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ZERO-HOPF BIFURCATION OF PERIODIC ORBITS IN THE GENERALIZED RÖSSLER SYSTEM

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Abstract. We apply a technique of Llibre based on the averaging method to a Rössler-type system and we prove the existence of a periodic orbit. The system studied is a three-dimensional quadratic system and has the form

$$\begin{cases} \dot{x} = -y - z + kx, \\ \dot{y} = x + ay, \\ \dot{z} = bx - cz + xz, \end{cases}$$

where a, b, c and k are real arbitrary parameters.

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Key words. Rössler system, periodic solution, averaging theory, zero-Hopf bifurcation.

1. INTRODUCTION

The Rössler system

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x + ay, \\ \dot{z} = bx - cz + xz, \end{cases} \tag{1}$$

where a, b and c are real arbitrary parameters, was obtained by O. E. Rössler in [9].

In [6] J. Llibre studied the zero-Hopf bifurcation using averaging theory of first order, and applied it to the system (1). This method can be applied to any differential system in $\mathbb{R}^n$, $n \geq 3$. Recently, it was successfully applied for other interesting models (see for example [1, 2, 3, 4, 5, 7, 8] and references therein).

Our objective in this paper is to study the zero-Hopf bifurcation using averaging theory of first order, and apply it to the generalized Rössler system. We use the computer software MAPLE to perform the tedious computations.

2. LIMIT CYCLES VIA AVERAGING THEORY

The averaging theory of first order for studying periodic orbits can be found in Verhulst [10].

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Consider the differential equation

$$\dot{x} = \varepsilon F(t,x) + \varepsilon^2 G(t,x,\varepsilon), \quad x(0) = x_0, \tag{2}$$

with $x \in D$, where D is an open subset of $\mathbb{R}^n$, $t \geq 0$. Moreover, we assume that both $F(t,x)$ and $G(t,x,\varepsilon)$ are T-periodic in t. We also consider in D the averaged differential equation

$$\dot{y} = \varepsilon f(y), \quad y(0) = x_0, \tag{3}$$

where

$$f(y) = \frac{1}{T}\int_0^T F(t,y)\mathrm{d}t. \tag{4}$$

Under certain conditions, equilibrium solutions of the averaged equation (3) turn out to correspond with T-periodic solutions of equation (2).

THEOREM 2.1. *Consider the two initial value problems* (2) *and* (3). *Suppose*

(i) *F, its Jacobian $\dfrac{\partial F}{\partial x}$, its Hessian $\dfrac{\partial^2 F}{\partial x^2}$, G and its Jacobian $\dfrac{\partial G}{\partial x}$ are defined, continuous and bounded by a constant independent of ε in $[0,\infty[\times D$ and $\varepsilon \in]0,\varepsilon_0]$.*

(ii) *F and G are T-periodic in t (T independent of ε).*

Then the following statements hold:

(a) *If p is an equilibrium point of the averaged equation* (3) *and*

$$\det\left(\frac{\partial f}{\partial y}\right)\bigg|_{y=p} \neq 0, \tag{5}$$

then there exists a T-periodic solution $\varphi(t,\varepsilon)$ of equation (2) *such that*

$$\varphi(0,\varepsilon) \to p \text{ as } \varepsilon \to 0.$$

(b) *The stability or instability of the limit cycle $\varphi(t,\varepsilon)$ is given by the stability or instability of the equilibrium point p of the averaged system* (3). *In fact the singular point p has the stability behavior of the Poincaré map associated to the limit cycle $\varphi(t,\varepsilon)$.*

Our objective in this paper, is to study analytically the periodic solutions of the zero-Hopf bifurcation for the generalized Rössler differential system.

3. MAIN RESULTS

In this paper, using Llibre method [6], we study the zero-Hopf bifurcation for the generalized Rössler system in $\mathbb{R}^3$:

$$(S) \qquad \begin{cases} \dot{x} = -y - z + kx, \\ \dot{y} = x + ay, \\ \dot{z} = bx - cz + xz, \end{cases}$$

where a, b, c and k are real arbitrary parameters. System (S) possesses the equilibrium points $p_1 = (0,0,0)$ and

$$p_2 = \left(\frac{c+ack-ab}{1+ak}, -\frac{c+ack-ab}{a(1+ak)}, \frac{c+ack-ab}{a}\right),$$

with $c+ack-ab \neq 0$ and $a(1+ak) \neq 0$.

PROPOSITION 3.1. *There is one two-parameter families of system (S) for which the origin of coordinates is a zero-Hopf equilibrium point. Namely:*

$$c = a+k \ , \ b = \frac{(a+k)(1+ak)}{a} \quad , \ \frac{-a^3+2a+k}{a} > 0 \ \textit{and} \ a \neq 0.$$

The next result gives sufficient conditions for the bifurcation of a limit cycle from the origin when it is a zero-Hopf equilibrium.

THEOREM 3.2. *Let*

$$(a,b,c,k) = \left(\overline{a}+\varepsilon\alpha, \frac{(a+k)(1+ak)}{a} + \varepsilon\beta, a+k+\varepsilon\gamma, aw^2+a^3-2a+\varepsilon\delta\right)$$

be with $a \neq 0, w \neq 0$ and ε a sufficiently small parameter. Assume that

$$\overline{a}(\overline{a}^2+w^2-1)(2\overline{a}^4+(2w^2-4)\overline{a}^2-w^2+2) > 0 \quad \textit{and} \quad \Gamma > 0 \tag{6}$$

with

$$\begin{aligned}\Gamma \ = \ & (\overline{a}^8\gamma+(2w^2-4)\gamma\overline{a}^6-\beta\overline{a}^5+\gamma(w^4-4w^2+6)\overline{a}^4 \\ & +(-\beta w^2+2\beta)\overline{a}^3+(2w^2-4)\gamma\overline{a}^2+(\beta w^2-\beta)\overline{a}+\gamma) \\ & (\overline{a}^4\gamma+\gamma(w^2-2)\overline{a}^2-\beta\overline{a}+(-w^2+1)\gamma).\end{aligned}$$

Then, the system (S) has a zero-Hopf bifurcation at the equilibrium point localized at the origin of coordinates, and a periodic orbit exists at this equilibrium when $\varepsilon = 0$, and it exists for $\varepsilon > 0$ sufficiently small. Moreover, the stability or instability of this periodic orbit is given by the eigenvalues

$$\frac{A \pm \sqrt{B}}{4w((\overline{a}^2-\frac{1}{2})w^2+(\overline{a}^2-1)^2)}, \tag{7}$$

where $A = (-\gamma\overline{a}^2w^2+(-\overline{a}^4\gamma+2\gamma\overline{a}^2-\beta\overline{a}-\gamma))w^2$ and

$$\begin{aligned}B = & -8\overline{a}^4\gamma^2\left(\overline{a}^4-\frac{3}{2}\overline{a}^2+\frac{3}{8}\right)w^8-32\overline{a}\left(\overline{a}\left(\overline{a}^4-\frac{9}{8}\overline{a}^2+\frac{3}{16}\right)(\overline{a}+1)^2\right. \\ & \left.(\overline{a}-1)^2\gamma-\frac{\beta}{2}\left(\overline{a}^6-\frac{3}{2}\overline{a}^4+\frac{9}{8}\overline{a}^2-\frac{1}{4}\right)\right)\gamma w^6+((-48\overline{a}^{12}+228\overline{a}^{10} \\ & -435\overline{a}^8+420\overline{a}^6-210\overline{a}^4+48\overline{a}^2-3)\gamma^2+48\beta\overline{a}(\overline{a}+1)^2(\overline{a}-1)^2 \\ & \gamma\left(\overline{a}^4-\overline{a}^2+\frac{3}{8}\right)-8\beta^2\overline{a}^2\left(\overline{a}^4-\frac{3}{2}\overline{a}^2+\frac{3}{8}\right)\Big)w^4-32(\gamma(\overline{a}^4-2\overline{a}^2+1) \\ & -\beta\overline{a})(\overline{a}+1)^2\left(\left(\overline{a}-\frac{3}{8}\right)(\overline{a}+1)^2(\overline{a}-1)^2\gamma-\frac{\beta\overline{a}}{2}\left(\overline{a}^2-\frac{3}{4}\right)\right)\end{aligned}$$

$$(\overline{a}-1)^2w^2-8((\overline{a}^4-2\overline{a}^2+1)\gamma-\beta\overline{a})^2(\overline{a}+1)^4(\overline{a}-1)^4.$$

Our main results, Proposition 3.1 and Theorem 3.2, are proved in the next section. Note that we can obtain similar results for the equilibrium p_2.

4. PROOFS

Proof of Proposition 3.1. The Jacobian of system (S) at the origin p_1 is

$$\begin{pmatrix} k & -1 & -1 \\ 1 & a & 0 \\ b & 0 & -c \end{pmatrix}.$$

The characteristic polynomial of the linear part of system (S) at p_1 is

$$p(\lambda)=-\lambda^3+(a+k-c)\lambda^2+(ac+ck-ak-1-b)\lambda+ab-c-ack.$$

In order to study the zero-Hopf bifurcation we force that

$$p(\lambda)=-\lambda(\lambda^2+w^2).$$

This occurs if and only if we have

$$\begin{cases} a+k-c=0, \\ ac+ck-ak-1-b+w^2=0, \\ ab-c-ack=0. \end{cases}$$

We obtain

(i) $c=a+k$,

(ii) $b=\dfrac{(a+k)(1+ak)}{a}$

(iii) $\dfrac{-a^3+2a+k}{a}=w^2$ and $a\neq 0$.

This completes the proof of Proposition 3.1. □

Proof of Theorem 3.2. If

$$(a,b,c,k)=(\overline{a}+\varepsilon\alpha,\frac{(a+k)(1+ak)}{a}+\varepsilon\beta,a+k+\varepsilon\gamma,aw^2+a^3-2a+\varepsilon\delta)$$

be with $k\in\mathbb{R}$, $a\neq 0$ and ε is a sufficiently small parameter, then the generalized Rössler system (S) becomes

$$(8)\quad \begin{cases} \dot{x}=-y-z+(\overline{a}^3+(w^2-2)\overline{a}+\varepsilon(\alpha w^2+3\overline{a}^2\alpha+3\overline{a}\varepsilon\alpha^2 \\ \quad +\varepsilon^2\alpha^3-2\alpha+\delta))x, \\ \dot{y}=x+(\overline{a}+\varepsilon\alpha)y, \\ \dot{z}=(\dfrac{(\overline{a}^3+(w^2-1)\overline{a}+\varepsilon(\alpha w^2+3\overline{a}^2\alpha+3\overline{a}\varepsilon\alpha^2+\varepsilon^2\alpha^3-\alpha}{\overline{a}+\varepsilon\alpha} \\ \quad +\dfrac{\delta))(1+(\overline{a}+\varepsilon\alpha))(\overline{a}^3+(w^2-2)\overline{a}+\varepsilon(\alpha w^2+3\overline{a}^2\alpha}{\overline{a}+\varepsilon\alpha} \\ \quad +\dfrac{3\overline{a}\varepsilon\alpha^2+\varepsilon^2\alpha^3-2\alpha+\delta))}{\overline{a}+\varepsilon\alpha}+\varepsilon\beta)x-(\overline{a}^3+(w^2-1)\overline{a} \\ \quad +\varepsilon(\alpha w^2+3\overline{a}^2\alpha+3\overline{a}\varepsilon\alpha^2+\varepsilon^2\alpha^3-\alpha+\delta+\gamma))z+xz. \end{cases}$$

By scaling the variables $(x, y, z) = (\varepsilon X, \varepsilon Y, \varepsilon Z)$, system (8) in the new variables (X, Y, Z) writes

$$(9)\quad \begin{cases} \dot{X} = -Y - Z + (\overline{a}^3 + (w^2-2)\overline{a})X + \varepsilon(\alpha w^2 + 3\overline{a}^2\alpha + 3\overline{a}\varepsilon\alpha^2 \\ \quad + \varepsilon^2\alpha^3 - 2\alpha + \delta)X, \\ \dot{Y} = X + \overline{a}Y + \varepsilon\alpha Y, \\ \dot{Z} = (\dfrac{(\overline{a}^3 + (w^2-1)\overline{a} + \varepsilon(\alpha w^2 + 3\overline{a}^2\alpha + 3\overline{a}\varepsilon\alpha^2 + \varepsilon^2\alpha^3}{\overline{a} + \varepsilon\alpha} \\ \quad + \dfrac{-\alpha + \delta))(1 + (\overline{a} + \varepsilon\alpha)(\overline{a}^3 + (w^2-2)\overline{a} + \varepsilon(\alpha w^2 + 3\overline{a}^2\alpha}{\overline{a} + \varepsilon\alpha} \\ \quad + \dfrac{3\overline{a}\varepsilon\alpha^2 + \varepsilon^2\alpha^3 - 2\alpha + \delta))}{\overline{a} + \varepsilon\alpha} + \varepsilon\beta)X - (\overline{a}^3 + (w^2-1)\overline{a} \\ \quad + \varepsilon(\alpha w^2 + 3\overline{a}^2\alpha + 3\overline{a}\varepsilon\alpha^2 + \varepsilon^2\alpha^3 - \alpha + \delta + \gamma))Z + \varepsilon XZ. \end{cases}$$

We need to write the linear part of system (9) at the equilibrium point $(0, 0, 0)$ when $\varepsilon = 0$ in its real Jordan normal form as $\begin{pmatrix} 0 & -w & 0 \\ w & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. In order to facilitate the application of the averaging theory, given in Theorem 2.1, we perform the change of variables $(X, Y, Z) \to (u, v, s)$, given by $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = B^{-1}\begin{pmatrix} u \\ v \\ s \end{pmatrix}$, where

$$B^{-1} = \begin{pmatrix} \dfrac{-1}{w} & \dfrac{\overline{a}}{w^2} & \dfrac{\overline{a}}{w^2} \\ 0 & \dfrac{-1}{w^2} & \dfrac{-1}{w^2} \\ \dfrac{-\overline{a}(-1 + w^2 + \overline{a}^2)}{w} & \dfrac{-2\overline{a}^2 + \overline{a}^2 w^2 + \overline{a}^4 + 1 - w^2}{w^2} & \dfrac{-2\overline{a}^2 + \overline{a}^2 w^2 + \overline{a}^4 + 1}{w^2} \end{pmatrix}$$

In the new variables (u, v, s) the differential system (9) writes

$$(10)\quad \begin{aligned} \dot{u} = &-w((\overline{a}w^2 + w^2\varepsilon\alpha + \overline{a}^3 + 3\overline{a}^2\varepsilon\alpha + 3\overline{a}\varepsilon^2\alpha^2 + \varepsilon^3\alpha^3 - 2\overline{a} - 2\varepsilon\alpha \\ &+ \varepsilon\delta)(\frac{-u}{w} + \overline{a}\frac{v}{w^2} + \overline{a}\frac{s}{w^2}) + \frac{v}{w^2} + \frac{s}{w^2} + \overline{a}(-1 + w^2 + \overline{a}^2)\frac{u}{w} \\ &- (-2\overline{a}^2 + w^2\overline{a}^2 + \overline{a}^4 + 1 - w^2)\frac{v}{w^2} - (-2\overline{a}^2 + w^2\overline{a}^2 + \overline{a}^4 + 1)\frac{s}{w^2}) \\ &- \overline{a}w(\frac{-u}{w} + \overline{a}\frac{v}{w^2} + \overline{a}\frac{s}{w^2} + \overline{a}(-\frac{v}{w^2} - \frac{s}{w^2}) + \varepsilon\alpha(-\frac{v}{w^2} - \frac{s}{w^2})), \end{aligned}$$

$$
\begin{aligned}
\dot{v} = {} & (\overline{a}^3 - \overline{a} + \overline{a}w^2)((\overline{a}w^2 + w^2\varepsilon\alpha + \overline{a}^3 + 3\overline{a}^2\varepsilon\alpha + 3\overline{a}\varepsilon^2\alpha^2 + \varepsilon^3\alpha^3 \\
& - 2\overline{a} - 2\varepsilon\alpha + \varepsilon\delta)(\frac{-u}{w} + \overline{a}\frac{v}{w^2} + \overline{a}\frac{s}{w^2}) + \frac{v}{w^2} + \frac{s}{w^2} \\
& + \overline{a}(-1 + w^2 + \overline{a}^2)\frac{u}{w} - (-2\overline{a}^2 + w^2\overline{a}^2 + \overline{a}^4 + 1 - w^2)\frac{v}{w^2} \\
& - (-2\overline{a}^2 + w^2\overline{a}^2 + \overline{a}^4 + 1)\frac{s}{w^2}) + (\overline{a}^2 - 1)(\frac{-u}{w} \\
& + \overline{a}\frac{v}{w^2} + \overline{a}\frac{s}{w^2} - \overline{a}(\frac{v}{w^2} + \frac{s}{w^2}) - \varepsilon\alpha(\frac{v}{w^2} + \frac{s}{w^2})) - ((-\overline{a} - \varepsilon\alpha \\
& + \overline{a}w^2 + w^2\varepsilon\alpha + \overline{a}^3 + 3\overline{a}^2\varepsilon\alpha + 3\overline{a}\varepsilon^2\alpha^2 + \varepsilon^3\alpha^3 + \varepsilon\delta)(1 + w^2\overline{a}^2 \\
& + 2\overline{a}w^2\varepsilon\alpha + \overline{a}^4 + 4\overline{a}^3\varepsilon\alpha + 6\overline{a}^2\varepsilon^2\alpha^2 + 4\overline{a}\varepsilon^3\alpha^3 - 2\overline{a}^2 \\
& - 4\overline{a}\varepsilon\alpha + \overline{a}\varepsilon\delta + \varepsilon^2\alpha^2w^2 + \varepsilon^4\alpha^4 - 2\varepsilon^2\alpha^2 + \varepsilon^2\alpha\delta) \\
& \frac{1}{\overline{a} + \varepsilon\alpha} + \varepsilon\beta)(\frac{-u}{w} + \overline{a}\frac{v}{w^2} + \overline{a}\frac{s}{w^2}) + (-\overline{a} - \varepsilon\alpha + \overline{a}w^2 \\
& + w^2\varepsilon\alpha + \overline{a}^3 + 3\overline{a}^2\varepsilon\alpha + 3\overline{a}\varepsilon^2\alpha^2 + \varepsilon^3\alpha^3 + \varepsilon\delta \\
& + \varepsilon\gamma)(-\overline{a}(-1 + w^2 + \overline{a}^2)\frac{u}{w} + (-2\overline{a}^2 + w^2\overline{a}^2 \\
& + \overline{a}^4 + 1 - w^2)\frac{v}{w^2} + (-2\overline{a}^2 + w^2\overline{a}^2 + \overline{a}^4 + 1)\frac{s}{w^2}) \\
& - \varepsilon(\frac{-u}{w} + \overline{a}\frac{v}{w^2} + \overline{a}\frac{s}{w^2})(-\overline{a}(-1 + w^2 + \overline{a}^2)\frac{u}{w} \\
& + (-2\overline{a}^2 + w^2\overline{a}^2 + \overline{a}^4 + 1 - w^2)\frac{v}{w^2} + (-2\overline{a}^2 + w^2\overline{a}^2 \\
& + \overline{a}^4 + 1)\frac{s}{w^2}),
\end{aligned}
$$

$$
\begin{aligned}
\dot{s} = {} & -\overline{a}(-1 + w^2 + \overline{a}^2)((\overline{a}w^2 + w^2\varepsilon\alpha + \overline{a}^3 + 3\overline{a}^2\varepsilon\alpha + 3\overline{a}\varepsilon^2\alpha^2 \\
& + \varepsilon^3\alpha^3 - 2\overline{a} - 2\varepsilon\alpha + \varepsilon\delta)(\frac{-u}{w} + \overline{a}\frac{v}{w^2} + \overline{a}\frac{s}{w^2}) + \frac{v}{w^2} + \frac{s}{w^2} \\
& + \overline{a}(-1 + w^2 + \overline{a}^2)\frac{u}{w} - (-2\overline{a}^2 + w^2\overline{a}^2 + \overline{a}^4 + 1 - w^2)\frac{v}{w^2} \\
& - (-2\overline{a}^2 + w^2\overline{a}^2 + \overline{a}^4 + 1)\frac{s}{w^2}) + (1 - w^2 - \overline{a}^2)(\frac{-u}{w} + \overline{a}\frac{v}{w^2} \\
& + \overline{a}\frac{s}{w^2} - \overline{a}(\frac{v}{w^2} + \frac{s}{w^2}) - \varepsilon\alpha(\frac{v}{w^2} + \frac{s}{w^2})) + ((-\overline{a} - \varepsilon\alpha \\
& + \overline{a}w^2 + w^2\varepsilon\alpha + \overline{a}^3 + 3\overline{a}^2\varepsilon\alpha + 3\overline{a}\varepsilon^2\alpha^2 + \varepsilon^3\alpha^3 + \varepsilon\delta)(1 + w^2\overline{a}^2 \\
& + 2\overline{a}w^2\varepsilon\alpha + \overline{a}^4 + 4\overline{a}^3\varepsilon\alpha + 6\overline{a}^2\varepsilon^2\alpha^2 + 4\overline{a}\varepsilon^3\alpha^3 \\
& - 2\overline{a}^2 - 4\overline{a}\varepsilon\alpha + \overline{a}\varepsilon\delta + \varepsilon^2\alpha^2w^2 + \varepsilon^4\alpha^4 - 2\varepsilon^2\alpha^2 + \varepsilon^2\alpha\delta)\frac{1}{\overline{a} + \varepsilon\alpha} \\
& + \varepsilon\beta)(\frac{-u}{w} + \overline{a}\frac{v}{w^2} + \overline{a}\frac{s}{w^2}) - (-\overline{a} - \varepsilon\alpha + \overline{a}w^2 + w^2\varepsilon\alpha + \overline{a}^3
\end{aligned}
$$

$$
\begin{aligned}
&+3\overline{a}^2\varepsilon\alpha+3\overline{a}\varepsilon^2\alpha^2+\varepsilon^3\alpha^3+\varepsilon\delta+\varepsilon\gamma)(-\overline{a}(-1+w^2+\overline{a}^2)\frac{u}{w}\\
&+(-2\overline{a}^2+w^2\overline{a}^2+\overline{a}^4+1-w^2)\frac{v}{w^2}+(-2\overline{a}^2+w^2\overline{a}^2+\overline{a}^4\\
&+1)\frac{s}{w^2}+\varepsilon(\frac{-u}{w}+\overline{a}\frac{v}{w^2}+\overline{a}\frac{s}{w^2})(-\overline{a}(-1+w^2+\overline{a}^2)\frac{u}{w}+(-2\overline{a}^2\\
&+w^2\overline{a}^2+\overline{a}^4+1-w^2)\frac{v}{w^2}+(-2\overline{a}^2+w^2\overline{a}^2+\overline{a}^4+1)\frac{s}{w^2})).
\end{aligned}
$$

Consider the cylindrical coordinates (r,θ,s) defined by $u=r\cos\theta, v=r\sin\theta$ and $s=s$. Then, the differential system (10) becomes

$$
\begin{aligned}
\dot{r}=&\frac{1}{r}\Big(\frac{r\cos(\theta)}{w}(-r\sin(\theta)w^2-w^2\varepsilon\alpha\overline{a}r\sin(\theta)+3\overline{a}^2\varepsilon\alpha r\cos(\theta)w\\
&+3\overline{a}\varepsilon^2\alpha^2r\cos(\theta)w-3\overline{a}^3\varepsilon\alpha s-3\overline{a}^2\varepsilon^2\alpha^2s-\overline{a}\varepsilon^3\alpha^3s+3\overline{a}\varepsilon\alpha s\\
&-\overline{a}\varepsilon\delta s+w^3\varepsilon\alpha r\cos(\theta)-w^2\varepsilon\alpha\overline{a}s-3\overline{a}^3\varepsilon\alpha r\sin(\theta)-3\overline{a}^2\varepsilon^2\alpha^2r\sin(\theta)\\
&+\varepsilon^3\alpha^3rw\cos(\theta)-\overline{a}\varepsilon^3\alpha^3r\sin(\theta)-2\varepsilon\alpha rw\cos(\theta)+3\varepsilon\alpha\overline{a}r\sin(\theta)\\
&+\varepsilon\delta rw\cos(\theta)-\varepsilon\delta\overline{a}r\sin(\theta))+r\sin(\theta)((\overline{a}^3-\overline{a}+\overline{a}w^2)((\overline{a}w^2+w^2\varepsilon\alpha\\
&+\overline{a}^3+3\overline{a}^2\varepsilon\alpha+3\overline{a}\varepsilon^2\alpha^2+\varepsilon^3\alpha^3-2\overline{a}-2\varepsilon\alpha+\varepsilon\delta)(\frac{-r\cos(\theta)}{w}\\
&+\frac{\overline{a}r\sin(\theta)}{w^2}+\frac{\overline{a}s}{w^2})+\frac{r\sin(\theta)}{w^2}+\frac{s}{w^2}+\overline{a}(-1+w^2+\overline{a}^2)r\frac{\cos(\theta)}{w}\\
&-(-2\overline{a}^2+w^2\overline{a}^2+\overline{a}^4+1-w^2)r\frac{\sin(\theta)}{w^2}-(-2\overline{a}^2+w^2\overline{a}^2+\overline{a}^4+1)\frac{s}{w^2})\\
&+(\overline{a}^2-1)(\frac{-r\cos(\theta)}{w}+\frac{\overline{a}r\sin(\theta)}{w^2}+\overline{a}\frac{s}{w^2}+c(-r\frac{\sin(\theta)}{w^2}-\frac{s}{w^2})\\
&+\varepsilon\alpha(-r\frac{\sin(\theta)}{w^2}-\frac{s}{w^2}))-((-\overline{a}-\varepsilon\alpha+\overline{a}w^2+w^2\varepsilon\alpha+\overline{a}^3\\
&+3\overline{a}^2\varepsilon\alpha+3\overline{a}\varepsilon^2\alpha^2+\varepsilon^3\alpha^3+\varepsilon\delta)(1+w^2\overline{a}^2+2\overline{a}w^2\varepsilon\alpha+\overline{a}^4\\
&+4\overline{a}^3\varepsilon\alpha+6\overline{a}^2\varepsilon^2\alpha^2+4\overline{a}\varepsilon^3\alpha^3-2\overline{a}^2-4\overline{a}\varepsilon\alpha+\overline{a}\varepsilon\delta+\varepsilon^2\alpha^2w^2\\
&+\varepsilon^4\alpha^4-2\varepsilon^2\alpha^2+\varepsilon^2\alpha\delta)\frac{1}{\overline{a}+\varepsilon\alpha}+\varepsilon\beta)(\frac{-r\cos(\theta)}{w}+\frac{\overline{a}r\sin(\theta)}{w^2}+\frac{\overline{a}s}{w^2})\\
&+(-\overline{a}-\varepsilon\alpha+\overline{a}w^2+w^2\varepsilon\alpha+\overline{a}^3+3\overline{a}^2\varepsilon\alpha+3\overline{a}\varepsilon^2\alpha^2+\varepsilon^3\alpha^3+\varepsilon\delta+\varepsilon\gamma)\\
&(-\overline{a}(-1+w^2+\overline{a}^2)\frac{r\cos(\theta)}{w}+(-2\overline{a}^2+w^2\overline{a}^2+\overline{a}^4+1-w^2)\frac{r\sin(\theta)}{w^2}\\
&+(-2\overline{a}^2+w^2\overline{a}^2+\overline{a}^4+1)\frac{s}{w^2})-\varepsilon(\frac{-r\cos(\theta)}{w}+\frac{\overline{a}r\sin(\theta)}{w^2}+\frac{\overline{a}s}{w^2})\\
&(-\overline{a}(-1+w^2+\overline{a}^2)\frac{r\cos(\theta)}{w}+(-2\overline{a}^2+w^2\overline{a}^2+\overline{a}^4+1-w^2)\frac{r\sin(\theta)}{w^2}\\
&+(-2\overline{a}^2+w^2\overline{a}^2+\overline{a}^4+1)\frac{s}{w^2}))\Big),
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\dot{\theta} &= \frac{1}{r^2}(r\cos(\theta)((\overline{a}^3 - \overline{a} + \overline{a}w^2)((\overline{a}w^2 + w^2\varepsilon\alpha + \overline{a}^3 + 3\overline{a}^2\varepsilon\alpha \\
&+ 3\overline{a}\varepsilon^2\alpha^2 + \varepsilon^3\alpha^3 - 2\overline{a} - 2\varepsilon\alpha + \varepsilon\delta)(\frac{-r\cos(\theta)}{w} + \frac{\overline{a}r\sin(\theta)}{w^2} + \frac{\overline{a}s}{w^2}) \\
&+ \frac{r\sin(\theta)}{w^2} + \frac{s}{w^2} + \overline{a}(-1 + w^2 + \overline{a}^2)r\frac{\cos(\theta)}{w} - (-2\overline{a}^2 + w^2\overline{a}^2 \\
&+ \overline{a}^4 + 1 - w^2)r\frac{\sin(\theta)}{w^2} - (-2\overline{a}^2 + w^2\overline{a}^2 + \overline{a}^4 + 1)\frac{s}{w^2}) \\
&+ (\overline{a}^2 - 1)(\frac{-r\cos(\theta)}{w} + \frac{\overline{a}r\sin(\theta)}{w^2} + \overline{a}\frac{s}{w^2} + c(-r\frac{\sin(\theta)}{w^2} - \frac{s}{w^2}) \\
&+ \varepsilon\alpha(-r\frac{\sin(\theta)}{w^2} - \frac{s}{w^2})) - ((-\overline{a} - \varepsilon\alpha + \overline{a}w^2 + w^2\varepsilon\alpha + \overline{a}^3 \\
&+ 3\overline{a}^2\varepsilon\alpha + 3\overline{a}\varepsilon^2\alpha^2 + \varepsilon^3\alpha^3 + \varepsilon\delta)(1 + w^2\overline{a}^2 + 2\overline{a}w^2\varepsilon\alpha + \overline{a}^4 \\
&+ 4\overline{a}^3\varepsilon\alpha + 6\overline{a}^2\varepsilon^2\alpha^2 + 4\overline{a}\varepsilon^3\alpha^3 - 2\overline{a}^2 - 4\overline{a}\varepsilon\alpha + \overline{a}\varepsilon\delta + \varepsilon^2\alpha^2w^2 \\
&+ \varepsilon^4\alpha^4 - 2\varepsilon^2\alpha^2 + \varepsilon^2\alpha\delta)\frac{1}{\overline{a} + \varepsilon\alpha} + \varepsilon\beta)(\frac{-r\cos(\theta)}{w} \\
&+ \frac{\overline{a}r\sin(\theta)}{w^2} + \frac{\overline{a}s}{w^2}) + (-\overline{a} - \varepsilon\alpha + \overline{a}w^2 + w^2\varepsilon\alpha + \overline{a}^3 \\
&+ 3\overline{a}^2\varepsilon\alpha + 3\overline{a}\varepsilon^2\alpha^2 + \varepsilon^3\alpha^3 + \varepsilon\delta + \varepsilon\gamma)(-\overline{a}(-1 + w^2 + \overline{a}^2)\frac{r\cos(\theta)}{w} \\
&+ (-2\overline{a}^2 + w^2\overline{a}^2 + \overline{a}^4 + 1 - w^2)\frac{r\sin(\theta)}{w^2} + (-2\overline{a}^2 + w^2\overline{a}^2 + \overline{a}^4 + 1)\frac{s}{w^2}) \\
&- \varepsilon(\frac{-rcos(\theta)}{w} + \frac{\overline{a}r\sin(\theta)}{w^2} + \frac{\overline{a}s}{w^2})(-\overline{a}(-1 + w^2 + \overline{a}^2)\frac{r\cos(\theta)}{w} \\
&+ (-2\overline{a}^2 + w^2\overline{a}^2 + \overline{a}^4 + 1 - w^2)\frac{r\sin(\theta)}{w^2} + (-2\overline{a}^2 + w^2\overline{a}^2 + \overline{a}^4 + 1)\frac{s}{w^2})) \\
&- \frac{1}{w}(r\sin(\theta)(-r\sin(\theta)w^2 - w^2\varepsilon\alpha\overline{a}r\sin(\theta) + 3\overline{a}^2\varepsilon\alpha rw\cos(\theta) \\
&+ 3\overline{a}\varepsilon^2\alpha^2rw\cos(\theta) - 3\overline{a}^3\varepsilon\alpha s - 3\overline{a}^2\varepsilon^2\alpha^2s - \varepsilon^3\alpha^3\overline{a}s + 3\varepsilon\alpha\overline{a}s - \varepsilon\delta\overline{a}s \\
&+ w^3\varepsilon\alpha r\cos(\theta) - w^2\varepsilon\alpha\overline{a}s - 3\overline{a}^3\varepsilon\alpha r\sin(\theta) - 3\overline{a}^2\varepsilon^2\alpha^2r\sin(\theta) \\
&+ \varepsilon^3\alpha^3rw\cos(\theta) - \varepsilon^3\alpha^3\overline{a}r\sin(\theta) - 2\varepsilon\alpha rw\cos(\theta) + 3\varepsilon\alpha\overline{a}r\sin(\theta) \\
&+ \varepsilon\delta rw\cos(\theta) - \varepsilon\delta\overline{a}r\sin(\theta)), \\
\dot{s} &= -\overline{a}(-1 + w^2 + \overline{a}^2)((\overline{a}w^2 + w^2\varepsilon\alpha + \overline{a}^3 + 3\overline{a}^2\varepsilon\alpha + 3\overline{a}\varepsilon^2\alpha^2 \\
&+ \varepsilon^3\alpha^3 - 2\overline{a} - 2\varepsilon\alpha + \varepsilon\delta)(\frac{-r\cos(\theta)}{w} + \frac{\overline{a}r\sin(\theta)}{w^2} + \frac{\overline{a}s}{w^2}) \\
&+ \frac{r\sin(\theta)}{w^2} + \frac{s}{w^2} + \overline{a}(-1 + w^2 + \overline{a}^2)\frac{r\cos(\theta)}{w} - (-2\overline{a}^2 + w^2\overline{a}^2 \\
&+ \overline{a}^4 + 1 - w^2)\frac{r\sin(\theta)}{w^2} - (-2\overline{a}^2 + w^2\overline{a}^2 + \overline{a}^4 + 1)\frac{s}{w^2}) \\
&+ (1 - w^2 - \overline{a}^2)(\frac{-r\cos(\theta)}{w} + \frac{\overline{a}r\sin(\theta)}{w^2} + \frac{\overline{a}s}{w^2}
\end{aligned}
$$

$$+\overline{a}(-\frac{r\sin(\theta)}{w^2}-\frac{s}{w^2})+\varepsilon\alpha(-\frac{v}{w^2}-\frac{s}{w^2}))+((-\overline{a}-\varepsilon\alpha$$
$$+\overline{a}w^2+w^2\varepsilon\alpha+\overline{a}^3+3\overline{a}^2\varepsilon\alpha+3\overline{a}\varepsilon^2\alpha^2+\varepsilon^3\alpha^3+\varepsilon\delta)(1+w^2\overline{a}^2$$
$$+2\overline{a}w^2\varepsilon\alpha+\overline{a}^4+4\overline{a}^3\varepsilon\alpha+6\overline{a}^2\varepsilon^2\alpha^2+4\overline{a}\varepsilon^3\alpha^3-2\overline{a}^2-4\overline{a}\varepsilon\alpha$$
$$+\overline{a}\varepsilon\delta+\varepsilon^2\alpha^2w^2+\varepsilon^4\alpha^4-2\varepsilon^2\alpha^2+\varepsilon^2\alpha\delta)\frac{1}{\overline{a}+\varepsilon\alpha}$$
$$+\varepsilon\delta)(\frac{-r\cos(\theta)}{w}+\frac{\overline{a}r\sin(\theta)}{w^2}+\frac{\overline{a}s}{w^2})-(-\overline{a}-\varepsilon\alpha+\overline{a}w^2$$
$$+w^2\varepsilon\alpha\overline{a}^3+3\overline{a}^2\varepsilon\alpha+3\overline{a}\varepsilon^2\alpha^2+\varepsilon^3\alpha^3+\varepsilon\alpha+\varepsilon\delta)(-\overline{a}(-1$$
$$+w^2+\overline{a}^2)\frac{r\cos(\theta)}{w}+(-2\overline{a}^2+w^2\overline{a}^2+\overline{a}^4+1-w^2)\frac{r\sin(\theta)}{w^2}$$
$$+(-2\overline{a}^2+w^2\overline{a}^2+\overline{a}^4+1)\frac{s}{w^2})+\varepsilon(\frac{-r\cos(\theta)}{w}+\frac{\overline{a}r\sin(\theta)}{w^2}$$
$$+\frac{\overline{a}s}{w^2})(-\overline{a}(-1+w^2+\overline{a}^2)\frac{r\cos(\theta)}{w}+(-2\overline{a}^2+w^2\overline{a}^2$$
$$+\overline{a}^4+1-w^2)\frac{r\sin(\theta)}{w^2}+(-2\overline{a}^2+w^2\overline{a}^2+\overline{a}^4+1)\frac{s}{w^2}).$$

Therefore, taking θ as the new independent variable of the differential system (11), its solutions in the region $\dot{\theta}>0$ can be studied by analyzing the solution of the differential system

$$
\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}\theta} &= \varepsilon(\frac{-1}{w^5\overline{a}}(-2r\sin\theta\cos\theta w^3\overline{a}^3s-2r\sin\theta\cos\theta w\overline{a}^5s \\
&-r\sin\theta\cos\theta w\overline{a}s-\sin\theta w^3\overline{a}^2\gamma r\cos(\theta)+\sin\theta\overline{a}^2w^5\gamma r\cos \\
&+\sin\theta\overline{a}^4w^3\gamma r\cos\theta+2\overline{a}^2sr+2\overline{a}^3w^2\gamma r-\overline{a}w^2\gamma r-\sin\theta r^2w^2\overline{a}^2 \\
&+\sin\theta r^2w^2\overline{a}^4+w^2\beta\overline{a}^2r+2\overline{a}^4rw^2s+w^6\overline{a}\alpha r+3w^4\overline{a}^3\alpha r \\
&+w^4\overline{a}\delta r-2w^4\overline{a}\alpha r+\sin\theta\overline{a}^4s^2w^2+\cos^3\theta\overline{a}r^2w \\
&-3\cos^3\theta r^2w\overline{a}^3-\cos^3\theta r^2w^3\overline{a}+2\cos^3\theta r^2w^3\overline{a}^3+2\cos^3\theta r^2w\overline{a}^5 \\
&-2\overline{a}^6rs\cos^2\theta+4\overline{a}^4rs\cos^2\theta-2\overline{a}^2sr\cos^2\theta+r^2\overline{a}w^3\cos\theta \\
&-r^2\overline{a}w\cos\theta-w^4\overline{a}^3\gamma r-\overline{a}^5w^2\gamma r+w^4\overline{a}\gamma r-\overline{a}^6r^2\sin\theta\cos^2\theta \\
&-\overline{a}^2r^2\sin\theta\cos^2\theta+2\overline{a}^4r^2\sin\theta\cos^2\theta-2r^2\cos\theta w^3\overline{a}^3 \\
&+3r^2w\overline{a}^3\cos\theta-2r^2w\overline{a}^5\cos\theta+6\cos\theta w^2\overline{a}^4\alpha r\sin\theta \\
&+2\cos\theta w^3\overline{a}^2\delta r\sin\theta-6\cos\theta w^3\overline{a}^2\alpha r\sin\theta-\sin\theta w^3\beta\overline{a}r\cos\theta \\
&+3\sin\theta\cos\theta rw\overline{a}^3s-w^2\beta\overline{a}^2r\cos^2\theta-2\overline{a}^4rw^2s\cos^2\theta \\
&+\overline{a}^2srw^2\cos^2\theta+\overline{a}^5w^2\gamma r\cos^2\theta-w^4\overline{a}\gamma r\cos^2\theta \\
&+\overline{a}w^2\gamma r\cos^2\theta-2\overline{a}^3w^2\gamma r\cos^2\theta+w^4\overline{a}^3\gamma r\cos^2\theta
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&- 2\cos^2\theta w^6\overline{a}\alpha r + \cos\theta w^5\overline{a}^2\alpha s - 6\cos^2\theta w^4\overline{a}^3\alpha r\\
&+ 3\cos\theta w^3\overline{a}^4\alpha s - 2\cos^2\theta w^4\overline{a}\delta r + \cos\theta w^3\overline{a}^2\delta s\\
&+ 4\cos^2\theta w^4\overline{a}\alpha r - 3\cos\theta w^3\overline{a}^2\alpha s + 2\sin\theta\overline{a}^3 w^2\gamma s\\
&- \sin\theta\overline{a}^5 w^2\gamma s - \sin\theta w^3\delta r\cos\theta + \sin\theta w^2\beta\overline{a}^2 s\\
&- \sin\theta w^4\alpha\overline{a}s + \sin\theta r^2\cos^2\theta w^4\overline{a}^2 - \sin\theta\overline{a}^3 w^4\gamma s\\
&- \sin\theta\overline{a}w^2\gamma s + 2\cos\theta w^5\overline{a}^2\alpha r\sin\theta - \overline{a}^2 srw^2\\
&- 2\sin\theta\overline{a}^4 s^2 + \sin\theta\overline{a}^6 s^2 + \sin\theta\overline{a}^2 s^2 + 2\overline{a}^6 rs - 4\overline{a}^4 rs\\
&+ \overline{a}^6 r^2\sin\theta + \overline{a}^2 r^2\sin\theta - 2\overline{a}^4 r^2\sin\theta) + O(\varepsilon^2),\\
&= \varepsilon F_1(\theta, r, s) + O(\varepsilon^2),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}s}{\mathrm{d}\theta} &= \varepsilon(-\frac{1}{w^5\overline{a}}(\overline{a}^3 w^4\gamma s - \overline{a}^6 s^2 - \overline{a}^2 s^2 + 2\overline{a}^4 r^2 + 2\overline{a}^4 s^2\\
&- r^2 w^2\overline{a}^4 - \overline{a}^4 s^2 w^2 + r^2 w^2\overline{a}^2 + \overline{a}^5 w^2\gamma s + 2r\cos\theta w\overline{a}^5 s\\
&+ \overline{a}r\cos\theta ws + \overline{a}^2 w^3\gamma r\cos\theta - \overline{a}^2 w^5\gamma r\cos\theta - \overline{a}^4 w^3\gamma r\cos\theta)\\
&- r^2\sin\theta\overline{a}w^3\cos\theta - 3\overline{a}^3 w^4\alpha r\sin\theta - w^2\beta\overline{a}^2 r\sin\theta + w^4\alpha\overline{a}r\sin\theta\\
&- 3r^2\cos\theta w\overline{a}^3\sin\theta + 2r^2\cos\theta w^3\overline{a}^3\sin\theta + 2r^2\cos\theta w\overline{a}^5\sin\theta\\
&- 2\overline{a}^4 r\sin\theta w^2 s + \overline{a}^2 r\sin\theta s w^2 + \overline{a}r^2\cos\theta w\sin\theta + \overline{a}^3 w^4\gamma r\sin\theta\\
&- \overline{a}w^6\alpha r\sin\theta - \overline{a}w^4\delta r\sin\theta - 2\overline{a}^3 w^2\gamma r\sin\theta + \overline{a}^5 w^2\gamma r\sin\theta\\
&- \overline{a}w^4\gamma r\sin\theta + \overline{a}w^2\gamma r\sin\theta + 3\overline{a}^2 w^3\alpha r\cos\theta - 3\overline{a}^4 w^3\alpha r\cos\theta\\
&- \overline{a}^2 w^3\delta r\cos\theta - \overline{a}^2 w^5\alpha r\cos\theta + w^3\beta\overline{a}r\cos\theta - 3r\cos\theta w\overline{a}^3\varpi\\
&+ 2r\cos\theta w^3\overline{a}^3 s + \overline{a}^2 r^2 cos^2\theta + \overline{a}^6 r^2 cos^2\theta - 2\overline{a}^4 r^2 cos^2\theta\\
&+ 4\overline{a}^4 r\sin\theta s - 2\overline{a}^3 w^2\gamma s - \overline{a}^6 r^2 - \overline{a}^2 r^2 - 2\overline{a}^6 r\sin\theta s\\
&- w^2\beta\overline{a}^2 s + w^3\delta r\cos\theta - r^2 cos^2\theta w^4\overline{a}^2 - 2\overline{a}^2 r\sin\theta s + \overline{a}w^2\gamma s) + O(\varepsilon^2)\\
&= \varepsilon F_2(\theta, r, s) + O(\varepsilon^2).
\end{aligned}
$$

We shall apply the averaging theory described in Theorem 2.1 to the differential system (12). Using the notations of Section 2, we have

$$t = \theta,\ T = 2\pi,\ X = (r, s)^T,$$

$$F(\theta, r, w) = \begin{pmatrix} F_1(\theta, r, s) \\ F_2(\theta, r, s) \end{pmatrix},$$

and

$$f(r, s) = \begin{pmatrix} f_1(r, s) \\ f_2(r, s) \end{pmatrix}.$$

It is readily checked that system (12) satisfies all the assumptions of Theorem 2.1.

Now, we compute the integrals (4), i.e.

$$\begin{cases} f_1(r,s) = \frac{1}{2\pi}\int\limits_0^{2\pi} F_1(\theta,r,s)\mathrm{d}\theta \\ \qquad = \dfrac{r}{2w^5}(4\overline{a}^3 s - 2\overline{a}s - 2\overline{a}^5 s + \overline{a}^4 w^2\gamma + \overline{a}w^2 s \\ \qquad - w^4\gamma - w^2\beta\overline{a} + w^2\gamma + \overline{a}^2 w^4\gamma - 2\overline{a}^2 w^2\gamma - 2\overline{a}^3 w^2 s) \\ f_2(r,s) = \frac{1}{2\pi}\int\limits_0^{2\pi} F_2(\theta,r,s)\mathrm{d}\theta \\ \qquad = -\dfrac{1}{2w^5}(2\overline{a}r^2w^2 - 2\overline{a}^3 s^2 w^2 - 2\overline{a}^3 r^2 w^2 + 2\gamma s w^2 \\ \qquad - \overline{a}^5 r^2 + 4\overline{a}^3 s^2 + 2\overline{a}^3 r^2 - 2\overline{a}s^2 - 2\overline{a}^5 s^2 - \overline{a}r^2 w^4 \\ \qquad + 2\overline{a}^4\gamma s w^2 + 2\overline{a}^2\gamma s w^4 - 4\overline{a}^2\gamma s w^2 - 2\overline{a}\beta s w^2 - \overline{a}r^2) \end{cases}$$

The system $f_1(r,s) = f_2(r,s) = 0$ has a unique solution (r^*, s^*) with $r^* > 0$, given by

$$r^* = \frac{2w^2\sqrt{\Gamma}}{\sqrt{2\overline{a}}(\overline{a}^2 + w^2 - 1)(2\overline{a}^4 + (2w^2 - 4)\overline{a}^2 - w^2 + 2)},$$

$$s^* = \frac{w^2(\overline{a}^4\gamma + \gamma(w^2 - 2)\overline{a}^2 - \beta\overline{a} + (-w^2 + 1)\gamma)}{\overline{a}(2\overline{a}^4 + (2w^2 - 4)\overline{a}^2 - w^2 + 2)},$$

provided that Condition (6) holds. We note that the Jacobian (5) at (r^*, s^*) takes the value

$$\frac{1}{w^6(2\overline{a}^4 + (2w^2 - 4)\overline{a}^2 - w^2 + 2)}(\overline{a}^8\gamma + (2w^2 - 4)\gamma\overline{a}^6 - \beta\overline{a}^5 \\ +\gamma(w^4 - 4w^2 + 6)\overline{a}^4 + (-\beta w^2 + 2\beta)\overline{a}^3 + (2w^2 - 4)\gamma\overline{a}^2 \\ +(\beta w^2 - \beta)\overline{a} + \gamma)(\overline{a}^4\gamma + \gamma(w^2 - 2)\overline{a}^2 - \beta\overline{a} + (-w^2 + 1)\gamma).$$

The eigenvalues of the Jacobian matrix $\left.\frac{\partial(f_1,f_2)}{\partial(r,s)}\right|_{(r,s)=(r^*,s^*)}$ are the ones given in (7).

The rest of the proof of Theorem 3.2 follows immediately from Theorem 2.1 if we show that the periodic solution corresponding to (r^*, s^*) provides a periodic orbit bifurcating from the origin of coordinates of the differential system (8) at $\varepsilon = 0$.

Theorem 2.1 guarantees, for $\varepsilon \neq 0$ sufficiently small, the existence of a periodic solution $(r(\theta,\varepsilon), s(\theta,\varepsilon))$ of system (12) such that

$$(R(\theta,\varepsilon), V(\theta,\varepsilon)) \to (R^*, V^*) \text{ when } \varepsilon \to 0.$$

That is, system (10) has the periodic solution

$$(u(\theta,\varepsilon), v(\theta,\varepsilon), s(\theta,\varepsilon)) = (r(\theta,\varepsilon)\cos\theta, r(\theta,\varepsilon)\sin\theta, s(\theta,\varepsilon)), \tag{13}$$

for $\varepsilon > 0$ sufficiently small. Consequently, system (9) has the periodic solution $(X(\theta), Y(\theta), Z(\theta))$ obtained from (13) through the change of variables

$$(X, Y, Z)^T = B^{-1}(u, v, s)^T.$$

Finally, for $\varepsilon > 0$ sufficiently small system (8) has a periodic solution

$$(x(\theta), y(\theta), z(\theta)) = (\varepsilon X(\theta), \varepsilon Y(\theta), \varepsilon Z(\theta))$$

which tends to the origin of coordinates when $\varepsilon \to 0$. Therefore, it is a periodic solution starting at the zero-Hopf equilibrium point, located at the origin of coordinates when $\varepsilon = 0$. This completes the proof of Theorem 3.2. □

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