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EXISTENCE OF SOLUTION FOR THIRD-ORDER THREE-POINT BOUNDARY VALUE PROBLEM

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Abstract. By imposing some conditions on the nonlinear term f, we construct a lower solution and an upper solution to prove the existence of a solution for a type of nonlinear third-order nonlocal boundary value problem. Our main tools are the upper and the lower solution method and the Schauder fixed point theorem. The results are illustrated by an example.

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1. INTRODUCTION

The purpose of this paper is to establish the existence of a solution for the nonlinear third-order differential equations with nonlocal boundary conditions (BVP for short)

$$u'''(t) + f\left(t, u(t), u'(t), u''(t)\right) = 0,\ t \in [0,1], \tag{1}$$

$$u(0) = \gamma u(\eta),\ u'(1) = 0,\ u''(0) = 0, \tag{2}$$

where $\eta \in [0,1]$, $\gamma \in (0,1)$ and $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous.

Third-order differential equations arise in a variety of different areas of applied mathematics and physics, in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on, see [7]. Recently, third-order and higher-order two-point or three-point boundary value problems have received much attention from many authors, see [1, 2, 3, 4, 8] and the references therein.

Equation $u'''(t) = f(t, u(t))$, $t \in (0,1)$, satisfying the three-point condition $u(0) = u'(\eta) = u''(1) = 0$ with $\eta > \frac{1}{2}$ has been studied by many authors, see [11, 20, 21]. In 2009, Y. Sun [17], studied the existence and nonexistence of positive solutions to the third-order three-point nonhomogeneous boundary value problem

$$\begin{cases} u'''(t) + a(t) f(u(t)) = 0, & t \in (0,1), \\ u(0) = u'(0) = 0,\ u'(1) - \alpha u'(\eta) = \lambda, \end{cases}$$

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where $\eta \in (0,1)\,,\ \alpha \in \left[0, \frac{1}{\eta}\right)$ are constants and $\lambda \in (0,\infty)$ is a parameter. In 2013, F. J. Torres [18], studied the existence of positive solutions to a nonlinear third-order three-point boundary value problem

$$\begin{cases} u'''(t) + a(t) f(u(t)) = 0, \quad t \in (0,1)\,, \\ u(0) = 0,\ u'(0) = u'(1) = \alpha u(\eta)\,, \end{cases}$$

where $\eta \in (0,1)\,,\ \alpha \in \left[0, \frac{1}{\eta}\right)$ are constants. In 2016, A. Rezaiguia et al. [13], studied the existence of positive solutions to a nonlinear third-order three-point boundary value problem

$$\begin{cases} u'''(t) + a(t) f(u(t)) = 0,\ \ t \in (0,1)\,, \\ u'(0) = u'(1) = \alpha u(\eta)\,,\ u(0) = \beta u(\eta)\,, \end{cases}$$

where α, β and η are constants with $\alpha \in \left[0, \frac{1}{\eta}\right)$ and $\eta \in (0,1)$. Some results concerning the existence of solutions for third-order differential equations can be found in related studies [6, 9, 10, 12, 15, 16, 19, 22].

In this paper, we shall introduce some auxiliary functions that play a fundamental role in our analysis. By imposing some conditions on the nonlinear term f, we construct a lower solution and an upper solution to prove the existence of a solution to the above boundary value problem. Our mains tools are the upper and lower solution method and the Schauder fixed point theorem. The current paper is organized as follows. In section 2, we introduce some definitions, theorems and preliminary results that will be used in the next section. In section 3, we deal with the main result and we give an example to illustrate our result.

2. PRELIMINARY RESULTS

We present some fundamental definitions, lemmas and theorems.

Definition 2.1. If $\alpha \in C^3[0,1]$ satisfies

$$\begin{cases} \alpha'''(t) + f(t, \alpha(t), \alpha'(t), \alpha''(t)) \geq 0,\ t \in [0,1]\,, \\ \alpha(0) = \gamma \alpha(\eta)\,,\ \alpha'(1) = 0,\ \alpha''(0) \geq 0, \end{cases} \tag{3}$$

then α is called a lower solution of the BVP $(1) - (2)$.

Definition 2.2. If $\beta \in C^3[0,1]$ satisfies

$$\begin{cases} \beta'''(t) + f(t, \beta(t), \beta'(t), \beta''(t)) \leq 0,\ t \in [0,1]\,, \\ \beta(0) = \gamma \beta(\eta)\,,\ \beta'(1) = 0,\ \beta''(0) \leq 0, \end{cases} \tag{4}$$

then β is called an upper solution of the BVP $(1) - (2)$.

Definition 2.3 ([5, p. 67]). Consider two Banach spaces X and Y, a subset Ω of X and a map $T : \Omega \to Y$. The mapping T is said to be compact if it is cotinuous and $T(\Omega)$ is relatively compact.

Definition 2.4. An operator is called completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Theorem 2.5 (Schauder [15, p. 110]). *Let T be a mapping which maps a closed convex subset D of a Banach space X into it self. If $T(D)$ is contained in a compact subset of D, then T has at least one fixed point.*

The classical tool to verify the conditions of Schauder's fixed point theorem for the space of continuous functions $C[a,b]$ is the Arzela-Ascoli Theorem.

Theorem 2.6 (Arzela-Ascoli [15]). *A necessary and sufficient condition for a family of continuous functions defined on the compact interval $[a,b]$ to be compact in $C[a,b]$ is that this family is uniformly bounded and equicontinuous.*

Lemma 2.7. *For any $h \in C[0,1]$, the problem*

$$\begin{cases} -u''(t) = h(t), \ t \in [0,1], \\ u(0) = \gamma u(\eta), \ u'(1) = 0, \ \eta \in [0,1], \gamma \in [0,1], \end{cases} \tag{5}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)\, h(s)\, \mathrm{d}s, \ t \in [0,1], \tag{6}$$

where

$$G(t,s) = K(t,s) + \frac{\gamma}{1-\gamma} K(\eta, s), \quad (t,s) \in [0,1] \times [0,1],$$

here

$$K(t,s) = \begin{cases} s, \ 0 \le s \le t \le 1, \\ t, \ 0 \le t \le s \le 1. \end{cases}$$

Proof. From $u''(t) = -h(t)$, we have

$$u'(t) = u'(0) - \int_0^t h(s)\, \mathrm{d}s, \ t \in [0,1].$$

Integrating over $[0,t]$, we have

$$u(t) = u(0) + tu'(0) - \int_0^t (t-s)\, h(s)\, \mathrm{d}s. \tag{7}$$

By the boundary conditions in (5), we have

$$u(0) = \gamma u(0) + \gamma\eta u'(0) - \gamma \int_0^\eta (\eta - s)\, h(s)\, \mathrm{d}s,$$

and $u'(1) = u'(0) - \int_0^1 h(s)\,\mathrm{d}s$. Thus

$$(1-\gamma)\,u(0) = \gamma\eta u'(0) - \gamma \int_0^\eta (\eta - s)\,h(s)\,\mathrm{d}s,$$

and $u'(0) = \int_0^1 h(s)\,\mathrm{d}s$. Therefore

$$u(0) = \frac{\eta\gamma}{1-\gamma}\int_0^1 h(s)\,\mathrm{d}s - \frac{\gamma}{1-\gamma}\int_0^\eta (\eta - s)\,h(s)\,\mathrm{d}s.$$

Replacing, these expressions in (7), we have

$$\begin{aligned}
u(t) &= \frac{\eta\gamma}{1-\gamma}\int_0^1 h(s)\,\mathrm{d}s - \frac{\gamma}{1-\gamma}\int_0^\eta (\eta - s)\,h(s)\,\mathrm{d}s \\
&\quad + t\int_0^1 h(s)\,\mathrm{d}s - \int_0^t (t-s)\,h(s)\,\mathrm{d}s, \\
&= \frac{\eta\gamma}{1-\gamma}\int_0^\eta h(s)\,\mathrm{d}s + \frac{\eta}{1-\gamma}\int_\eta^1 h(s)\,\mathrm{d}s - \frac{\gamma}{1-\gamma}\int_0^\eta (\eta - s)\,h(s)\,\mathrm{d}s \\
&\quad + t\int_0^t h(s)\,\mathrm{d}s + t\int_t^1 h(s)\,\mathrm{d}s - t\int_0^t h(s)\,\mathrm{d}s + \int_0^t s h(s)\,\mathrm{d}s, \\
&= \frac{\eta\gamma}{1-\gamma}\int_\eta^1 h(s)\,\mathrm{d}s + \frac{\gamma}{1-\gamma}\int_0^\eta s h(s)\,\mathrm{d}s + t\int_0^t h(s)\,\mathrm{d}s + t\int_t^1 h(s)\,\mathrm{d}s \\
&\quad - t\int_0^t h(s)\,\mathrm{d}s + \int_0^t s h(s)\,\mathrm{d}s, \\
&= \frac{\eta\gamma}{1-\gamma}\int_\eta^1 h(s)\,\mathrm{d}s + \frac{\gamma}{1-\gamma}\int_0^\eta s h(s)\,\mathrm{d}s + t\int_t^1 h(s)\,\mathrm{d}s + \int_0^t s h(s)\,\mathrm{d}s, \\
&= \int_0^1 K(t,s)\,h(s)\,\mathrm{d}s + \frac{\gamma}{1-\gamma}\int_0^1 K(\eta,s)\,h(s)\,\mathrm{d}s,\ t \in [0,1].
\end{aligned} \tag{8}$$

□

LEMMA 2.8. *For all* $(t,s) \in [0,1]\times[0,1]$, *we have*

$$0 \le G(t,s) \le G(s,s).$$

Proof. The proof is evident, we omit it. □

3. MAIN RESULT

Let $E = C[0,1]$ be equipped with the norm $\|v\|_\infty = \max\limits_{0\leq t\leq 1} |v(t)|$ and $K = \{v \in E : v(t) \geq 0, \ for\ t \in [0,1]\}$. Then K is a cone in E and (E, K) is an ordered Banach space. For convenience, let $\lambda = \int_0^1 G(s,s)\, s\mathrm{d}s$.

Lemma 3.1. *If there exist a constant $N \geq 0$ such that*

$$f(t,s,r,l) \leq N, \ for\,(t,s,r,l) \in [0,1] \times [0, \lambda N] \times \left[0, \frac{N}{2}\right] \times [-N, 0], \tag{9}$$

then the BVP $(1)-(2)$ has an upper solution.

Proof. By Lemma 2.1, if we let $v(t) = -u''(t)$, $t \in [0,1]$, we have

$$u(t) = \int_0^1 G(t,s)\, v(s)\, \mathrm{d}s, \tag{10}$$

and

$$u'(t) = \int_t^1 v(s)\, \mathrm{d}s. \tag{11}$$

Define the operators A and $B : E \to E$ as follows:

$$(Av)(t) = \int_0^1 G(t,s)\, v(s)\, \mathrm{d}s, \ t \in [0,1],$$

and

$$(Bv)(t) = \int_t^1 v(s)\, \mathrm{d}s, \ t \in [0,1].$$

Obviously, A and B are increasing on E. Then BVP $(1)-(2)$ is equivalent to the following problem

$$\begin{cases} v'(t) = f(t, (Av)(t), (Bv)(t), -v(t)), \\ v(0) = 0. \end{cases} \tag{12}$$

If we let $y(t) = -\beta''(t) = Nt$, $t \in [0,1]$, then it follows from (9) that

$$\begin{cases} y'(t) - f(t, (Ay)(t), (By)(t), -y(t)) \geq 0, \quad t \in [0,1], \\ y(0) \geq 0. \end{cases}$$

which implies that β is an upper solution of the BVP $(1)-(2)$. □

Lemma 3.2. *If there exists a constant $M \leq 0$ such that*

$$M \leq f(t,s,r,l), \ for\ \ (t,s,r,l) \in [0,1] \times [\lambda M, 0] \times \left[\frac{M}{2}, 0\right] \times [0, -M], \tag{13}$$

then the BVP $(1)-(2)$ *has a lower solution.*

Proof. By Lemma 2.1, if we let $v(t)=-u''(t)$, we have

$$u(t)=\int\limits_0^1 G(t,s)\,v(s)\,\mathrm{d}s,$$

and

$$u'(t)=\int\limits_t^1 v(s)\,\mathrm{d}s.$$

Then the BVP $(1)-(2)$ is equivalent to the following problem

$$\begin{cases} v'(t)=f(t,(Av)(t),(Bv)(t),-v(t)), \\ v(0)=0, \end{cases}$$

where A and B are defined above. If we let $x(t)=-\alpha''(t)=Mt,\ t\in[0,1]$, then it follows from (13) that

$$\begin{cases} x'(t)-f(t,(Ax)(t),(Bx)(t),-x(t))\leq 0, \quad t\in[0,1], \\ x(0)\leq 0 \end{cases}$$

which implies that α is a lower solution of the BVP $(1)-(2)$. □

Theorem 3.3. *Let α and β be, respectively, a lower and an upper solution of the BVP* $(1)-(2)$ *such that $\alpha''\geq\beta''$ on* $[0,1]$. *Assume that conditions (9) and (13) are satisfied. Then the BVP* $(1)-(2)$ *has a solution u_0, which satisfies*

$$\alpha(t)\leq u_0(t)\leq\beta(t)\ \textit{and}\ \beta''(t)\leq u_0''(t)\leq\alpha''(t),\ \textit{for}\ t\in[0,1],$$

where $\alpha(t)=M\int_0^1 G(t,s)\,s\mathrm{d}s$ *and* $\beta(t)=N\int_0^1 G(t,s)\,s\mathrm{d}s,\ t\in[0,1]$.

Proof. Let $x(t)=-\alpha''(t)=Mt$ and $y(t)=-\beta''(t)=Nt,\ t\in[0,1]$, then it follows from (9) and (13) that

$$\begin{cases} y'(t)-f(t,(Ay)(t),(By)(t),-y(t))\geq 0, \quad t\in[0,1], \\ y(0)\geq 0, \end{cases}$$

and

$$\begin{cases} x'(t)-f(t,(Ax)(t),(Bx)(t),-x(t))\leq 0, \quad t\in[0,1], \\ x(0)\leq 0, \end{cases}$$

which implies that α and β are, respectively, a lower and an upper solution of the BVP $(1)-(2)$.

The condition $M\leq 0\leq N$ implies that $\alpha''(t)\geq\beta''(t)$. We consider the following auxiliary problem

$$(14)\qquad \begin{cases} v'(t)=F(t,(Av)(t),(Bv)(t),-v(t)), \quad t\in[0,1], \\ v(0)=0, \end{cases}$$

where

$$F(t,s,r,l)=\begin{cases} f_1(t,s,r,-x(t)), & l>-x(t),\\ f_1(t,s,r,l), & -y(t)\le l\le -x(t),\\ f_1(t,s,r,-y(t)), & l<-y(t),\end{cases}$$

$$f_1(t,s,r,l)=\begin{cases} f_2(t,s,(By)(t),l), & r>(By)(t),\\ f_2(t,s,r,l), & (Bx)(t)\le r\le (By)(t),\\ f_2(t,s,(Bx)(t),l), & r<(Bx)(t),\end{cases}$$

and

$$f_2(t,s,r,l)=\begin{cases} f(t,(Ay)(t),r,l), & s>(Ay)(t),\\ f(t,s,r,l), & (Ax)(t)\le s\le (Ay)(t),\\ f(t,(Ax)(t),r,l), & s<(Ax)(t).\end{cases}$$

If we define an operator $T:E\to E$ by

$$(Tv)(t)=\int_0^t F(s,(Av)(s),(Bv)(s),-v(s))\,\mathrm{d}s,\ t\in[0,1],$$

then it is obvious that the fixed points of T are solutions of the auxiliary problem (14). Now, we will apply Schauder's fixed point theorem to prove that the operator T has a fixed point.

Let $B_N=\{v\in E:\|v\|_\infty\le N\}$. Then B_N is a bounded, closed and convex set. Since $|M|=\|x\|_\infty\le\|y\|_\infty=N$, we have $x,y\in B_N$.

First, we prove that $T:B_N\to B_N$. For any $v\in B_N$, we consider the following four cases:

Case 1. $y(t)<v(t)\le N,\ t\in[0,1]$.

Case 2 . $0\le v(t)\le y(t),\ t\in[0,1]$.

Case 3 . $x(t)<v(t)\le 0,\ t\in[0,1]$.

Case 4 . $-N<v(t)\le x(t),\ t\in[0,1]$.

We can verify in Cases 1 and 2 that

$$0\le F(t,(Av)(t),(Bv)(t),-v(t))\le N, \tag{15}$$

and in Cases 3 and 4 that

$$M\le F(t,(Av)(t),(Bv)(t),-v(t))\le 0. \tag{16}$$

Since that the proof is similar, we only consider Case 1. In this case, by the definition of F, we obtain

$$\begin{aligned} F(t,(Av)(t),(Bv)(t),-v(t)) &= f_1(t,(Av)(t),(Bv)(t),-y(t)),\\ &= f_2(t,(Av)(t),(By)(t),-y(t)),\\ &= f(t,(Ay)(t),(By)(t),-y(t)),\end{aligned}$$

which together with (9) indicates that (15) is fulfilled. Since $N\ge -M$, it follows from (15) and (16) that for any $v\in B_N$

$$|F(t,(Av)(t),(Bv)(t),-v(t))|\le N,\ t\in[0,1],$$

which implies that

$$\begin{aligned}|(Tv)(t)| &= \left|\int_0^1 F(s,(Av)(s),(Bv)(s),-v(s)\,\mathrm{d}s)\right|,\\ &\le \int_0^1 |F(s,(Av)(s),(Bv)(s),-v(s))|\,\mathrm{d}s,\\ &\le N.\end{aligned}$$

This shows that $T: B_N \to B_N$.

Next, we prove that $T: B_N \to B_N$ is completely continuous. Since the continuity of T is obvious (because T is bounded), we only need to prove that T is compact. Let X be a bounded subset in B_N. Then $T(X) \subseteq B_N$, which implies that $T(X)$ is uniformly bounded. Now, we shall prove that $T(X)$ is equicontinuous. For any $\epsilon > 0$, we choose $\delta = \frac{\epsilon}{N+1}$. Then, for any $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$, we have

$$\begin{aligned}|(Tv)(t_1) - (Tv)(t_2)| &= \left|\int_0^{t_1} F(s,(Av)(s),(Bv)(s),-v(s))\,\mathrm{d}s\right.\\ &\qquad \left. - \int_0^{t_2} F(s,(Av)(s),(Bv)(s),-v(s))\,\mathrm{d}s\right|\end{aligned}$$

$$\begin{aligned}&\le \left|\int_{t_2}^{t_1} F(s,(Av)(s),(Bv)(s),-v(s))\,\mathrm{d}s\right|,\\ &\le N|t_1 - t_2|,\\ &< \epsilon,\end{aligned}$$

which shows that $T(X)$ is equicontinuous. By the Arzela-Ascoli Theorem, we know that $T: B_N \to B_N$ is a compact mapping. From the Schauder fixed point theorem we deduce that the operator T has a fixed point v^*, which solves the auxiliary problem (14).

Finally, we prove that v^*, is a solution of the problem (12). To this end, we only need to verify that $x(t) \le v^*(t) \le y(t)$, for $t \in [0,1]$. Since the proof of $v^*(t) \le y(t)$, for $t \in [0,1]$, is similar, we only prove that $x(t) \le v^*(t)$, for $t \in [0,1]$. Suppose on the contrary that there exists $\bar{t} \in [0,1]$ such that $v^*(\bar{t}) < x(\bar{t})$. Obviously $\bar{t} \in (0,1)$. By the continuity of v^* and x and since $v^*(0) = 0 = x(0)$, we know that there exists $t^* \in (0,1)$ such that $v^*(t^*) =$

$x(t^*)$ and $v^*(t) < x(t)$, for $t \in [t^*, \bar{t}]$. Therefore

$$
\begin{aligned}
(v^*)'(t) &= F(t, (Av^*)(t), (Bv^*)(t), -v^*(t)), \\
&= f_1(t, (Av^*)(t), (Bv^*)(t), -x(t)), \\
&= f_2(t, (Av^*)(t), (Bx)(t), -x(t)), \\
&= f((Ax)(t), (Bx)(t), -x(t)),
\end{aligned} \tag{17}
$$

for $t \in [t^*, \bar{t}]$. Let $m(t) = v^*(t) - x(t)$, $t \in [t^*, \bar{t}]$. Since α is a lower solution of the BVP $(1) - (2)$, one has

$$
\begin{aligned}
x'(t) = -\alpha'''(t) &\le f\left(t, \alpha(t), \alpha'(t), \alpha''(t)\right), \\
&= f(t, (Ax)(t), (Bx)(t), -x(t)), \quad t \in [0,1].
\end{aligned} \tag{18}
$$

In view of (17) and (18), we have $m'(t) = (v^*)'(t) - x'(t) \ge 0$, $t \in (t^*, \bar{t}]$, which together with $m(t^*) = 0$ implies that $m(t) \ge 0$, for $t \in [t^*, \bar{t}]$, that is $v^*(t) \ge x(t)$, for $t \in [t^*, \bar{t}]$. This is a contradiction. Thus $x(t) \le v^*(t)$, for $t \in [t^*, \bar{t}]$.

We claim that the BVP $(1) - (2)$ has a solution. In fact, if we let $u^*(t) = \int_0^1 G(t,s) v^*(s)\,\mathrm{d}s$, $t \in [0,1]$, then u^* is the desired solution of the BVP $(1) - (2)$ satisfying $\alpha(t) \le u^*(t) \le \beta(t)$ and $\beta''(t) \le (u^*)''(t) \le \alpha''(t)$, for $t \in [0,1]$. □

We construct an example to illustrate the applicability of the results presented.

Example 3.4. Consider the following boundary value problem

$$
u'''(t) + \frac{1}{6}\left(t + 32(u(t))^2 + 36u'(t) - 3u''(t)\right) = 0, \tag{19}
$$

$$
u(0) = \frac{1}{2} u\left(\frac{1}{2}\right), \ u'(1) = 0, \ u''(1) = 0, \tag{20}
$$

where $f(t, u, u', u'') = \frac{1}{6}\left(t + 32u^2 + 36u' - 3u''\right)$ and $\gamma = \eta = \frac{1}{2}$.

There exists $N = 3$ such that, for $(t, s, r, l) \in [0,1] \times \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{12}\right] \times [-3, 0]$, we have

$$
f(t, s, r, l) = \frac{1}{6}\left(t + 32s^2 + 36r - 3l\right) \le 3.
$$

There exists $M = 0$ such that, for $t = s = r = l = 0$, we have

$$
f(t, s, r, l) = \frac{1}{6}\left(t + 32s^2 + 36r - 3l\right) \ge 0.
$$

Hence, all the assumptions of Theorem 3.1 are satisfied, which implies that the boundary value problem (19)-(20) has a solution u_0 which satisfies $\alpha(t) \le u_0(t) \le \beta(t)$ and $\beta''(t) \le u_0''(t) \le \alpha''(t)$.

It is easy to construct the lower solution and the upper solution; we obtain

$$
\alpha(t) = \frac{N}{2}\left(t - \frac{1}{3}t^3 + \frac{1}{2(1-\gamma)}\left(\eta - \frac{1}{3}\eta^3\right)\right),
$$

and

$$\beta(t) = \frac{M}{2}\left(t - \frac{1}{3}t^3 + \frac{1}{2(1-\gamma)}\left(\eta - \frac{1}{3}\eta^3\right)\right).$$

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