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NEARLY PARTIAL TERNARY CUBIC DERIVATIONS ON BANACH TERNARY ALGEBRAS

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Abstract. Let $A_1, A_2, ..., A_n$ be normed ternary algebras over the complex field $\mathbb{C}$ and let B be a Banach ternary algebra over $\mathbb{C}$. A mapping δ_k from $A_1 \times \cdots \times A_n$ into B is called a *k-th partial ternary cubic derivation* if there exists a cubic mapping $g_k : A_k \to B$ such that

$$\delta_k(x_1, \cdots, [a_k b_k c_k], \cdots, x_n) = [g_k(a_k)g_k(b_k)\delta_k(x_1, \cdots, c_k, \cdots, x_n)]$$
$$+[g_k(a_k)\delta_k(x_1, \cdots, b_k, \cdots, x_n)g_k(c_k)] + [\delta_k(x_1, \cdots, a_k, \cdots, x_n)g_k(b_k)g_k(c_k)],$$

and

$$\delta_k(x_1, \cdots, 2a_k + b_k, \cdots, x_n) + \delta_k(x_1, \cdots, 2a_k - b_k, \cdots, x_n)$$
$$=2\delta_k(x_1, \cdots, a_k + b_k, \cdots, x_n)$$
$$+2\delta_k(x_1, \cdots, a_k - b_k, \cdots, x_n) + 12\delta_k(x_1, \cdots, a_k, \cdots, x_n),$$

for all $a_k, b_k, c_k \in A_k$ and all $x_i \in A_i$ $(i \neq k)$. We prove the generalized Hyers-Ulam-Rassias stability of the partial ternary cubic derivations on Banach ternary algebras.

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Key words. Hyers-Ulam-Rassias stability, generalized Hyers-Ulam-Rassias, Banach ternary algebra, partial ternary cubic derivation.

1. INTRODUCTION

Ternary algebraic operations were considered in the 19 th century by several mathematicians such as A. Cayley [5] who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990 ([30]). The comments on physical applications of ternary structures can be found in [3, 4, 7, 33, 37].

A ternary (associative) algebra $(A, [\ ])$ is a linear space A over a scalar field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ equipped with a linear mapping, the so-called ternary product, $[\ ] : A \times A \times A \to A$ such that $[[abc]de] = [a[bcd]e] = [ab[cde]]$ for all $a, b, c, d, e \in A$. This notion is a natural generalization of the binary case. Indeed if $(A, \odot)$ is a usual (binary) algebra then $[abc] := (a \odot b) \odot c$ induced a ternary product making A in to a ternary algebra which will be called trivial. By a Banach ternary algebra we mean a ternary algebra equipped with a complete norm $\| \cdot \|$ such that $\|[abc]\| \leq \|a\|\|b\|\|c\|$, for all $a, b, c \in A$. The study of stability problems for functional equations is related to a question of Ulam [36] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [28]. Subsequently, the result of Hyers was generalized by Aoki [1] for additive mappings and by Th. M. Rassias [35] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias has provided a lot of influence in the development of what we now call a generalized Hyers-Ulam stability of functional equations. We refer the interested readers for more information on such problems to the papers [24, 25, 27, 31, 32] and [34].

The cubic function $f(x) = ax^3$ satisfies the functional equation [29]

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{$*$}$$

(See [8]–[23], [26] and [31]–[34]). Recently, the stability of derivations has been investigated by some authors; see [2, 6, 18, 23] and references therein. For more detailed definitions of such terminologies, we can refer to [3, 11, 13, 16, 22] and [33].

2. MAIN RESULTS

Let $A_1, A_2, ..., A_n$ be normed ternary algebras over the complex field $\mathbb{C}$ and let B be a Banach ternary algebra over $\mathbb{C}$. A mapping δ_k from $A_1 \times \cdots \times A_n$ into B is called a *k-th partial ternary cubic derivation* if there exists a cubic mapping $g_k : A_k \to B$ such that

$$\delta_k(x_1,\cdots,[a_kb_kc_k],\cdots,x_n) = [g_k(a_k)g_k(b_k)\delta_k(x_1,\cdots,c_k,\cdots,x_n)]$$
$$[g_k(a_k)\delta_k(x_1,\cdots,b_k,\cdots,x_n)g_k(c_k)] + [\delta_k(x_1,\cdots,a_k,\cdots,x_n)g_k(b_k)g_k(c_k)]$$

and

$$\begin{aligned}&\delta_k(x_1,\cdots,2a_k+b_k,\cdots,x_n)+\delta_k(x_1,\cdots,2a_k-b_k,\cdots,x_n)\\=&2\delta_k(x_1,\cdots,a_k+b_k,\cdots,x_n)\\+&2\delta_k(x_1,\cdots,a_k-b_k,\cdots,x_n)+12\delta_k(x_1,\cdots,a_k,\cdots,x_n),\end{aligned}$$

for all $a_k, b_k, c_k \in A_k$ and all $x_i \in A_i$ $(i \neq k)$. We denote that $0_k, 0_B$ are zero elements of A_k, B, respectively.

THEOREM 2.1. *Let $p \geq 0$ be given with $p < 3$ and let θ be nonnegative real numbers. Let $F_k : A_1 \times \cdots \times A_n \to B$ be a mapping with*

$$F_k(x_1,\cdots,0_k,\cdots,x_n) = 0_B.$$

Suppose that there exist a cubic mapping $g_k : A_k \to B$ such that

$$\begin{aligned}&\|F_k(x_1,\cdots,2a_k+b_k,\cdots,x_n)+F_k(x_1,\cdots,2a_k-b_k,\cdots,x_n)\\&-2F_k(x_1,\cdots,a_k+b_k,\cdots,x_n)-2F_k(x_1,\cdots,a_k-b_k,\cdots,x_n)\\&-12F_k(x_1,\cdots,a_k,\cdots,x_n)\| \leq \theta(\|a_k\|^p+\|b_k\|^p),\end{aligned} \tag{1}$$

$$\begin{aligned}&\|F_k(x_1,\cdots,[a_kb_kc_k],\cdots,x_n)-[g_k(a_k)g_k(b_k)F_k(x_1,\cdots,c_k,\cdots,x_n)]\\&-[g_k(a_k)F_k(x_1,\cdots,b_k,\cdots,x_n)g_k(c_k)]\\&-[F_k(x_1,\cdots,a_k,\cdots,x_n)g_k(b_k)g_k(c_k)]\| \leq \theta(\|a_k\|^p+\|b_k\|^p+\|c_k\|^p),\end{aligned} \tag{2}$$

for all $a_k, b_k, c_k \in A_k$, $x_i \in A_i$ $(i \neq k)$. *Then there exists a unique k-th partial ternary cubic derivation* $\delta_k : A_1 \times \cdots \times A_n \to B$ *such that*

$$\|F_k(x_1, \cdots, x_n) - \delta_k(x_1, \cdots, x_n)\| \leq \frac{\theta}{2(8-2^p)}\|x_k\|^p \tag{3}$$

holds for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$.

Proof. In (1), putting $a_k = x_k$ and $b_k = 0_k$, we have

$$\|2F_k(x_1, \cdots, 2x_k, \cdots, x_n) - 16F_k(x_1, \cdots, x_k, \cdots, x_n)\| \leq \theta\|x_k\|^p, \tag{4}$$

that is,

$$\|F_k(x_1, \cdots, x_k, \cdots, x_n) - \frac{1}{8}F_k(x_1, \cdots, 2x_k, \cdots, x_n)\| \leq \frac{\theta}{16}\|x_k\|^p \tag{5}$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$. One can use induction on m to show that

$$\|F_k(x_1, \cdots, x_k, \cdots, x_n) - \frac{1}{2^{3m}}F_k(x_1, \cdots, 2^m x_k, \cdots, x_n)\|$$
$$\leq \frac{\theta}{16}\sum_{i=0}^{m-1} 2^{i(p-3)}\|x_k\|^p, \tag{6}$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$ and all non-negative integers m. Hence

$$\|\frac{1}{2^{3j}}F_k(x_1, \cdots, 2^j x_k, \cdots, x_n) - \frac{1}{2^{3(m+j)}}F_k(x_1, \cdots, 2^{(m+j)}x_k, \cdots, x_n)\|$$
$$\leq \frac{\theta}{16}\sum_{i=j}^{m+j-1} 2^{i(p-3)}\|x_k\|^p, \tag{7}$$

for all non-negative integers m and j with $m \geq j$ and all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$.

It follows from $p < 3$ that the sequence $\{\frac{1}{2^{3m}}F_k(x_1, \cdots, 2^m x_k, \cdots, x_n)\}$ is Cauchy. Due to the completeness of B, this sequence is convergent. So one can define the mapping $\delta_k : A_1 \times \cdots \times A_n \to B$ given by

$$\delta_k(x_1, \cdots, x_n) = \lim_{m\to\infty} \frac{1}{2^{3m}}F_k(x_1, \cdots, 2^m x_k, \cdots, x_n), \tag{8}$$

for all $x_i \in A_i$ $(i = 1, \cdots, n)$. In (1), replacing a_k, b_k with $2^m a_k, 2^m b_k$, respectively, we obtain that

$$\|\frac{1}{2^{3m}}F_k(x_1, \cdots, 2^m(2a_k + b_k), \cdots, x_n) + \frac{1}{2^{3m}}F_k(x_1, \cdots, 2^m(2a_k - b_k), \cdots, x_n)$$
$$- \frac{2}{2^{3m}}F_k(x_1, \cdots, 2^m(a_k - b_k), \cdots, x_n) - \frac{2}{2^{3m}}F_k(x_1, \cdots, 2^m(a_k - b_k), \cdots, x_n)$$
$$- \frac{12}{2^{3m}}F_k(x_1, \cdots, 2^m a_k, \cdots, x_n)\| \leq \theta \cdot 2^{m(p-3)}(\|a_k\|^p + \|b_k\|^p),$$

which tends to zero as $m \to \infty$. Thus we obtain

$$\begin{aligned}
&\delta_k(x_1, \cdots, 2a_k + b_k, \cdots, x_n) + \delta_k(x_1, \cdots, 2a_k - b_k, \cdots, x_n)\\
=&2\delta_k(x_1, \cdots, a_k + b_k, \cdots, x_n)\\
+&2\delta_k(x_1, \cdots, a_k - b_k, \cdots, x_n) + 12\delta_k(x_1, \cdots, a_k, \cdots, x_n),
\end{aligned} \tag{9}$$

for all $a_k, b_k \in A_k$ and all $x_i \in A_i (i \neq k)$. Hence δ_k is cubic with respect to the k-th variable. It follows from (7) that

$$\|F_k(x_1, \cdots, x_k, \cdots, x_n) - \delta_k(x_1, \cdots, x_k, \cdots, x_n)\| \leq \frac{\theta}{2(8 - 2^p)}\|x_k\|^p,$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$. Replacing in (2) the elements a_k, b_k, c_k with $2^m a_k, 2^m b_k, 2^m c_k$, respectively, and dividing both sides of the inequality by 2^{9m}, we obtain for all $a_k, b_k, c_k \in A_k$ that

$$\begin{aligned}
&\|\frac{1}{2^{9m}}F_k(x_1, \cdots, 2^{3m}[a_k b_k c_k], \cdots, x_n)\\
-&\frac{1}{2^{9m}}[2^{3m}g_k(a_k)2^{3m}g_k(b_k)F_k(x_1, \cdots, 2^m c_k, \cdots, x_n)]\\
-&\frac{1}{2^{9m}}[2^{3m}g_k(a_k)F_k(x_1, \cdots, 2^m b_k, \cdots, x_n)2^{3m}g_k(c_k)]\\
-&\frac{1}{2^{9m}}[F_k(x_1, \cdots, 2^m a_k, \cdots, x_n)2^{3m}g_k(b_k)2^{3m}g_k(c_k)]\|\\
\leq&2^{m(p-9)} \cdot \theta(\|a_k\|^p + \|b_k\|^p + \|c_k\|^p).
\end{aligned}$$

Passing the limit $m \to \infty$ in the above inequality, we get

$$\begin{aligned}
&\delta_k(x_1, \cdots, [a_k b_k c_k], \cdots, x_n) = [g_k(a_k)g_k(b_k)\delta_k(x_1, \cdots, c_k, \cdots, x_n)]\\
&+[g_k(a_k)\delta_k(x_1, \cdots, b_k, \cdots, x_n)g_k(c_k)] + [\delta_k(x_1, \cdots, a_k, \cdots, x_n)g_k(b_k)g_k(c_k)],
\end{aligned}$$

for all $a_k, b_k, c_k \in A_k$ and all $x_i \in A_i$ $(i \neq k)$.

Finally, to prove the uniqueness of δ_k, let $\delta_k' : A_1 \times \cdots \times A_n \to B$ be another k-th partial ternary cubic derivation satisfying (3). Then we have

$$\begin{aligned}
&\|\delta_k(x_1, \cdots, x_n) - \delta_k'(x_1, \cdots, x_n)\|\\
=&\frac{1}{2^{3m}}\|\delta_k(x_1, \cdots, 2^m x_k, \cdots, x_n) - \delta_k'(x_1, \cdots, 2^m x_k, \cdots, x_n)\|\\
\leq&\frac{1}{2^{3m}}(\|\delta_k(x_1, \cdots, 2^m x_k, \cdots, x_n) - F_k(x_1, \cdots, 2^m x_k, \cdots, x_n)\|\\
+&\|F_k(x_1, \cdots, 2^m x_k, \cdots, x_n) - \delta_k'(x_1, \cdots, 2^m x_k, \cdots, x_n)\|)\\
\leq&\theta \sum_{i=m}^{\infty} 2^{i(p-3)}\|x_k\|^p,
\end{aligned}$$

which tends to zero as $m \to \infty$ for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$. So we conclude that $\delta_k(x_1, \cdots, x_n) = \delta_k'(x_1, \cdots, x_n)$. This proves the uniqueness of δ. □

THEOREM 2.2. *Let $p > 3$ be and let θ be nonnegative real numbers. Let $F_k : A_1 \times \cdots \times A_n \to B$ be a mapping with $F_k(x_1, \cdots, 0_k, \cdots, x_n) = 0_B$. Suppose that there exist a cubic mapping $g_k : A_k \to B$ such that satisfying* (1) *and* (2) *for all $a_k, b_k, c_k \in A_k$, $x_i \in A_i$ $(i \neq k)$. Then there exists a unique k-th partial ternary cubic derivation $\delta_k : A_1 \times \cdots \times A_n \to B$ such that*

$$\|F_k(x_1, \cdots, x_n) - \delta_k(x_1, \cdots, x_n)\| \leq \frac{2\theta}{2^p - 4}\|x_k\|^p \tag{10}$$

holds for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$.

Proof. In (1), putting $a_k = \frac{x_k}{2}$ and $b_k = 0_k$, we have

$$\|F_k(x_1, \cdots, x_k, \cdots, x_n) - 8F_k(x_1, \cdots, \tfrac{x_k}{2}, \cdots, x_n)\| \leq \tfrac{\theta}{2.2^p}\|x_k\|^p, \tag{11}$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$. One can use induction on m to show that

$$\begin{aligned} &\|F_k(x_1, \cdots, x_k, \cdots, x_n) - 2^{3m}F_k(x_1, \cdots, \frac{x_k}{2^m}, \cdots, x_n)\| \\ \leq\ & \frac{\theta}{2 \cdot 2^p}\sum_{i=0}^{m-1} 2^{i(3-p)}\|x_k\|^p, \end{aligned} \tag{12}$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$ and all non-negative integers m. Hence

$$\begin{aligned} &\|2^{3j}F_k(x_1, \cdots, \frac{x_k}{2^j}, \cdots, x_n) - 2^{3(m+j)}F_k(x_1, \cdots, \frac{x_k}{2^{(m+j)}}, \cdots, x_n)\| \\ \leq\ & \frac{\theta}{2 \cdot 2^p}\sum_{i=j}^{m+j-1} 2^{i(3-p)}\|x_k\|^p, \end{aligned} \tag{13}$$

for all non-negative integers m and j with $m \geq j$ and all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$. Since $p > 3$, the sequence $\{2^{3m}F_k(x_1, \cdots, \frac{x_k}{2^m}, \cdots, x_n)\}$ is Cauchy. Due to the completeness of B, this sequence is convergent. So one can define the mapping $\delta_k : A_1 \times \cdots \times A_n \to B$ given by

$$\delta_k(x_1, \cdots, x_n) = \lim_{m\to\infty} 2^{3m}F_k(x_1, \cdots, \frac{x_k}{2^m}, \cdots, x_n), \tag{14}$$

for all $x_i \in A_i$ $(i = 1, \cdots, n)$. In (1), replacing a_k, b_k with $\frac{a_k}{2^m}, \frac{b_k}{2^m}$, respectively, we obtain that

$$\begin{aligned} &\|2^{3m}F_k(x_1, \cdots, \frac{2a_k + b_k}{2^m}, \cdots, x_n) + 2^{3m}F_k(x_1, \cdots, \frac{2a_k - b_k}{2^m}, \cdots, x_n) \\ &-2 \cdot 2^{3m}F_k(x_1, \cdots, \frac{a_k + b_k}{2^m}, \cdots, x_n) - 2 \cdot 2^{3m}F_k(x_1, \cdots, \frac{a_k - b_k}{2^m}, \cdots, x_n) \\ &-12 \cdot 2^{3m}F_k(x_1, \cdots, \frac{a_k}{2^m}, \cdots, x_n)\| \leq \theta \cdot 2^{m(3-p)}(\|a_k\|^p + \|b_k\|^p), \end{aligned}$$

which tends to zero as $m \to \infty$. Thus we obtain

$$\begin{aligned} & \delta_k(x_1, \cdots, 2a_k + b_k, \cdots, x_n) + \delta_k(x_1, \cdots, 2a_k - b_k, \cdots, x_n) \\ =\ & 2\delta_k(x_1, \cdots, a_k + b_k, \cdots, x_n) \\ +\ & 2\delta_k(x_1, \cdots, a_k - b_k, \cdots, x_n) + 12\delta_k(x_1, \cdots, a_k, \cdots, x_n), \end{aligned} \tag{15}$$

for all $a_k, b_k \in A_k$ and all $x_i \in A_i (i \neq k)$. Hence δ_k is cubic with respect to the k-th variable. It follows from (12) that

$$\|F_k(x_1, \cdots, x_k, \cdots, x_n) - \delta_k(x_1, \cdots, x_k, \cdots, x_n)\| \leq \frac{\theta}{2(2^p - 8)} \|x_k\|^p,$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$.
Replacing in (2) the elements a_k, b_k, c_k with $\frac{a_k}{2^m}, \frac{b_k}{2^m}, \frac{c_k}{2^m}$, respectively, and multiplying both sides of the inequality by 2^{9m}, we obtain, for all $a_k, b_k, c_k \in A_k$,

$$\begin{aligned}
&\|2^{9m} F_k(x_1, \cdots, \frac{[a_k b_k c_k]}{2^{3m}}, \cdots, x_n) - 2^{9m}[\frac{g_k(a_k)}{2^{3m}} \frac{g_k(b_k)}{2^{3m}} F_k(x_1, \cdots, \frac{c_k}{2^m}, \cdots, x_n)] \\
&- 2^{9m}[\frac{g_k(a_k)}{2^{3m}} F_k(x_1, \cdots, \frac{b_k}{2^m}, \cdots, x_n) \frac{g_k(c_k)}{2^{3m}}] \\
&- 2^{9m}[F_k(x_1, \cdots, \frac{a_k}{2^m}, \cdots, x_n) \frac{g_k(b_k)}{2^{3m}} \frac{g_k(c_k)}{2^{3m}}]\| \\
&\leq 2^{m(9-p)} \cdot \theta(\|a_k\|^p + \|b_k\|^p + \|c_k\|^p).
\end{aligned}$$

Passing the limit $m \to \infty$ in the above inequality, we obtain

$$\begin{aligned}
&\delta_k(x_1, \cdots, [a_k b_k c_k], \cdots, x_n) = [g_k(a_k) g_k(b_k) \delta_k(x_1, \cdots, c_k, \cdots, x_n)] \\
&+ [g_k(a_k) \delta_k(x_1, \cdots, b_k, \cdots, x_n) g_k(c_k)] + [\delta_k(x_1, \cdots, a_k, \cdots, x_n) g_k(b_k) g_k(c_k)],
\end{aligned}$$

for all $a_k, b_k, c_k \in A_k$ and all $x_i \in A_i$ $(i \neq k)$.

Finally, to prove the uniqueness of δ_k, let $\delta_k' : A_1 \times \cdots \times A_n \to B$ be another k-th partial ternary cubic derivation satisfying (10). Then we have

$$\begin{aligned}
&\left\|\delta_k(x_1, \cdots, x_n) - \delta_k'(x_1, \cdots, x_n)\right\| \\
&= 2^{3m} \left\|\delta_k(x_1, \cdots, \frac{x_k}{2^m}, \cdots, x_n) - \delta_k'(x_1, \cdots, \frac{x_k}{2^m}, \cdots, x_n)\right\| \\
&\leq 2^{3m} \left(\left\|\delta_k(x_1, \cdots, \frac{x_k}{2^m}, \cdots, x_n) - F_k(x_1, \cdots, \frac{x_k}{2^m}, \cdots, x_n)\right\|\right. \\
&+ \left.\left\|F_k(x_1, \cdots, \frac{x_k}{2^m}, \cdots, x_n) - \delta_k'(x_1, \cdots, \frac{x_k}{2^m}, \cdots, x_n)\right\|\right) \\
&\leq \theta \sum_{i=m}^{\infty} 2^{i(3-p)} \|x_k\|^p,
\end{aligned}$$

which tends to zero as $m \to \infty$ for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$. So we conclude that $\delta_k(x_1, \cdots, x_n) = \delta_k'(x_1, \cdots, x_n)$. This proves the uniqueness of δ. □

By Theorems 2.1 and 2.2 we solve the following Hyers-Ulam stability problem.

Corollary 2.3. *Let ϵ be nonnegative real numbers and let $F_k : A_1 \times \cdots \times A_n \to B$ be a mapping with $F_k(x_1, \cdots, 0_k, \cdots, x_n) = 0_B$. Assume that there* *exist a cubic mapping* $g_k : A_k \to B$ *such that*

$$\begin{aligned} &\|F_k(x_1,\cdots,2a_k+b_k,\cdots,x_n)+F_k(x_1,\cdots,2a_k-b_k,\cdots,x_n)\\ &-2F_k(x_1,\cdots,a_k+b_k,\cdots,x_n)-2F_k(x_1,\cdots,a_k-b_k,\cdots,x_n)\\ &-12F_k(x_1,\cdots,a_k,\cdots,x_n)\|\le\epsilon, \end{aligned} \tag{16}$$

$$\begin{aligned} &\|F_k(x_1,\cdots,[a_kb_kc_k],\cdots,x_n)-[g_k(a_k)g_k(b_k)F_k(x_1,\cdots,c_k,\cdots,x_n)]\\ &-[g_k(a_k)F_k(x_1,\cdots,b_k,\cdots,x_n)g_k(c_k)]\\ &-[F_k(x_1,\cdots,a_k,\cdots,x_n)g_k(b_k)g_k(c_k)]\|\le\epsilon, \end{aligned} \tag{17}$$

for all $a_k, b_k, c_k \in A_k$, $x_i \in A_i$ $(i \neq k)$. *Then there exists a unique* k*-th partial ternary cubic derivation* $\delta_k : A_1 \times \cdots \times A_n \to B$ *such that*

$$\|F_k(x_1,\cdots,x_n)-\delta_k(x_1,\cdots,x_n)\|\le\frac{\epsilon}{14} \tag{18}$$

holds for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$.

Proof. Put $p := 0$, $\theta := \epsilon$, and apply Theorem 2.1. □

THEOREM 2.4. *Let* $F_k : A_1 \times \cdots \times A_n \to B$ *be a mapping with*

$$F_k(x_1,\cdots,0_k,\cdots,x_n)=0_B.$$

Assume that there exist a function $\varphi_k : A_k \times A_k \times A_k \to [0,\infty)$ *and a cubic mapping* $g_k : A_k \to B$ *such that*

$$\lim_{m\to\infty}\frac{1}{2^{3m}}\varphi_k(2^m a_k,2^m b_k,2^m c_k)=0,$$

$$\tilde{\varphi_k}(a_k,b_k,c_k):=\sum_{m=0}^{\infty}\frac{1}{2^{3m}}\varphi_k(2^m a_k,2^m b_k,2^m c_k)<\infty,$$

$$\begin{aligned} &\|F_k(x_1,\cdots,2a_k+b_k,\cdots,x_n)+F_k(x_1,\cdots,2a_k-b_k,\cdots,x_n)\\ &-2F_k(x_1,\cdots,a_k+b_k,\cdots,x_n)\\ &-2F_k(x_1,\cdots,a_k-b_k,\cdots,x_n)-12F_k(x_1,\cdots,a_k,\cdots,x_n)\|\\ &\le\varphi_k(a_k,b_k,0_k) \end{aligned} \tag{19}$$

and

$$\begin{aligned} &\|F_k(x_1,\cdots,[a_kb_kc_k],\cdots,x_n)-[g_k(a_k)g_k(b_k)F_k(x_1,\cdots,c_k,\cdots,x_n)]\\ &-[g_k(a_k)F_k(x_1,\cdots,b_k,\cdots,x_n)g_k(c_k)]\\ &-[F_k(x_1,\cdots,a_k,\cdots,x_n)g_k(b_k)g_k(c_k)]\|\le\varphi_k(a_k,b_k,c_k), \end{aligned} \tag{20}$$

for all $a_k, b_k, c_k \in A_k$, $x_i \in A_i$ $(i \neq k)$. *Then there exists a unique* k*-th partial cubic derivation* $\delta_k : A_1 \times \cdots \times A_n \to B$ *such that*

$$\|F_k(x_1,\cdots,x_n)-\delta_k(x_1,\cdots,x_n)\|\le\frac{\tilde{\varphi_k}(x_k,0_k,0_k)}{16}, \tag{21}$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$.

Proof. In (19), putting $a_k = x_k$ and $b_k = 0_k$, we have

$$\|2F_k(x_1, \cdots, 2x_k, \cdots, x_n) - 16F_k(x_1, \cdots, x_k, \cdots, x_n)\| \leq \varphi_k(x_k, 0_k, 0_k), \tag{22}$$

that is,

$$\|F_k(x_1, \cdots, x_k, \cdots, x_n) - \frac{1}{8}F_k(x_1, \cdots, 2x_k, \cdots, x_n)\| \leq \frac{\varphi_k(x_k, 0_k, 0_k)}{16}, \tag{23}$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$. One can use induction on m to show that

$$\begin{aligned} &\|F_k(x_1, \cdots, x_k, \cdots, x_n) - \frac{1}{2^{3m}}F_k(x_1, \cdots, 2^m x_k, \cdots, x_n)\| \\ \leq &\frac{1}{16}\sum_{i=0}^{m-1} \frac{\varphi_k(2^i x_k, 0_k, 0_k)}{2^{3i}}, \end{aligned} \tag{24}$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$ and all non-negative integers m. For any positive integer j, dividing the both sides by 2^{3j} and replacing x_k by $2^j x_k$ in (24), we have

$$\begin{aligned} &\|\frac{1}{2^{3j}}F_k(x_1, \cdots, 2^j x_k, \cdots, x_n) - \frac{1}{2^{3(m+j)}}F_k(x_1, \cdots, 2^{(m+j)}x_k, \cdots, x_n)\| \\ \leq &\frac{1}{16}\sum_{i=j}^{m+j-1} \frac{\varphi_k(2^i x_k, 0_k, 0_k)}{2^{3i}}, \end{aligned} \tag{25}$$

which tends to zero as $j \to \infty$. So the sequence $\{\frac{1}{2^{3m}}F_k(x_1, \cdots, 2^m x_k, \cdots, x_n)\}$ is a Cauchy sequence in B. By the completeness of B, this sequence is convergent and so we can define a mapping $\delta_k : A_1 \times \cdots \times A_n \to B$ given by

$$\delta_k(x_1, \cdots, x_n) = \lim_{m\to\infty} \frac{1}{2^{3m}}F_k(x_1, \cdots, 2^m x_k, \cdots, x_n), \tag{26}$$

for all $x_i \in A_i$ $(i = 1, \cdots, n)$. In (19), replacing a_k, b_k with $2^m a_k, 2^m b_k$, respectively, we obtain that

$$\begin{aligned} &\|\frac{1}{2^{3m}}F_k(x_1, \cdots, 2^m(2a_k + b_k), \cdots, x_n)\frac{1}{2^{3m}}F_k(x_1, \cdots, 2^m(2a_k - b_k), \cdots, x_n) \\ &- \frac{2}{2^{3m}}F_k(x_1, \cdots, 2^m(a_k - b_k), \cdots, x_n) \\ &- \frac{2}{2^{3m}}F_k(x_1, \cdots, 2^m(a_k - b_k), \cdots, x_n) - \frac{12}{2^{3m}}F_k(x_1, \cdots, 2^m a_k, \cdots, x_n)\| \\ &\leq \frac{\varphi_k(2^m a_k, 2^m b_k, 0_k)}{2^{3m}}, \end{aligned}$$

which tends to zero as $m \to \infty$. Thus we obtain

$$\begin{aligned} &\delta_k(x_1, \cdots, 2a_k + b_k, \cdots, x_n) + \delta_k(x_1, \cdots, 2a_k - b_k, \cdots, x_n) \\ =&2\delta_k(x_1, \cdots, a_k + b_k, \cdots, x_n) \\ &+2\delta_k(x_1, \cdots, a_k - b_k, \cdots, x_n) + 12\delta_k(x_1, \cdots, a_k, \cdots, x_n), \end{aligned} \tag{27}$$

for all $a_k, b_k \in A_k$ and all $x_i \in A_i (i \neq k)$. Hence δ_k is cubic with respect to the k-th variable. It follows from (24) that

$$\|F_k(x_1, \cdots, x_k, \cdots, x_n) - \delta_k(x_1, \cdots, x_k, \cdots, x_n)\| \leq \frac{\tilde{\varphi}_k(x_k, 0_k, 0_k)}{16},$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$.
Replacing in (20) the elements a_k, b_k, c_k with $2^m a_k, 2^m b_k, 2^m c_k$, respectively, and dividing both sides of the inequality by 2^{9m}, we obtain, for all $a_k, b_k, c_k \in A_k$,

$$\begin{aligned}
&\|\frac{1}{2^{9m}} F_k(x_1, \cdots, 2^{3m}[a_k b_k c_k], \cdots, x_n)\\
&-\frac{1}{2^{9m}}[2^{3m} g_k(a_k) 2^{3m} g_k(b_k) F_k(x_1, \cdots, 2^m c_k, \cdots, x_n)]\\
&-\frac{1}{2^{9m}}[2^{3m} g_k(a_k) F_k(x_1, \cdots, 2^m b_k, \cdots, x_n) 2^{3m} g_k(c_k)]\\
&-\frac{1}{2^{9m}}[F_k(x_1, \cdots, 2^m a_k, \cdots, x_n) 2^{3m} g_k(b_k) 2^{3m} g_k(c_k)]\|\\
&\leq \frac{\varphi_k(2^m a_k, 2^m b_k, 2^m c_k)}{2^{9m}}.
\end{aligned}$$

Passing the limit $m \to \infty$ in the above inequality, we get

$$\begin{aligned}
&\delta_k(x_1, \cdots, [a_k b_k c_k], \cdots, x_n) = [g_k(a_k) g_k(b_k) \delta_k(x_1, \cdots, c_k, \cdots, x_n)]\\
&+[g_k(a_k) \delta_k(x_1, \cdots, b_k, \cdots, x_n) g_k(c_k)] + [\delta_k(x_1, \cdots, a_k, \cdots, x_n) g_k(b_k) g_k(c_k)],
\end{aligned}$$

for all $a_k, b_k, c_k \in A_k$ and all $x_i \in A_i$ $(i \neq k)$.

Finally, to prove the uniqueness of δ_k, let $\delta_k^{'} : A_1 \times \cdots \times A_n \to B$ be another k-th partial ternary cubic derivation satisfying (21). Then we have

$$\begin{aligned}
&\|\delta_k(x_1, \cdots, x_n) - \delta_k^{'}(x_1, \cdots, x_n)\|\\
&=\frac{1}{2^{3m}}\|\delta_k(x_1, \cdots, 2^m x_k, \cdots, x_n) - \delta_k^{'}(x_1, \cdots, 2^m x_k, \cdots, x_n)\|\\
&\leq \frac{1}{2^{3m}}(\|\delta_k(x_1, \cdots, 2^m x_k, \cdots, x_n) - F_k(x_1, \cdots, 2^m x_k, \cdots, x_n)\|\\
&+\|F_k(x_1, \cdots, 2^m x_k, \cdots, x_n) - \delta_k^{'}(x_1, \cdots, 2^m x_k, \cdots, x_n)\|)\\
&\leq \frac{\tilde{\varphi}_k(2^m x_k, 0_k, 0_k)}{8 \cdot 2^{3m}},
\end{aligned}$$

which tends to zero as $m \to \infty$ for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$. So we conclude that $\delta_k(x_1, \cdots, x_n) = \delta_k^{'}(x_1, \cdots, x_n)$. This proves the uniqueness of δ. □

THEOREM 2.5. *Let $F_k : A_1 \times \cdots \times A_n \to B$ be a mapping with*

$$F_k(x_1, \cdots, 0_k, \cdots, x_n) = 0_B.$$

Assume that there exist a function $\varphi_k : A_k \times A_k \times A_k \to [0, \infty)$ *and a cubic mapping* $g_k : A_k \to B$ *such that satisfying* (19), (20),

$$\lim_{m\to\infty} 2^{3m}\varphi_k(\frac{a_k}{2^m}, \frac{b_k}{2^m}, \frac{c_k}{2^m}) = 0$$

and

$$\tilde{\varphi_k}(a_k, b_k, c_k) := \sum_{m=1}^{\infty} 2^{3m}\varphi_k(\frac{a_k}{2^m}, \frac{b_k}{2^m}, \frac{c_k}{2^m}) < \infty,$$

for all $a_k, b_k, c_k \in A_k$, $x_i \in A_i$ $(i \neq k)$. *Then there exists a unique k-th partial cubic derivation* $\delta_k : A_1 \times \cdots \times A_n \to B$ *such that*

$$\|F_k(x_1, \cdots, x_n) - \delta_k(x_1, \cdots, x_n)\| \leq \frac{\tilde{\varphi_k}(x_k, 0_k, 0_k)}{16}, \tag{28}$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$.

Proof. In (19), putting $a_k = \frac{x_k}{2}$ and $b_k = 0_k$, we have

$$\|F_k(x_1, \cdots, x_k, \cdots, x_n) - 8F_k(x_1, \cdots, \frac{x_k}{2}, \cdots, x_n)\| \leq \frac{1}{2}\varphi_k(\frac{x_k}{2}, 0_k, 0_k), \tag{29}$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$. One can use induction on m to show that

$$\|F_k(x_1, \cdots, x_k, \cdots, x_n) - 2^{3m}F_k(x_1, \cdots, \frac{x_k}{2^m}, \cdots, x_n)\|$$

$$\leq \frac{1}{16}\sum_{i=1}^{m} 8^i\varphi_k(\frac{x_k}{2^i}, 0_k, 0_k), \tag{30}$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$ and all non-negative integers m. Hence

$$\|2^{3j}F_k(x_1, \cdots, \frac{x_k}{2^j}, \cdots, x_n) - 2^{3(m+j)}F_k(x_1, \cdots, \frac{x_k}{2^{(m+j)}}, \cdots, x_n)\|$$

$$\leq \frac{1}{16}\sum_{i=j+1}^{m+j} 8^i\varphi_k(\frac{x_k}{2^i}, 0_k, 0_k), \tag{31}$$

which tends to zero as $j \to \infty$. So the sequence $\{2^{3m}F_k(x_1, \cdots, \frac{x_k}{2^m}, \cdots, x_n)\}$ is a Cauchy sequence in B. By the completeness of B, this sequence is convergent and so we can define a mapping $\delta_k : A_1 \times \cdots \times A_n \to B$ given by

$$\delta_k(x_1, \cdots, x_n) = \lim_{m\to\infty} 2^{3m}F_k(x_1, \cdots, \frac{x_k}{2^m}, \cdots, x_n), \tag{32}$$

for all $x_i \in A_i$ $(i = 1, \cdots, n)$. In (19), replacing a_k, b_k with $\frac{a_k}{2^m}, \frac{b_k}{2^m}$, respectively, we obtain that

$$\|2^{3m}F_k(x_1, \cdots, \frac{(2a_k + b_k)}{2^m}, \cdots, x_n) + 2^{3m}F_k(x_1, \cdots, \frac{(2a_k - b_k)}{2^m}, \cdots, x_n)$$
$$-2.2^{3m}F_k(x_1, \cdots, \frac{(a_k - b_k)}{2^m}, \cdots, x_n) - 2.2^{3m}F_k(x_1, \cdots, \frac{(a_k - b_k)}{2^m}, \cdots, x_n)$$
$$-12 \cdot 2^{3m}F_k(x_1, \cdots, \frac{a_k}{2^m}, \cdots, x_n)\| \leq 2^{3m}\varphi_k(\frac{a_k}{2^m}, \frac{b_k}{2^m}, 0_k),$$

which tends to zero as $m \to \infty$. Thus we obtain

$$\begin{aligned} &\delta_k(x_1, \cdots, 2a_k + b_k, \cdots, x_n) + \delta_k(x_1, \cdots, 2a_k - b_k, \cdots, x_n) \\ &= 2\delta_k(x_1, \cdots, a_k + b_k, \cdots, x_n) \\ &+ 2\delta_k(x_1, \cdots, a_k - b_k, \cdots, x_n) + 12\delta_k(x_1, \cdots, a_k, \cdots, x_n), \end{aligned} \tag{33}$$

for all $a_k, b_k \in A_k$ and all $x_i \in A_i (i \neq k)$. Hence δ_k is cubic with respect to the k-th variable. It follows from (30) that

$$\|F_k(x_1, \cdots, x_k, \cdots, x_n) - \delta_k(x_1, \cdots, x_k, \cdots, x_n)\| \leq \frac{\tilde{\varphi_k}(x_k, 0_k, 0_k)}{16},$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$.
Replacing in (20) the elements a_k, b_k, c_k with $\frac{a_k}{2^m}, \frac{b_k}{2^m}, \frac{c_k}{2^m}$, respectively, and multiplying both sides of the inequality by 2^{9m}, we get, for all $a_k, b_k, c_k \in A_k$,

$$\begin{aligned} &\|2^{9m} F_k(x_1, \cdots, \frac{[a_k b_k c_k]}{2^{3m}}, \cdots, x_n) \\ &-2^{9m}[\frac{g_k(a_k)}{2^{3m}} \frac{g_k(b_k)}{2^{3m}} F_k(x_1, \cdots, \frac{c_k}{2^m}, \cdots, x_n)] \\ &-2^{9m}[\frac{g_k(a_k)}{2^{3m}} F_k(x_1, \cdots, \frac{b_k}{2^m}, \cdots, x_n) \frac{g_k(c_k)}{2^{3m}}] \\ &-2^{9m}[F_k(x_1, \cdots, \frac{a_k}{2^m}, \cdots, x_n) \frac{g_k(b_k)}{2^{3m}} \frac{g_k(c_k)}{2^{3m}}]\| \\ &\leq 2^{9m} \varphi_k(\frac{a_k}{2^m}, \frac{b_k}{2^m}, \frac{c_k}{2^m}). \end{aligned}$$

Passing the limit $m \to \infty$ in the above inequality, we obtain

$$\begin{aligned} &\delta_k(x_1, \cdots, [a_k b_k c_k], \cdots, x_n) = [g_k(a_k) g_k(b_k) \delta_k(x_1, \cdots, c_k, \cdots, x_n)] \\ &+[g_k(a_k)\delta_k(x_1, \cdots, b_k, \cdots, x_n) g_k(c_k)] + [\delta_k(x_1, \cdots, a_k, \cdots, x_n) g_k(b_k) g_k(c_k)], \end{aligned}$$

for all $a_k, b_k, c_k \in A_k$ and all $x_i \in A_i$ $(i \neq k)$.

The uniqueness of δ follows as in the proof of Theorem 2.4. □

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