}_+, \mathbb{C}^L)$, we define the *affine system* generated by f by

$$A(f) = \{D_{p^j} T_k f : j \in \mathbb{Z}, k \in \mathbb{Z}_+\}.$$

For a finite subset F of $L^2(\mathbb{R}_+, \mathbb{C}^L)$, the *affine system* $A(F)$ generated by F can be defined similarly, that is,

$$A(F) = \{D_{p^j} T_k f : f \in F, j \in \mathbb{Z}, k \in \mathbb{Z}_+\}.$$

A nonzero closed subspace X of $L^2(\mathbb{R}_+, \mathbb{C}^L)$ is called a reducing subspace if $D_p X = X$ and $T_k X = X$ for each $k \in \mathbb{Z}_+$. Then we can obtain a result, which is similar as that on Euclidean subspaces, as follows.

PROPOSITION 2.1. *Let p be a positive integer greater than 1. A nonzero closed subspace X of $L^2(\mathbb{R}_+, \mathbb{C}^L)$ is a reducing subspace of $L^2(\mathbb{R}_+, \mathbb{C}^L)$ if and only if $X = FL^2(\Omega, \mathbb{C}^L)$ for some $\Omega \subset R_+$ with nonzero measure satisfying $\Omega = p\Omega$.*

By Proposition 2.1, to be specific, we denote a reducing subspace with $FL^2(\Omega, \mathbb{C}^L)$ instead of X. In particular, $FL^2(\mathbb{R}_+, \mathbb{C}^L) = L^2(\mathbb{R}_+, \mathbb{C}^L)$ and it is a reducing subspace of $L^2(\mathbb{R}_+, \mathbb{C}^L)$, and $FL^2((0, \infty), \mathbb{C}^L)$ is also a reducing subspace of $L^2(\mathbb{R}_+, \mathbb{C}^L)$. Let M be a closed subspace of $L^2(\mathbb{R}_+, \mathbb{C}^L)$. For $\psi, \tilde{\psi} \in L^2(\mathbb{R}_+, \mathbb{C}^L)$, $(A(\psi), X\tilde{\psi}))$ is called a p-adic affine bi-frame (ABF) for M if it is a bi-frame for M, i.e., $A(\psi)$ and $A(\tilde{\psi})$ are two frames for M satisfying

$$f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}_+} \langle f, D_{p^j} T_k \tilde{\psi} \rangle D_{p^j} T_k \psi \text{ for } f \in M; \tag{4}$$

$(A(\psi), A(\tilde{\psi}))$ is called a weak affine bi-frame for M if there exists a dense subset V of M such that

$$\langle f, g\rangle = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}_+}\langle f, D_{p^j}T_k\tilde{\psi}\rangle\langle D_{p^j}T_k\psi, g\rangle \text{ for } f, g \in V, \tag{5}$$

where the series in (4) converges unconditionally in $L^2(\mathbb{R}_+)$-norm, while, in (5),

$$\sum_{k\in\mathbb{Z}_+}\langle f, D_{p^j}T_k\tilde{\psi}\rangle\langle D_{p^j}T_k\psi, g\rangle$$

converges unconditionally, and

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}_+}\langle f, D_{p^j}T_k\tilde{\psi}\rangle\langle D_{p^j}T_k\psi, g\rangle\\ &= \lim_{(J',J)\to(\infty,\infty)}\sum_{j=-J'}^{J}\sum_{k\in\mathbb{Z}_+}\langle f, D_{p^j}T_k\tilde{\psi}\rangle\langle D_{p^j}T_k\psi, g\rangle.\end{aligned}$$

For two finite subsets $\Psi = \{\psi^{(n)} : 1 \le n \le N\}$, $\tilde{\Psi} = \{\tilde{\psi}^{(n)} : 1 \le n \le N\}$ of $L^2(\mathbb{R}_+, \mathbb{C}^L)$, we say that $(A(\Psi), A(\tilde{\Psi}))$ is a p-adic weak affine bi-frame for M if there exists a dense subset V of M such that

$$\langle f, g\rangle = \sum_{n=1}^{N}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}_+}\langle f, D_{p^j}T_k\tilde{\psi}^{(n)}\rangle\langle D_{p^j}T_k\psi^{(n)}, g\rangle \ for\ f, g \in V$$

with the convergence similar to the above.

Throughout the paper, we need the following sets

$$\widetilde{D} = \left\{f \in L^2(\mathbb{R}_+) : \hat{f} \in L^\infty(\mathbb{R}_+) \text{ and } \operatorname{supp}(\hat{f}) \subset K \text{ for some } K \subset \mathbb{R}_+ \setminus \{0\}\right\}$$

and

$$D = \left\{f \in L^2(\mathbb{R}_+, \mathbb{C}^L) : f_\ell \in \widetilde{D} \text{ for each } 1 \le \ell \le L\right\}.$$

3. CHARACTERIZATION OF VECTOR-VALUED WEAK AFFINE BI-FRAMES ON HALF LINE

This section is devoted to the characterization of vector valued p-adic weak affine bi-frames for reducing subspaces of $L^2(\mathbb{R}_+, \mathbb{C}^L)$. In order to prove the main result of the paper, we first state and prove following lemmas.

Lemma 3.1. *Suppose* $F, G \in L^2(\mathbb{R}_+)$, *and* $\operatorname{supp}(F)$, $\operatorname{supp}(G)$ *are bounded. Then*

$$\sum_{k\in\mathbb{Z}_+}\widetilde{F}(k)\overline{\widetilde{G}(k)} = \int_{\mathbb{R}_+}\left\{\sum_{m\in\mathbb{Z}_+}F(\xi\oplus m)\right\}\overline{G(\xi)}\mathrm{d}\xi.$$

Proof. Since F and G are periodic, we have

$$\widetilde{F}(\xi)=\sum_{m\in\mathbb{Z}_+}F(\xi\oplus m),\quad \text{and}\quad \widetilde{G}(\xi)=\sum_{m\in\mathbb{Z}_+}G(\xi\oplus m).$$

Clearly, $\widetilde{F},\widetilde{G}\in L^2(\mathbb{T})$, because only a finite number of terms contribute to the above sum. Since

$$\widehat{\widetilde{F}}(k)=\int_{\mathbb{T}}\widetilde{F}(\xi)\overline{\chi_k(\xi)}\,\mathrm{d}\xi=\widehat{F}(k),\ \ k\in\mathbb{Z}_+,$$

by the Plancherel formula we obtain

$$\int_{\mathbb{R}_+}\widetilde{F}(\xi)\overline{G(\xi)}\,\mathrm{d}\xi=\int_{\mathbb{T}}\widetilde{F}(\xi)\,\overline{\widetilde{G}(\xi)}\,\mathrm{d}\xi=\sum_{k\in\mathbb{Z}_+}\widehat{F}(k)\overline{\widehat{G}(k)}.$$

□

LEMMA 3.2. *For $\psi\in L^2(\mathbb{R}_+)$ and $J\in\mathbb{N}$, we have*

$$\int_{\mathbb{R}_+}\sum_{j=J}^{\infty}|\widetilde{\psi}(p^j\xi)|^2\mathrm{d}\xi<\infty \tag{6}$$

and thus

$$\sum_{j=J}^{\infty}|\widetilde{\psi}(p^j\xi)|^2<\infty\ \ a.e.\ \ on\ \mathbb{R}_+. \tag{7}$$

Proof. (7) immediately follows from (6). So we need to prove only (6).

$$\int_{\mathbb{R}_+}\sum_{j=J}^{\infty}|\widetilde{\psi}(p^j\xi)|^2\mathrm{d}\xi=\sum_{j=J}^{\infty}\int_{\mathbb{R}_+}|\widetilde{\psi}(p^j\xi)|^2\mathrm{d}\xi=\|\widehat{\psi}\|^2\sum_{j=J}^{\infty}p^{-j}=\frac{p^{-J+1}}{p-1}\|\widehat{\psi}\|^2<\infty.$$

□

LEMMA 3.3. *Let $\psi\in L^2(\mathbb{R}_+)$ and $f\in\widetilde{D}$. Then*

$$\lim_{j\to\infty}\sum_{j<-J}\sum_{k\in\mathbb{Z}_+}|\langle f,\psi_{j,k}\rangle|^2=0.$$

Proof. We have

$$\sum_{k\in\mathbb{Z}_+}|\langle f,\psi_{j,k}\rangle|^2=p^j\int_{\mathbb{R}_+}\overline{\widehat{f}(p^j\xi)}\widehat{\psi}(\xi)\sum_{m\in\mathbb{Z}_+}\widehat{f}\left(p^j(\xi\oplus m\right)\overline{\widehat{\psi}(\xi\oplus m)},$$

and thus

$$\begin{aligned}\sum_{j<-J}\sum_{k\in\mathbb{Z}_+}|\langle f,\psi_{j,k}\rangle|^2 &= \sum_{j<-J} p^j \int_{\mathbb{R}_+}\left|\widehat{f}(p^j\xi)\widehat{\psi}(\xi)\right|^2 \mathrm{d}\xi + P_J\\ &\le \|\widehat{f}\|_\infty^2\|\widehat{\psi}\|^2\sum_{j<-J}p^J + P_J\\ &= \frac{p^{-J}}{p-1}\|\widehat{f}\|_\infty^2\|\widehat{\psi}\|^2 + P_J,\end{aligned}$$

where

$$P_J = \sum_{j<-J} p^j \int_{\mathbb{R}_+}\overline{\widehat{f}(p^j\xi)}\widehat{\psi}(\xi)\sum_{m\in\mathbb{N}}\widehat{f}\left(p^j(\xi\oplus m)\right)\overline{\widehat{\psi}(\xi\oplus m)}.$$

Now we proceed to estimate P_J.

$$\begin{aligned}|P_J| &\le \frac{1}{2}\sum_{j<-J}p^j\sum_{m\in\mathbb{N}}\int_{\mathbb{R}_+}\left|\widehat{f}\left(p^j(\xi\oplus m)\right)\widehat{f}(p^j\xi)\right|\left(|\widehat{\psi}(\xi)|^2+|\widehat{\psi}(\xi\oplus m)|^2\right)\mathrm{d}\xi\\ &= \int_{\mathbb{R}_+}\sum_{j<-J}p^j\sum_{m\in\mathbb{N}}\left|\widehat{f}\left(p^j(\xi\oplus m)\right)\widehat{f}(p^j\xi)\right||\widehat{\psi}(\xi)|^2\,\mathrm{d}\xi.\end{aligned}$$

By Lemma 3.1,

$$\lim_{J\to\infty}\sum_{j<-J}p^j\sum_{m\in\mathbb{N}}\left|\widehat{f}\left(p^j(\xi\oplus m)\right)\widehat{f}(p^j\xi)\right| = 0$$

and the integral is dominated by $M\|\widehat{f}\|_\infty^2|\widehat{\psi}(\xi)|^2$ which belongs to $L^1(\mathbb{R}_+)$.

Therefore, by invoking the Lebesgue dominated convergence theorem we obtain $\lim_{J\to\infty}P_J = 0$ and by (3), we get

$$\lim_{j\to\infty}\sum_{j<-J}\sum_{k\in\mathbb{Z}_+}|\langle f,\psi_{j,k}\rangle|^2 = 0.$$

□

LEMMA 3.4 ([27, Lemma 3.5]). *Let $f,g\in\widetilde{D}$. Then*

$$\int_{\mathbb{R}_+}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}_+}|\widehat{f}(\xi\oplus p^jk)h_1(p^{-j}\xi\oplus k)\widehat{g}(\xi)h_2(p^{-j}\xi)|\mathrm{d}\xi = C_{h_1,h_2} < \infty$$

for $h_1,h_2\in L^2(\mathbb{R}_+)$.

LEMMA 3.5. *Assume that $\psi,\widetilde{\psi}\in L^2(\mathbb{R}_+,\mathbb{C}^L)$. Then*

$$\begin{aligned}&\sum_{k\in\mathbb{Z}_+}\langle f, D_{p^j}T_k\widetilde{\psi}\rangle\langle D_{p^j}T_k\widetilde{\psi}, g\rangle\\ &= \int_{\mathbb{R}_+}\left\{\sum_{k\in\mathbb{Z}_+}\sum_{\ell=1}^{L}\widehat{f}_\ell(\xi\oplus p^jk)\overline{\widehat{\widetilde{\psi}_\ell}(p^{-j}\xi\oplus k)}\right\}\left\{\sum_{\ell=1}^{L}\widehat{\psi_\ell}(p^{-j}\xi)\overline{\widehat{g_\ell}(\xi)}\right\}\mathrm{d}\xi\end{aligned}$$

for $f,g\in D$ and a fixed $j\in\mathbb{Z}$.

LEMMA 3.6. *Given $J \in \mathbb{N}$, assume that $\psi, \widetilde{\psi} \in L^2(\mathbb{R}_+, \mathbb{C}^L)$. Then*

$$\begin{aligned}
&\sum_{j<J}\sum_{k\in\mathbb{Z}_+}\langle f, D_{p^j}T_k\widetilde{\psi}\rangle\langle D_{p^j}T_k\psi, g\rangle \\
&= \int_{\mathbb{R}_+}\sum_{l\le \ell_1,\ell_2\le L}\widehat{f}_{\ell_1}(\xi)\overline{\widehat{g}_{\ell_2}(\xi)}\sum_{j>-J}\overline{\widehat{\widetilde{\psi}}_{\ell_1}(p^j\xi)}\widehat{\psi}_{\ell_2}(p^j\xi)\mathrm{d}\xi \\
&\quad + \int_{\mathbb{R}_+}\sum_{\gamma\in\mathbb{Z}_+\setminus q\mathbb{Z}_+}\sum_{j\in\mathbb{Z}}\sum_{l\le \ell_1,\ell_2\le L}\widehat{f}_{\ell_1}(\xi\oplus p^j\gamma)\overline{\widehat{g}_{\ell_2}(\xi)} \\
&\qquad \sum_{m=0}^{\infty}\overline{\widehat{\widetilde{\psi}}_{\ell_1}(p^m(p^{-j}\xi\oplus\gamma))}\widehat{\psi}_{\ell_2}(p^m.p^{-j}\xi)\mathrm{d}\xi,
\end{aligned}$$

for $f, g \in D$ and J sufficiently large.

Proof. Write

$$I_1(J) = \sum_{j<J}\int_{\mathbb{R}_+}\left\{\sum_{l=1}^{L}\widehat{f_{l_1}}(\xi)\overline{\widehat{\widetilde{\psi}}_l(p^{-j}\xi)}\right\}\left\{\sum_{l=1}^{L}\widehat{\psi}_l(p^{-j}\xi)\overline{\widehat{g_l}(\xi)}\right\}\mathrm{d}\xi,$$

and

$$\begin{aligned}
&I_2(J) \\
&= \sum_{j<J}\int_{\mathbb{R}_+}\left\{\sum_{k\in\mathbb{Z}_+}\sum_{l=1}^{L}\widehat{f_{l_1}}(p^j(\xi\oplus p^jk))\overline{\widehat{\widetilde{\psi_{l_1}}}(p^{-j}\xi\oplus k)}\right\}\left\{\sum_{l=1}^{L}\widehat{\psi}_l(p^{-j}\xi)\overline{\widehat{g_l}(\xi)}\right\}\mathrm{d}\xi.
\end{aligned}$$

Then

$$\sum_{j<J}\sum_{k\in\mathbb{Z}_+}\langle f, D_{p^j}T_k\widetilde{\psi}\rangle\langle D_{p^j}T_k\psi, g\rangle = I_1(J) + I_2(J) \tag{8}$$

Since $|\widehat{\widetilde{\psi_{\ell_1}}}(p^j\xi)\widehat{\widetilde{\psi_{\ell_2}}}(p^j\xi)| \le \frac{1}{2}\left(|\widehat{\widetilde{\psi_{\ell_1}}}(p^j\xi)|^2 + |\widehat{\widetilde{\psi_{\ell_2}}}(p^j\xi)|^2\right)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}_+}\sum_{j<J}\sum_{l\le \ell_1,\ell_2\le L}|\widehat{f_{\ell_1}}(\xi)\overline{\widehat{g_{\ell_2}}(\xi)}\overline{\widehat{\widetilde{\psi_{\ell_1}}}(p^j\xi)}\widehat{\psi_{\ell_2}}(p^j\xi)|\mathrm{d}\xi \\
&\le \sum_{l\le \ell_1,\ell_2\le L}\|\widehat{f_{\ell_1}}\|_\infty\|\widehat{g_{\ell_2}}\|_\infty\int_{\mathbb{R}_+}\sum_{j>-J}|\widehat{\widetilde{\psi_{\ell_1}}}(p^j\xi)\widehat{\psi_{\ell_2}}(p^j\xi)|\mathrm{d}\xi \\
&\le \sum_{l\le \ell_1,\ell_2\le L}\|\widehat{f_{\ell_1}}\|_\infty\|\widehat{g_{\ell_2}}\|_\infty\int_{\mathbb{R}_+}\sum_{j>-J}\left\{|\widehat{\widetilde{\psi_{\ell_1}}}(p^j\xi)|^2 + |\widehat{\psi_{\ell_2}}(p^j\xi)|^2\right\}\mathrm{d}\xi \\
&< \infty
\end{aligned}$$

by Lemma 3.2. Then we can rewrite $I_1(J)$ as

$$I_1(J) = \int_{\mathbb{R}_+}\sum_{l\le \ell_1,\ell_2\le L}\widehat{f_{\ell_1}}(\xi)\overline{\widehat{g_{\ell_2}}(\xi)}\sum_{j>-J}\overline{\widehat{\widetilde{\psi_{\ell_1}}}(p^j\xi)}\widehat{\psi_{\ell_2}}(p^j\xi)\mathrm{d}\xi. \tag{9}$$

Now we turn to $I_2(J)$. Observing that, for an arbitrarily fixed $j \in \mathbb{Z}$, there are at most finitely many $k \in \mathbb{Z}_+$ such that $\widehat{f}_{\ell_1}(\xi \oplus p^j k)\,\overline{\widehat{g}_{\ell_2}(\xi)} \neq 0$, we have

$$\begin{aligned} I_2(J) &= \sum_{j<J} \int_{\mathbb{R}_+} \sum_{k\in\mathbb{Z}_+} \sum_{l\leq \ell_1,\ell_2 \leq L} \widehat{f_{\ell_1}}(\xi \oplus p^j k)\overline{\widehat{g_{\ell_2}}(\xi)}\overline{\widehat{\widehat{\psi}_{\ell_1}}(p^{-j}\xi \oplus k)}\widehat{\psi_{\ell_2}}(p^{-j}\xi)\mathrm{d}\xi \\ &= \sum_{j<J} \int_{\mathbb{R}_+} \sum_{l\leq \ell_1,\ell_2 \leq L} \sum_{k\in\mathbb{Z}_+} \widehat{f_{\ell_1}}(\xi \oplus p^j k)\overline{\widehat{g_{\ell_2}}(\xi)}\overline{\widehat{\widehat{\psi}_{\ell_1}}(p^{-j}\xi \oplus k)}\widehat{\psi_{\ell_2}}(p^{-j}\xi)\mathrm{d}\xi. \end{aligned}$$

Since $f_\ell, g_\ell \in \widetilde{D}$ for $1 \leq \ell \leq L$, we have

$$\int_{\mathbb{R}_+} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}_+} |\widehat{f}_{\ell_1}(\xi \oplus p^j k)\overline{\widehat{\widehat{\psi}_{\ell_1}}(p^{-j}\xi \oplus k)}\widehat{g_{\ell_2}}(\xi)\widehat{\psi_{\ell_2}}(p^{-j}\xi)|\mathrm{d}\xi < \infty$$

for $1 \leq \ell_1, \ell_2 \leq L$ by Lemma 3.4, which implies that

$$\int_{\mathbb{R}_+} \sum_{j<J} \sum_{k\in\mathbb{Z}_+} |\widehat{f_{\ell_1}}(\xi \oplus p^j k)\overline{\widehat{\widehat{\psi}_{\ell_1}}(p^{-j}\xi \oplus k)}\widehat{g_{\ell_2}}(\xi)\widehat{\psi_{\ell_2}}(p^{-j}\xi)|\mathrm{d}\xi < \infty$$

for $1 \leq \ell_1, \ell_2 \leq L$. So

$$\begin{aligned} &I_2(J) \\ &= \int_{\mathbb{R}_+} \sum_{l\leq \ell_1,\ell_2\leq L} \sum_{\gamma\in\mathbb{Z}_+\setminus q\mathbb{Z}_+} \sum_{m=0}^{\infty} \sum_{j<J+m} \widehat{f_{\ell_1}}(\xi \oplus p^j\gamma) \\ &\qquad \overline{\widehat{\widehat{\psi}_{\ell_1}}(p^{m-j}\xi \oplus p^m\gamma)}\overline{\widehat{g_{\ell_2}}(\xi)}\widehat{\psi_{\ell_2}}(p^{m-j}\xi)\mathrm{d}\xi. \end{aligned} \tag{10}$$

The equality (10) is obtained by the fact that $\mathbb{Z}_+ = \bigcup_{m=0}^{\infty} p^m(\mathbb{Z}_+ \setminus p\mathbb{Z}_+)$. Since if J large enough, $\widehat{f_{l_1}}(\xi \oplus p^j\gamma)\overline{\widehat{g_{l_1}}(\xi)} = 0$ for all $j \geq J$ and $\gamma \in \mathbb{Z}_+ \setminus p\mathbb{Z}_+$. It leads to

$$\begin{aligned} &I_2(J) \\ &= \int_{\mathbb{R}_+} \sum_{l\leq \ell_1,\ell_2\leq L} \sum_{\gamma\in\mathbb{Z}_+\setminus q\mathbb{Z}_+} \sum_{m=0}^{\infty} \sum_{j\in\mathbb{Z}} \widehat{f_{\ell_1}}(\xi \oplus p^j\gamma) \\ &\qquad \overline{\widehat{\widehat{\psi}_{\ell_1}}(p^m(p^{-j}\xi \oplus \gamma))}\overline{\widehat{g_{\ell_2}}(\xi)}\widehat{\psi_{\ell_2}}(p^{m-j}\xi)\mathrm{d}\xi \\ &= \int_{\mathbb{R}_+} \sum_{\gamma\in\mathbb{Z}_+\setminus q\mathbb{Z}_+} \sum_{j\in\mathbb{Z}} \sum_{l\leq \ell_1,\ell_2\leq L} \widehat{f_{\ell_1}}(\xi \oplus p^j\gamma)\overline{\widehat{g_{l_2}}(\xi)} \\ &\qquad \sum_{m=0}^{\infty} \overline{\widehat{\widehat{\psi}_{l_1}}(p^m(p^{-j}\xi \oplus \gamma))}\widehat{\psi_{l_2}}(p^m.p^{-j}\xi)\mathrm{d}\xi \end{aligned} \tag{11}$$

for J sufficiently large. The proof is finished by (8), (9) and (11). $\square$

THEOREM 3.7. *Let $FL^2(\Omega,\mathbb{C}^L)$ be a reducing subspace of $L^2(\mathbb{R}_+,\mathbb{C}^L)$, and $\psi,\widetilde{\psi}\in FL^2(\Omega,\mathbb{C}^L)$. Then we have that $(A(\psi),A(\widetilde{\psi}))$ is a WABF associated with $D\cap FL^2(\Omega,\mathbb{C}^L)$ if and only if*

$$\lim_{j\to\infty}\sum_{j>-J}\overline{\widehat{\widetilde{\psi}}_{\ell_1}(p^j\xi)}\widehat{\psi}_{\ell_2}(p^j\xi)=\delta_{\ell_1,\ell_2}\chi_\Omega(\xi)\ \textit{for}\ 1\le\ell_1,\ell_2\le L \tag{12}$$

weakly in $L^1(\mathbb{R}_+)$, for all compact $K\subset\mathbb{R}_+\setminus\{0\}$, and

$$\sum_{j=0}^{\infty}\overline{\widehat{\widetilde{\psi}}_{\ell_1}(p^j(\xi\oplus\gamma))}\widehat{\psi}_{\ell_2}(p^j\xi)=0 \tag{13}$$

for $1\le\ell_1,\ell_2\le L,\gamma\in\mathbb{Z}_+\setminus p\mathbb{Z}_+$ and a.e. $\xi\in\Omega$.

Proof. It is easy to check that if $(A(\psi),A(\widetilde{\psi}))$ is a WABF associated with $D\cap FL^2(\Omega,\mathbb{C}^L)$, then $(A(\psi_\ell),A(\widetilde{\psi_\ell}))$ is a WABF associated with $\widetilde{D}\cap FL^2(\Omega)$ for every $1\le\ell\le L$. So, by Lemma 3.5, we may as well assume that (12) holds for $1\le\ell_1=\ell_2\le L$.

Next we prove the theorem under this assumption. By Lemma 3.3 and the Cauchy-Schwarz inequality, the series in (12) and (13) are absolutely convergent, and by Lemma 3.3, we have

$$\lim_{j\to\infty}\sum_{j<-J}\sum_{k\in\mathbb{Z}_+}\langle f,D_{p^j}T_k\widetilde{\psi}\rangle\langle D_{p^j}T_k\widetilde{\psi},g\rangle=0\ \text{for}\ f,g\in D.$$

Then $(A(\psi),A(\widetilde{\psi}))$ is a WABF associated with $D\cap FL^2(\Omega,\mathbb{C}^L)$ if and only if

$$\lim_{j\to\infty}\sum_{j<J}\sum_{k\in\mathbb{Z}_+}\langle f,D_{p^j}T_k\widetilde{\psi}\rangle\langle D_{p^j}T_k\widetilde{\psi},g\rangle=\langle f,g\rangle, \tag{14}$$

for $f,g\in D\cap FL^2(\Omega,\mathbb{C}^L)$.

By Lemma 3.8, we see that (14) is equivalent to

$$\lim_{j\to\infty}(I_1(J)+I_2(J))=\langle f,g\rangle\ \text{for}\ f,g\in D\cap FL^2(\Omega,\mathbb{C}^L), \tag{15}$$

where

$$I_1(J)=\int_{\mathbb{R}_+}\sum_{l\le\ell_1,\ell_2\le L}\widehat{f_{\ell_1}}(\xi)\overline{\widehat{g_{\ell_2}}(\xi)}\sum_{j>-J}\overline{\widehat{\widetilde{\psi}}_{\ell_1}(p^j\xi)}\widehat{\psi}_{\ell_2}(p^j\xi)\mathrm{d}\xi \tag{16}$$

and

$$\begin{aligned}I_2(J)=\int_{\mathbb{R}_+}\sum_{\gamma\in\mathbb{Z}_+\setminus p\mathbb{Z}_+}\sum_{j\in\mathbb{Z}}\sum_{l\le\ell_1,\ell_2\le L}\widehat{f}_{\ell_1}(\xi\oplus p^j\gamma)\overline{\widehat{g}_{\ell_2}(\xi)}&\\ \times\sum_{m=0}^{\infty}\overline{\widehat{\widetilde{\psi}}_{\ell_1}(p^m(p^{-j}\xi\oplus\gamma))}\widehat{\psi}_{\ell_2}(p^m.p^{-j}\xi)\mathrm{d}\xi,&\end{aligned} \tag{17}$$

for J large enough.

Now we prove (15) is equivalent to both (12) and (13) holding, to finish the proof. First suppose (12) and (13) hold. Then $I_2(J)=0$ for J large enough, and thus

$$\begin{aligned}\lim_{j\to\infty}(I_1(J)+I_2(J)) &= \lim_{j\to\infty} I_1(J)\\ &= \lim_{j\to\infty}\int_{\mathbb{R}_+}\sum_{\ell=1}^{L}\widehat{f_\ell}(\xi)\overline{\widehat{g_\ell}(\xi)}\sum_{j>-J}\overline{\widehat{\widehat{\psi_\ell}}(p^j\xi)}\widehat{\psi_\ell}(p^j\xi)\mathrm{d}\xi.\end{aligned}\tag{18}$$

Also observe that for every $1\le l\le L$, $\widehat{f_\ell}(\xi)\widehat{g_\ell}(\xi)\in L^\infty(K)$ with some compact $K\subset\mathbb{R}_+\setminus\{0\}$ if $f,g\in D\cap FL^2(\Omega,\mathbb{C}^L)$. It follows that

$$\lim_{j\to\infty}\int_R\widehat{f_\ell}(\xi)\overline{\widehat{g_\ell}(\xi)}\sum_{j>-J}\overline{\widehat{\widehat{\psi_\ell}}(p^j\xi)}\widehat{\psi_\ell}(p^j\xi)\mathrm{d}\xi=\int_{\mathbb{R}_+}\widehat{f_\ell}(\xi)\overline{\widehat{g_\ell}(\xi)}\mathrm{d}\xi$$

for $1\le\ell\le L$. So we have

$$\lim_{j\to\infty}(I_1(J)+I_2(J))=\int_{\mathbb{R}_+}\widehat{f_\ell}(\xi)\overline{\widehat{g_\ell}(\xi)}\mathrm{d}\xi=\langle f,g\rangle$$

by (18) and the Plancherel theorem. Therefore, (15) holds.

Next we prove the converse implication. Suppose (15) holds. First we prove (14) for $1\le\ell_1,\ell_2\le L$ with $\ell_1\ne\ell_2$.

Fix $\gamma_0\in\mathbb{Z}_+\setminus p\mathbb{Z}_+$ and $1\le\ell_1,\ell_2\le L$ with $\ell_1\ne\ell_2$. Define t_γ by

$$t_\gamma=\sum_{m=0}^{\infty}\overline{\widehat{\widehat{\psi_{\ell_1}}}(p^m(\xi\oplus\gamma))}\widehat{\psi_{\ell_2}}(p^m\xi),$$

for $\gamma\in\mathbb{Z}_+\setminus p\mathbb{Z}_+$.

By Lemma 3.2 and the Cauchy-Schwarz inequality, $t_\gamma\in L^1(\mathbb{R}_+)$ for each $\gamma\in\mathbb{Z}_+\setminus p\mathbb{Z}_+$. So almost every $\xi\in\mathbb{R}_+$ is a Lebesgue point of $t_\gamma(\xi)\chi_{\Omega\cap(\Omega-\gamma)}(\xi)$ for all $\gamma\in\mathbb{Z}_+\setminus p\mathbb{Z}_+$.

Arbitrarily fix such a point $\xi_0\ne0$. Without loss of generality, we assume that $\xi_0>0$. Take $f,g\in D\cap FL^2(\Omega,\mathbb{C}^L)$ in (7) such that $f_\ell(\xi)=0$ for $1\le\ell\le L$ with $\ell\ne\ell_1$, $g_\ell(\xi)=0$ for $1\le\ell\le L$ with $\ell\ne\ell_2$, and

$$\widehat{f}_{\ell_1}(\xi\oplus\gamma_0)=\widehat{g}_{\ell_2}(\xi)=\frac{1}{\sqrt{2\varepsilon}}\chi_{\xi_0+\varepsilon_1\xi_0+\varepsilon}(\xi)\chi_{\Omega\cap(\Omega+\gamma_0)}(\xi),$$

with $0<\varepsilon<\min\left\{\frac12,\frac{a-1}{a+1}\xi_0\right\}$.

Then we have

$$\begin{aligned}0=&I_2(J)=\int_{\mathbb{R}_+}\sum_{\gamma\in\mathbb{Z}_+\setminus p\mathbb{Z}_+}\sum_{j\in\mathbb{Z}}\widehat{f_{l_1}}(\xi\oplus p^j\gamma)\overline{\widehat{g_{l_2}}(\xi)}\\ &\sum_{m=0}^{\infty}\overline{\widehat{\widehat{\psi_{l_1}}}(p^m(p^{-j}\xi\oplus\gamma))}\widehat{\psi_{l_2}}(p^m.p^{-j}\xi)\mathrm{d}\xi=I_3(\varepsilon)+I_4(\varepsilon),\end{aligned}\tag{19}$$

for J large enough. We have

$$I_3(\varepsilon)=\frac{1}{2\varepsilon}\int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon}\chi_{\Omega\cap(\Omega-\gamma_0)}(\xi)t_{\gamma_0}(\xi)\mathrm{d}\xi,$$

$$I_4(\varepsilon)=\int_{\mathbb{R}_+}\sum_{(\gamma,j)\in((\mathbb{Z}_+\setminus p\mathbb{Z}_+)\times\mathbb{Z}_+)\setminus\{(\gamma_0,0)\}}p^j\widehat{f_{\ell_1}}(p^j(\xi\oplus\gamma))\overline{\widehat{g_{\ell_2}}(p^j\xi)}t_\gamma(\xi)\mathrm{d}\xi.$$

Observing that

$$|t_\gamma(\xi)|\le\frac{1}{2}\left[\sum_{m=0}^{\infty}|\widehat{\widetilde{\psi}}_{\ell_1}(p^m(\xi\oplus\gamma))|^2+\sum_{m=0}^{\infty}|\widehat{\psi}_{\ell_2}(p^m\xi)|^2\right],$$

we have

$$|I_4(\varepsilon)|\le I_5(\varepsilon)+I_6(\varepsilon),\tag{20}$$

$$I_5(\varepsilon)=\int_{\mathbb{R}_+}\sum_{(\gamma,j)\in((\mathbb{Z}_+\setminus p\mathbb{Z}_+)\times\mathbb{Z}_+)\setminus\{(\gamma_0,0)\}}p^j|\widehat{f_{\ell_1}}(p^j(\xi\oplus\gamma))\widehat{g_{\ell_2}}(p^j\xi)|\sum_{m=0}^{\infty}|\widehat{\psi_{\ell_2}}(p^m\xi)|^2\mathrm{d}\xi,$$

$$I_6(\varepsilon)=\int_{\mathbb{R}_+}\sum_{(\gamma,j)\in((\mathbb{Z}_+\setminus p\mathbb{Z}_+)\times\mathbb{Z}_+)\setminus\{(\gamma_0,0)\}}p^j|\widehat{f_{\ell_1}}(p^j(\xi\oplus\gamma))\widehat{g_{\ell_2}}(p^j\xi)|\sum_{m=0}^{\infty}|\widehat{\psi_{\ell_1}}(p^m\xi)|^2\mathrm{d}\xi.$$

Let us first estimate $I_5(\varepsilon)$. Our argument is borrowed from the proof of [27, Theorem 2.4]. But, for reader's convenience, we state it here.

If $j>0, |p^j\gamma-\gamma_0|\ge 1>2\varepsilon$. If $j<0$, take $j_0=\max\{j\in Z: a_j\le 2\varepsilon\}$, then $j_0<0$, and $|p^j\gamma-\gamma_0|=p^j\gamma-p^{-j}\gamma_0\le p^j>2\varepsilon$.

Thus $\widehat{f_{\ell_1}}(p^j(\xi\oplus\gamma))\widehat{g_{\ell_2}}(p^j\xi)=0$ for $\gamma\in\mathbb{Z}_+\setminus p\mathbb{Z}_+$ and $j_0<j\in\mathbb{Z}$ by the definition of $\widehat{f_{\ell_1}}$ and $\widehat{g_{\ell_2}}$. It follows that

$$\begin{aligned}I_5(\varepsilon)&\le\frac{1}{2\varepsilon}\sum_{j=-\infty}^{j_0}\sum_{\gamma\in\mathbb{Z}_+\setminus p\mathbb{Z}_+}p^j\int_{p^{-j}(\xi_0-\varepsilon)}^{p^{-j}(\xi_0+\varepsilon)}\chi_{(\xi_0-\varepsilon,\xi_0\oplus\varepsilon)}(p^j\xi\oplus p^j\gamma-\gamma_0)\\&\qquad\sum_{m=0}^{\infty}|\widehat{\psi_{\ell_2}}(p^m\xi)|^2\mathrm{d}\xi\\&=\frac{1}{2\varepsilon}\sum_{j=-\infty}^{j_0}p^j\int_{p^{-j}(\xi_0-\varepsilon)}^{p^{-j}(\xi_0+\varepsilon)}\sum_{\gamma\in\mathbb{Z}_+\setminus p\mathbb{Z}_+}\chi_{(\xi_0-\varepsilon,\xi_0\oplus\varepsilon)}(p^j\xi\oplus p^j\gamma-\gamma_0)\\&\qquad\sum_{m=0}^{\infty}|\widehat{\psi_{\ell_2}}(p^m\xi)|^2\mathrm{d}\xi.\end{aligned}$$

It is easy to check that the cardinality of the set

$$\left\{\gamma \in \mathbb{Z}_+ \setminus p\mathbb{Z}_+ : \begin{array}{r} \chi_{(\xi_0-\varepsilon,\xi_0+\varepsilon)}(p^j\xi + p^j(\gamma-\gamma_0)) \neq 0, \\ \text{for some } \xi \in (p^{-j}(\xi_0-\varepsilon), p^{-j}(\xi_0+\varepsilon)) \end{array}\right\}$$

is less than $8p^{-j}\varepsilon$. So we have

$$\begin{aligned} I_5(\varepsilon) &\leq 4\sum_{j=-\infty}^{j_0}\int_{p^{-j}(\xi_0-\varepsilon)}^{p^{-j}(\xi_0+\varepsilon)}\sum_{m=0}^{\infty}|\widehat{\psi_{\ell_2}}(p^m\xi)|^2 \mathrm{d}\xi \\ &\leq 4\int_{p^{-j_0}(\xi_0-\varepsilon)}^{\infty}\sum_{m=0}^{\infty}|\widehat{\psi_{\ell_2}}(p^m\xi)|^2 \mathrm{d}\xi, \end{aligned} \tag{21}$$

where in the last inequality we used the fact that $\big(p^{-j}(\xi_0-\varepsilon), p^{-j}(\xi_0+\varepsilon)\big)$, $j \in \mathbb{Z}$ are mutually disjoint when $\varepsilon < \frac{a-1}{a+1}\xi_0$.

Also observing that

$$\int_{\mathbb{R}_+}\sum_{m=0}^{\infty}|\widehat{\psi_{\ell_2}}(p^m\xi)|^2 \mathrm{d}\xi = \sum_{m=0}^{\infty}p^{-m}|\widehat{\psi_{\ell_2}}|^2 < \infty,$$

and that $p^{-j_0}(\xi_0-\varepsilon) \geq \frac{\xi_0}{2\varepsilon} - \frac{1}{2}$, we have $\lim_{\varepsilon\to 0} T_5(\varepsilon) = 0$ by (21).

For $I_6(\varepsilon)$, we have

$$\begin{aligned} &I_6(\varepsilon) \\ &= \int_{\mathbb{R}_+}\sum_{(\gamma,j)\in((\mathbb{Z}_+\setminus p\mathbb{Z}_+)\times\mathbb{Z}_+)\setminus\{\gamma_0,0\}} p^j|\widehat{f_{\ell_1}}(p^j\xi)\widehat{g_{\ell_2}}(p^j(\xi-\gamma))|\sum_{m=0}^{\infty}|\widehat{\widehat{\psi_{\ell_1}}}(p^m\xi)|^2 d\xi \end{aligned}$$

by a change of variables. Then, by an argument similar to above, we can prove that $\lim_{\varepsilon\to\infty} I_6(\varepsilon) = 0$. So we have $I_4(\varepsilon) = 0$ by (20), and thus $I_3(\varepsilon) = 0$. by (19), that is, $\chi_{\Omega\cap(\Omega-\gamma_0)}(\xi_0)t_{\gamma_0}(\xi_0) = 0$.

It follows that $\chi_{\Omega\cap(\Omega-\gamma)}(\xi)t_\gamma(\xi) = 0$ a.e. on $\mathbb{R}_+$ for $\gamma \in \mathbb{Z}_+ \setminus p\mathbb{Z}_+$ by the arbitrariness of ξ_0 and γ_0. This is equivalent to $t_\gamma(\xi) = 0$ for $\gamma \in \mathbb{Z}_+ \setminus p\mathbb{Z}_+$ since $\mathrm{supp}(t_\gamma) \subset \Omega \cap (\Omega - \gamma)$.

Now we prove (13) for $1 \leq \ell_1, \ell_2 \leq L$ with $\ell_1 \neq \ell_2$. For $h \in L^\infty(K)$ with compact $K \subset \mathbb{R}_+ \setminus \{0\}$, define $f, g \in D \cap FL^2(\Omega, \mathbb{C}^L)$ such that $f_\ell(\cdot) = 0$ for $1 \leq \ell \leq L$ with $l \neq \ell_1, g_\ell(\cdot) = 0$ for $1 \leq \ell \leq L$ with $\ell \neq \ell_2$, and

$$\widehat{f_{\ell_1}} = \frac{|h|^{\frac{1}{2}}}{\arg(h)}\chi_\Omega \text{ and } \widehat{g_{\ell_2}} = |h|^{\frac{1}{2}}\chi_\Omega,$$

where

$$\arg(z) = \begin{cases} \frac{|z|}{z}, & \text{if } z \neq 0 \\ 0, & \text{if} z = 0. \end{cases}$$

for $z \in \mathbb{C}$. Then

$$\langle f, g\rangle = \sum_{\ell=1}^{L}\int_{\mathbb{R}_+} f_\ell(x)\overline{g_\ell(x)}\mathrm{d}x = 0.$$

From (13) and (15)-(17), we deduce that

$$\lim_{j\to\infty}\int_{\mathbb{R}_+}\left\{\sum_{j>-J}\overline{\widehat{\widetilde{\psi}_{\ell_1}}(p^j\xi)}\widehat{\psi_{\ell_2}}(p^j\xi)\right\}h(\xi)\mathrm{d}\xi=0.$$

This implies (12) for $\ell_1 \neq \ell_2$ by the arbitrariness of h. The proof is completed. □

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