MATHEMATICA, 64 (87), N° 1, 2022, pp. 75–82

FINITE p-GROUPS WHICH ARE NON-INNER NILPOTENT

MASOUMEH GANJALI, AHMAD ERFANIAN, and INTAN MUCHTADI-ALAMSYAH

Abstract. A group G is called a non-inner nilpotent group, whenever it is nilpotent with respect to a non-inner automorphism. In 2018, all finitely generated abelian non-inner nilpotent groups have been classified. Actually, the authors proved that a finitely generated abelian group G is a non-inner nilpotent group, if G is not isomorphic to cyclic groups $\mathbb{Z}_{p_1 p_2 \dots p_t}$ and $\mathbb{Z}$, for a positive integer t and distinct primes $p_1, p_2, \dots, p_t$. In this paper, we make this conjecture that all finite non-abelian p-groups are non-inner nilpotent and we prove this conjecture for finite p-groups of nilpotency class 2 or of co-class 2.

MSC 2010. 20F12; 20D45.

Key words. Central automorphism, inner automorphism, nilpotent group, non-inner nilpotent group.

1. INTRODUCTION

Let G be a group and $\alpha \in \operatorname{Aut}(G)$ be a fixed automorphism of G. An α-commutator of elements $x, y \in G$ is defined as $[x,y]_\alpha = x^{-1}y^{-1}xy^\alpha$. The subgroup

$$Z^\alpha(G) = \{x \in G : [y,x]_\alpha = 1, \forall y \in G\}$$

is called the α-center subgroup of G, is the intersection of subgroups $Z(G)$ and $\operatorname{Fix}(\alpha) = \{x \in G : x^\alpha = x\}$, so is a normal subgroup of G which is invariant under α. Now, assume that N is an arbitrary normal subgroup of G which is invariant under α and $\bar{\alpha}$ is an automorphism of quotient group G/N by the rule $gN^{\bar{\alpha}} = g^\alpha N$. Then the following normal series

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G,$$

is called a central α-series whenever $G_i^\alpha = G_i$ and $G_{i+1}/G_i \leq Z^{\bar{\alpha}}(G/G_i)$, for $0 \leq i \leq n-1$. An α-nilpotent group is a group which possesses at least a central α-series. Here, we recall the definition of two normal series that have been introduced by Barzegar et al. in [1], to find necessary and sufficient conditions for a given group G to be α-nilpotent, for an automorphism $\alpha \in \operatorname{Aut}(G)$. Put $Z_1^\alpha(G) = Z^\alpha(G)$ and define $Z^{\bar{\alpha}}(\frac{G}{Z_{i-1}^\alpha(G)}) = \frac{Z_i^\alpha(G)}{Z_{i-1}^\alpha(G)}$ for $i \in \mathbb{N}$, then the normal series

$$\{1\} = Z_0^\alpha(G) \trianglelefteq Z_1^\alpha(G) \trianglelefteq Z_2^\alpha(G) \trianglelefteq \dots$$

The authors thank the referee for his helpful comments and suggestions.

DOI: 10.24193/mathcluj.2022.1.09

is said to be an upper central α-series. For a group G and $\alpha \in \mathrm{Aut}(G)$, the subgroup $\Gamma_2^\alpha(G) = \langle [x,y]_\alpha : x, y \in G \rangle$ is called the α-commutator subgroup. The normal series

$$G = \Gamma_1^\alpha(G) \trianglerighteq \Gamma_2^\alpha(G) \trianglerighteq \cdots \trianglerighteq \Gamma_{n+1}^\alpha(G) \trianglerighteq \dots,$$

is called a lower central α-series, where

$$\Gamma_{n+1}^\alpha(G) = [G, \Gamma_n^\alpha(G)]_\alpha = \langle [x,y] : x \in G, y \in \Gamma_n^\alpha(G) \rangle$$

and $\Gamma_i^\alpha(G)^\alpha = \Gamma_i^\alpha(G)$. In [1], it has been proved that G is α-nilpotent if and only if there is a positive integer s such that $Z_s^\alpha(G) = G$, if and only if there exists a positive integer r such that $\Gamma_r^\alpha(G) = \{1\}$. It is not difficult to see that $Z_n^\alpha(G) \leq Z_n(G)$ for all $n \in \mathbb{N}$, so if a group G is α-nilpotent, then it is nilpotent. But the converse is not valid in general, for example, the cyclic group $\mathbb{Z}_{p_1 p_2 \dots p_t}$ is nilpotent only related to the identity automorphism, for distinct primes $p_1, p_2, \dots, p_t$. Therefore, it is important to discover some conditions that nilpotency and α-nilpotency are equivalent under such conditions, for a fixed automorphism α. Assume that $\mathrm{Inn}(G)$ contains all inner automorphisms of G and an element $\alpha_g \in \mathrm{Inn}(G)$ acts on $x \in G$ by $x^{\alpha_g} = g^{-1}xg$, for all $g \in G$. One can prove that $Z_n^{\alpha_g}(G) = Z_n(G)$, for all $n \geq 1$, so G is nilpotent if and only if is α_g-nilpotent. Therefore, the following question comes to the mind naturally:

Question. *Is there any non-inner automorphism α of a nilpotent group G such that G is α-nilpotent?*

A group G is called a non-inner nilpotent group, whenever it is nilpotent related to a non-inner automorphism. We studied non-inner nilpotency of finitely generated abelian groups and some families of finite non-abelian p-groups in [3] and [4], respectively. Actually, we classified all finitely generated abelian groups that are non-inner nilpotent, in this way we have shown that a finitely generated abelian group G is a non-inner nilpotent group, if G is not isomorphic to cyclic groups $\mathbb{Z}_{p_1 p_2 \dots p_t}$ and $\mathbb{Z}$, for a positive integer t and distinct primes $p_1, p_2, \dots, p_t$. In this paper, we make the following conjecture:

Conjecture. *All finite non-abelian p-groups are non-inner nilpotent.*

In paper [4], we proved that this conjecture is valid for finite non-abelian p-groups of order p^3 and for the group

$$M_n(p) = \langle x, y : x^{p^{n-1}} = y^p = 1,\ xy = yx^{p^{n-2}+1} \rangle,$$

where p is an odd prime number and $n \geq 3$. Furthermore, we proved that if a group G possesses a locally inner automorphism which is not inner, then it is a non-inner nilpotent group. Corollary 2.15 of [4], is about non-inner nilpotent p-groups of order p^5 which have a non-inner locally inner automorphism. Here, we investigate non-inner nilpotency of finite p-groups of order p^4 or p^5 where not have been studied in paper [4]. Actually, we prove the above conjecture for some families of groups of order p^5 where all their locally inner automorphisms are inner. For more details about locally inner automorphisms of some finite p-groups, see [5]. We also prove this conjecture for finite p-groups of nilpotency class 2 or of co-class 2.

An automorphism α of group G is called a central automorphism, if $x^{-1}x^{\alpha} \in Z(G)$ for all $x \in G$. We denote the subgroup of all central automorphisms of G by $\mathrm{Aut}_c(G)$. Assume that $C_{\mathrm{Aut}_c(G)}(Z(G))$ is the group of all central automorphisms of G fixing $Z(G)$ element-wise, then we see that if $\alpha \in C_{\mathrm{Aut}_c(G)}(Z(G))$, then $Z_n^{\alpha}(G) = Z_n(G)$ for all $n \geq 1$, and nilpotency and α-nilpotency are equivalent. Therefore, if $C_{\mathrm{Aut}_c(G)}(Z(G)) \nleq \mathrm{Inn}(G)$, then G is a non-inner nilpotent group, so we study the non-inner nilpotency of finite groups such that $C_{\mathrm{Aut}_c(G)}(Z(G)) \leq \mathrm{Inn}(G)$. For convenience, we denote $C_{\mathrm{Aut}_c(G)}(Z(G))$ by C^*.

2. MAIN RESULT

In this section, we recall from [1], the notion non-inner nilpotent group to study our conjecture for finite p-groups of maximal class, of nilpotency class 2 or of co-class 2. The structure of non-inner automorphisms and central automorphisms of finite p-groups help us to complete our investigation.

Definition 2.1. A group G is said to be non-inner nilpotent, whenever there exists a non-inner automorphism α of G such that G is α-nilpotent.

Definition 2.2. An automorphism α of a group G is called central if α commutes with every inner automorphism or equivalently if $g^{-1}g^{\alpha} \in Z(G)$, for all $g \in G$. The set of all central automorphisms of group G is denoted by $\mathrm{Aut}_c(G)$.

Let G be a group. The subgroup of $\mathrm{Aut}_c(G)$ which contains all central automorphisms of G fixing $Z(G)$ element-wise is denoted by C^*. The equality of subgroups C^*, $\mathrm{Aut}_c(G)$ and $\mathrm{Inn}(G)$ is one of the most interesting topics between authors who are working on automorphisms of groups. For instance, authors in [2] proved that if C^* is a subgroup of $\mathrm{Inn}(G)$, then $C^* = Z(\mathrm{Inn}(G))$. They also proved that if $C^* = \mathrm{Aut}_c(G) = Z(\mathrm{Inn}(G))$, then $Z(G) \leq G'$ and if $Z(\mathrm{Inn}(G))$ is a cyclic group, then $\mathrm{Aut}_c(G) \gneqq Z(\mathrm{Inn}(G))$. Therefore, if G is a p-group of maximal class, then $C^* = \mathrm{Aut}_c(G) \gneqq Z(\mathrm{Inn}(G))$. Now, the following result follows.

Theorem 2.3. *If G is a finite non-abelian p-group of maximal class, then it is non-inner nilpotent.*

Theorem 2.4 ([10]). *If G is a finite p-group, then $C^* = \mathrm{Inn}(G)$ if and only if G is abelian or G is nilpotent of class 2 and $Z(G)$ is cyclic.*

Here, we prove that if G is a p-group of nilpotency class 2 with non-cyclic center, then G is a non-inner nilpotent group.

Theorem 2.5. *Let G be a finite p-group of nilpotency class 2 such that $Z(G)$ is non-cyclic. Then G is a non-inner nilpotent group.*

Proof. We show that $\mathrm{Inn}(G) \leq C^*$ if and only if G is nilpotent of class 2. If G is nilpotent of class 2, then $Z_2(G) = G$ and if $\alpha_g \in \mathrm{Inn}(G)$, then $[x, g] = x^{-1}x^{\alpha_g} \in Z(G)$ for every $x \in G$. Clearly, α_g fixes $Z(G)$ element-wise, so $\alpha_g \in C^*$. Conversely, if $\mathrm{Inn}(G) \leq C^*$, then $x^{-1}x^{\alpha_g} \in Z(G)$ for all $x, g \in G$ and hence G is nilpotent of class 2. Now, since $Z(G)$ is non-cyclic, then by Theorem 2.4, $\mathrm{Inn}(G)$ is a proper subgroup of C^*. Choose $\alpha \in C^* \setminus \mathrm{Inn}(G)$, then G is α-nilpotent and the proof is completed. □

By Theorem 2.5, a finite p-group of nilpotency class 2 with non-cyclic center is a non-inner nilpotent group. The Example 2.8. of [4], shows that the group

$$G = M_n(p) = \langle x, y : x^{p^{n-1}} = y^p = 1,\ xy = yx^{p^{n-2}+1}\rangle$$

is a finite non-inner nilpotent p-group of class 2 such that its center is cyclic.

The central automorphisms of groups of order p^5 or p^6 have been studied in [12], actually, they find necessary and sufficient conditions for which $\mathrm{Aut}_c(G) = Z(\mathrm{Inn}(G))$. For instance, they proved Theorem 2.6 on p-groups of order p^5.

Theorem 2.6. *If $|G| = p^5$, p an odd prime number, and G is of nilpotency class* 3*, then* $\mathrm{Aut}_c(G) = Z(\mathrm{Inn}(G))$ *if and only if G is isomorphic to*

$$\Phi_8(32) = \langle a, b : a^{p^2} = 1 = b^{p^3}, a^{-1}ba = b^{p+1}\rangle.$$

Note that groups of order p^5, where p is an odd prime, are divided into ten isoclinism families in [7]. The group $\Phi_8(32)$ in the previous theorem, is the only group in the eighth family of nilpotency class 3 and $d(\Phi_8(32)) = 2$.

In [6] and [11], the equality of $\mathrm{Aut}_c(G)$ and C^* has been studied, next theorem is one of the results of [6].

Theorem 2.7. *If G is a non-abelian p-group and $exp(Z(G)) = p$, then* $\mathrm{Aut}_c(G) = C^*$ *if and only if $Z(G) \leq \Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of G.*

Finite non-inner nilpotent p-groups of order p^3, have been studied in [4]. Here, we prove our conjecture for a finite non-abelian p-group of order p^4 by using Theorem 2.5 and Theorem 2.7. The classification of p-groups of order p^4 which is used in the following theorem, has been investigated in [9].

Theorem 2.8. *If G is a finite non-abelian p-group of order p^4, $p > 3$, then G is a non-inner nilpotent group.*

Proof. Assume first that $|G'| = p^2$, then G is nilpotent of class 3, $|Z(G)| = p$ and $Z(G) \leq G'$, because $Z(G) \cap G' \neq \{1\}$. Since G is nilpotent, then $G' \leq \Phi(G)$ and so $Z(G) \leq \Phi(G)$ and by Theorem 2.7, $C^* = \mathrm{Aut}_c(G)$. Furthermore, $\mathrm{Inn}(G)$ is a non-abelian p-group of order p^3, so $|Z(\mathrm{Inn}(G))| = p$ and by Corollary 3.8. of [2], $C^* = \mathrm{Aut}_c(G) \gneq Z(\mathrm{Inn}(G))$. Hence, G is a non-inner nilpotent group. Now, if $|G'| = p$, then G is nilpotent of class 2, $|Z(G)| = p^2$ and there exist only 6 groups of order p^4. If $Z(G)$ is non-cyclic, then by Theorem 2.5 G is a non-inner nilpotent group. If $Z(G)$ is cyclic, then G is isomorphic to one of the following groups

$$G_1 = \langle a, b : a^{p^3} = b^p = 1, ba = a^{1+p^2} b\rangle,$$

$$G_2 = \langle a, b, c : a^{p^2} = b^p = c^p = 1, cb = a^p bc, ab = ba, ac = ca\rangle.$$

Obviously, $G_1 \cong M_4(p)$, which is non-inner nilpotent by Example 2.8. of [4]. If $G \cong G_2$, then we consider α as $a^\alpha = a$, $b^\alpha = a^p b$ and $c^\alpha = cb$, so we see that $\Gamma_2^\alpha(G) = \langle a^p, b\rangle$, $\Gamma_3^\alpha(G) = \langle a^p\rangle$, $\Gamma_4^\alpha(G) = \{1\}$, $\alpha \notin \mathrm{Inn}(G)$ and G is a non-inner nilpotent group. □

The next theorem provides the condition that under which, every finite non-abelian p-group of co-class 2 is a non-inner nilpotent group.

Theorem 2.9. *If G is a finite non-abelian p-group of co-class 2, p is odd prime, such that $|Z(G)| \neq p$, then G is a non-inner nilpotent group.*

Proof. Assume that $|G| = p^n$. Since G is of co-class 2, then $Z_{n-2}(G) = G$. If the nilpotency class of G is 2, then by assumption we should have $n = 4$ and by Theorem 2.8 G is a non-inner nilpotent group. So assume that the nilpotency class of G is equal or greater than 3. If $|Z(G)| = p^3$, then $|Z_2(G)| = p^4$ and $|\frac{Z_2(G)}{Z(G)}| = |Z(\frac{G}{Z(G)})| = p$ and $Z(\mathrm{Inn}(G))$ is a cyclic group. We know that C^* is a non-cyclic group, thus $Z(\mathrm{Inn}(G))$ is a proper subgroup of C^* and G is non-inner nilpotent. Now, if $|Z(G)| = p^2$, then $|\frac{G}{Z(G)}| = p^{n-2}$ and $\frac{G}{Z(G)}$ is of maximal class. Here, we should note that if $\frac{G}{Z(G)}$ is abelian, then $Z_2(G) = G$ and G is of nilpotency class less than 2, which is a contradiction. Hence, $Z(\frac{G}{Z(G)})$ is a proper subgroup of C^* of order p and we are done. □

Now, we prove that there exists a finite non-inner nilpotent group of co-class 2 with the center of prime order p, actually we study the non-inner nilpotency of group $\Phi_8(32)$. By Theorem 2.6, we can not use the subgroup C^* for investigating non-inner nilpotency of $\Phi_8(32)$. Also, by the Theorem 2.14. of [4], all locally inner automorphisms of $\Phi_8(32)$ are inner, so we should study the non-inner nilpotency of this group by the structure of its automorphisms, directly. At first, we present a lemma which deduce some properties of $\Phi_8(32)$.

Lemma 2.10. *Let p be an odd prime number. If $G \cong \Phi_8(32)$, then for $m, n, r \in \mathbb{Z}$, we have*

(i) $b^n a^m = a^m b^{n(p+1)^m}$,

(ii) $(a^m b^n)^r = a^{rm} b^{n(\frac{(p+1)^{rm}-1}{(p+1)^m-1})}$ *and, in the special case, we have* $(a^p b)^n = a^{np} b^{\frac{n(n-1)}{2}p^2+n}$,

(ii) $(a^m b^n)^r = a^{rm} b^{n(\frac{(p+1)^{rm}-1}{(p+1)^m-1})}$ *and, in the special case, we have* $(a^p b)^n = a^{np} b^{\frac{n(n-1)}{2}p^2+n}$,

(iii) *the automorphism group of G is*

$$\mathrm{Aut}(G)=\{\alpha_{z,\omega,\mu} : a^{\alpha_{z,\omega,\mu}} = a(a^z b^\omega)^\mu, b^{\alpha_{z,\omega,\mu}} = a^z b^\omega;\ zp \overset{p^2}{\equiv} 0,\ \omega \overset{p}{\not\equiv} 0,\ \mu p^2 \overset{p^3}{\equiv} 0\}.$$

Proof. One can prove parts (i) and (ii) by induction on m, n and r for positive integers m, n and r. If m, n or r is a negative integer, then use a relation between a and b as $a^{-1}ba = b^{p+1}$. Since G is a p-group with cyclic commutator subgroup, then part (iii) is done by the main result of [8]. □

Theorem 2.11. *If G is a p-group which is defined in Theorem* 2.6 *and $\alpha = \alpha_{p,1,\mu} \in \mathrm{Aut}(G)$. Then G is an α-nilpotent group.*

Proof. Since $\mu p^2 \overset{p^3}{\equiv} 0$, then there exists an integer $k \in \mathbb{Z}$ such that $\mu = pk$. An automorphism α acts on generators of G as $a^\alpha = a(a^p b)^\mu$ and $b^\alpha = a^p b$. By Lemma 2.10, we have

$$(a^p b)^\mu = a^{p\mu} b^{\frac{\mu(\mu-1)}{2}p^2+\mu},$$

from where we conclude that $(a^p b)^\mu = b^\mu$, because $\mu = pk$ and $a^{p^2} = b^{p^3} = 1$. Also, the part (ii) of Lemma 2.10 shows that $(ab^\mu)^p = a^p b^{kp^2}$. It is not difficult to prove that $b^{(p+1)^p} = b^{p^2+1}$. Now, we conclude that the α-commutator subgroup of G, $\Gamma_2^\alpha(G)$, is generated by the following α-commutators

$$[a,a]_\alpha = b^\mu,\ [a,b]_\alpha = a^p b^{-(p^2+p)},\ [b,a]_\alpha = b^{p+\mu},\ [b,b]_\alpha = a^p b^{-p^2}.$$

Also, we deduce the generators of $\Gamma_3^\alpha(G)$ as

$$[a, b^\mu]_\alpha = b^{-kp^2}, [a, a^p b^{-p^2-p}]_\alpha = b^{p^2(k+1)}, [a, b^{p+\mu}]_\alpha = b^{-p^2(k+1)},$$

$$[a, a^p b^{-p^2}]_\alpha = b^{kp^2}, [b, a^p b^{-p^2-p}]_\alpha = b^{(k+1)p^2},$$

and

$$[b, a^p b^{-p^2}]_\alpha = b^{p^2(k+1)}, [b, b^\mu]_\alpha = [b, b^{\mu+p}]_\alpha = 1.$$

Now, we have

$$[a, b^{kp^2}]_\alpha = [a, b^{(k+1)p^2}]_\alpha = [b, b^{kp^2}]_\alpha = [b, b^{(k+1)p^2}]_\alpha = 1.$$

Hence, $\Gamma_4^\alpha(G) = \{1\}$ and G is an α-nilpotent group. □

Theorem 2.12. *Assume that $G \cong \Phi_8(32)$ and $\alpha = \alpha_{0,\omega,\mu} \in \mathrm{Aut}(G)$. Then G is α-nilpotent if and only if $\omega \overset{p}{\equiv} 1$.*

Proof. Let $\mu = pk$, for some integer $k \in \mathbb{Z}$. We see that

$$[a, b^m]_\alpha = a^{-1}b^{-m}ab^{m\omega} = a^{-1}ab^{-m(p+1)}b^{m\omega} = b^{m(\omega-p-1)}$$

and $[b, b^n]_\alpha = b^{n(\omega-1)}$, $m, n \in \mathbb{N}$. Also, $[a,a]_\alpha = b^{p\omega k}$, $[a,b]_\alpha = b^{\omega-p-1}$, $[b,a]_\alpha = b^{-1}a^{-1}bab^{p\omega k} = b^{p(\omega k+1)}$, $[b,b]_\alpha = b^{\omega-1}$. Therefore,

$$\Gamma_2^\alpha(G) = \langle b^{p\omega k}, b^{\omega-p-1}, b^{p(\omega k+1)}, b^{\omega-1}\rangle.$$

As we mentioned at the first part of the proof of theorem, $[a, b^m]_\alpha = b^{m(\omega-p-1)}$ and $[b, b^n]_\alpha = b^{n(\omega-1)}$, therefore we conclude that

$$\Gamma_4^\alpha(G) = \langle b^{p(\omega-p-1)^2}, b^{p(\omega-p-1)(\omega-1)}, b^{(\omega-p-1)^3}, b^{p(\omega-1)^2}, b^{(\omega-p-1)^2(\omega-1)}, b^{(\omega-p-1)(\omega-1)^2}, b^{(\omega-1)^3} \rangle.$$

Now, if $\omega \overset{p}{\equiv} 1$, then $\Gamma_4^\alpha(G) = \{1\}$ and G is an α-nilpotent group. Otherwise, if $\omega \overset{p}{\not\equiv} 1$, then $b^{(\omega-1)^n} \in \Gamma_{n+1}^\alpha(G)$ and $\Gamma_{n+1}^\alpha(G)$ is a non-trivial subgroup of G for all $n \in \mathbb{N}$ and G can not be α-nilpotent. $\square$

THEOREM 2.13. *Let $G \cong \Phi_8(32)$. Then G is a non-inner nilpotent group.*

Proof. Define $\alpha \in \mathrm{Aut}(G)$ as $a^\alpha = a$, $b^\alpha = a^p b$, then by Theorem 2.11, G is an α-nilpotent. The automorphism α is a non-inner automorphism of G, because if there exists $x = a^i b^j \in G$ such that $\alpha = \alpha_x$ then $b^\alpha = b^{\alpha_x}$. Now, we have

$$a^p b = x^{-1} b x = (a^i b^j)^{-1} b (a^i b^j) = b^{-j} a^{-i} b a^i b^j = b^{(p+1)^i},$$

therefore, $a^p b = b^{(p+1)^i}$ and so

$$a^p b = b^{p^i + \binom{i}{1} p^{i-1} + \binom{i}{2} p^{i-2} + \cdots + \binom{i}{i-1} p},$$

by the relation $ba^p = a^p b^{(p+1)^p}$, we can see that $b = b^{(p+1)^p}$ and $p^3 \mid (p+1)^p - 1$. But

$$(p+1)^p - 1 = \binom{p}{0} p^p + \binom{p}{1} p^{p-1} + \binom{p}{2} p^{p-2} + \cdots + \binom{p}{p-1} p$$

and we should have $p^3 \mid \binom{p}{0} p^p + \cdots + p^2$, which is a contradiction. Hence, α is a non-inner automorphism of G. $\square$

REFERENCES

[1] R. Barzegar and A. Erfanian, *Nilpotency and solubility of groups relative to an automorphism*, Caspian J. Math. Sci., **4** (2015), 271–283.

[2] M.J. Curran, *Finite groups with central automorphism group of minimal order*, Math. Proc. R. Ir. Acad., **104A** (2004), 223–229.

[3] A. Erfanian and M. Ganjali, *Nilpotent groups related to an automorphism*, Proc. Indian Acad. Sci. Math. Sci., **128** (2018), 1–12.

[4] M. Ganjali, A. Erfanian and I. Muchtadi-Alamsyah, *Some notes on non-inner nilpotent groups*, Southeast Asian Bull. Math., **44** (2020), 797–802.

[5] D. Gumber and M. Sharma, *Class-preserving automorphisms of some finite p-groups*, Proc. Indian Acad. Sci. Math. Sci., **125** (2015), 181–186.

[6] S.H. Jafari, *Central automorphism groups fixing the center element-wise*, Int. Electron. J. Algebra., **9** (2011), 167–170.

[7] R. James, *The groups of order p^6 (p an odd prime)*, Math. Comp., **34** (1980), 613–637.

[8] F. Menegazzo, Aut*omorphisms of p-groups with cyclic commutator subgroup*, Rend. Semin. Mat. Univ. Padova, **90** (1993), 81–101.

[9] G. Sædén Ståhl, J. Laine and G. Behm, *On p-groups of low power order*, Bachelor Thesis, KTH Royal Institute of Technology, 2010, pp. 1–124.

[10] M. Shabani Attar, *On central automorphisms that fix the centre element-wise*, Arch. Math., **89** (2007), 296–297.

[11] M. Shabani Attar, *Finite p-groups in which each central automorphism fixes center element-wise*, Comm. Algebra, **40** (2012), 1096–1102.
[12] M. Sharma and D. Gumber, *On central automorphisms of finite p-groups*, Comm. Algebra, **41** (2013), 1117–1122.

Received November 9, 2020
Accepted May 5, 2021

Ferdowsi University of Mashhad
Department of Mathematics
Mashhad, Iran
E-mail: `m.ganjali20@yahoo.com`
`https://orcid.org/0000-0003-4445-3290`
E-mail: `erfanian@math.um.ac.ir`
`https://orcid.org/0000-0002-9637-1417`

Institut Teknologi Bandung
Faculty of Mathematics and Natural Sciences
Bandung, Indonesia
E-mail: `ntan@math.itb.ac.id`
`https://orcid.org/0000-0001-7059-3196`