MATHEMATICA, Tome 54 (77), N° 1, 2012, pp. 78–83

THE MOMENTS OF BERNSTEIN-STANCU OPERATORS REVISITED

DAN MICLĂUŞ

Abstract. In the present paper the author establishes a general formula concerning calculation of the test functions by Bernstein-Stancu operators and also gives in this general case, an appropriate application.

MSC 2010. 41A10, 41A36.

Key words. Bernstein operators, Bernstein-Stancu operators, test functions, falling factorials, Pochhammer symbol, Stirling numbers of second kind.

1. INTRODUCTION

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The operators $B_n : C([0,1]) \to C([0,1])$ given by

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1}$$

where $p_{n,k}$ are the fundamental Bernstein's polynomials defined by

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \tag{2}$$

for any $x \in [0,1]$, $k \in \{0,1,\ldots,n\}$ and $n \in \mathbb{N}$, are called Bernstein operators. These operators were first introduced by S. N. Bernstein in [2].

Let α be a non-negative parameter, which may depend only on the natural number n. The operators $P_n^{\langle\alpha\rangle} : C([0,1]) \to C([0,1])$ given by

$$P_n^{\langle\alpha\rangle}(f;x) = \sum_{k=0}^{n} p_{n,k}^{\langle\alpha\rangle}(x) f\left(\frac{k}{n}\right), \tag{3}$$

where $p_{n,k}^{\langle\alpha\rangle}$ is a polynomial, which can be expressed by means of the factorial power $t^{[n,h]} = t(t-h)\cdot\ldots\cdot(t-\overline{n-1}h)$, (the n^{th} factorial power of t with increment h), defined by

$$p_{n,k}^{\langle\alpha\rangle}(x) = \frac{1}{1^{[n,-\alpha]}}\binom{n}{k} x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}, \tag{4}$$

for any $x \in [0,1]$, $k \in \{0,1,\ldots,n\}$ and $n \in \mathbb{N}$, are called Bernstein-Stancu operators. These operators were first introduced by D. D. Stancu in [6]. He

The author gratefully acknowledges the helpful remarks of referee.

has investigated this linear polynomial operator of Bernstein type, in order to use it in the theory of uniform approximation of functions.

In the case $\alpha = 0$, the operators (3) reduce, obviously, to the classical Bernstein operators.

The purpose of this paper is to establish a general result concerning calculation of the moments for Bernstein-Stancu operators, similar with the result proved first in 1970, by S. Karlin and Z. Ziegler [3], for Bernstein operators. To reach this aim, we give two ways starting from some results established in [6], by D. D. Stancu.

Remark. Some properties and results concerning the Bernstein-Stancu operators can also be found in the monograph [1].

2. PRELIMINARIES

Of the greatest utility in the calculus of finite differences, in number theory, in the summation of series, in the calculation of the Bernstein polynomials are the numbers introduced in 1730 by J. Stirling in his *Methodus differentialis* [7], subsequently called "Stirling numbers" of the first and second kind.

For any $x \in \mathbb{R}$ and any $n \in \mathbb{N}_0$, let $(x)_n := \prod_{i=0}^{n-1} (x-i)$, where $(x)_0 := 1$ be the falling factorial denoted by Pochhammer symbol. The Stirling numbers of second kind have the following properties

$$S(j,i) := \begin{cases} 1, & \text{if } j=i=0;\ j=i \textit{ or } j>1,\ i=1; \\ S(j-1,i-1) + i \cdot S(j-1,i), & \text{if } j,i>1, \end{cases} \tag{5}$$

which implies

$$S(j,i) = \frac{1}{i!} \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} k^j. \tag{6}$$

Let $e_j(x) = x^j$, $j \in \mathbb{N}$, be the test functions. The moments of Bernstein operators have been established in terms of the Stirling numbers of second kind,

$$B_n(e_j;x) = \frac{1}{n^j} \sum_{i=1}^{j} S(j,i)(n)_i x^j, \tag{7}$$

for any $n, j \in \mathbb{N}$, by S. Karlin and Z. Ziegler [3].

Later, in [5] the authors proved another result concerning calculation of the test function by Bernstein operators. Before to mention the result we set that, for any $n \in \mathbb{N}_0$ and $k \in \mathbb{Z}\backslash\{0,1,\ldots,n\}$, the $\binom{n}{k} = 0$ and $A_n^k = 0$, where A_n^k are arrangements of n taken k.

THEOREM 1 ([5]). *For any $j,n \in \mathbb{N}$ and any $x \in [0,1]$, the following holds*

$$B_n(e_j;x) = \frac{1}{n^j}\sum_{i=0}^{j-1} a_j^{(i)} A_n^{j-i} x^{j-i}, \tag{8}$$

where

$$a_j^{(i)} > 0, \quad i = \overline{1, j-2}, \qquad a_j^{(0)} = a_j^{(j-1)} = 1 \tag{9}$$

and

$$a_{j+1}^{(i)} = (j-i+1)a_j^{(i-1)} + a_j^{(i)}, \quad 1 \le i \le j-1. \tag{10}$$

Remark. In [4] we established that the result proved in Theorem 1 does not differ from the result given at (7) and in addition, between (7), respectively (8) there exists a relationship given by

$$B_n(e_j;x) = \frac{1}{n^j}\sum_{i=0}^{j-1} S(j,j-i)(n)_{j-i}x^{j-i}. \tag{11}$$

For the Bernstein-Stancu operators the following result holds.

THEOREM 2 ([6]). a) *The operators $P_n^{\langle\alpha\rangle}$ can be represented by*

$$P_n^{\langle\alpha\rangle}(f;x) = f(0) + \sum_{i=1}^{n}\binom{n}{i}\frac{x(x+\alpha)\dots(x+\overline{i-1}\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+\overline{i-1}\alpha)}\Delta^i_{\frac{1}{n}}f(0), \tag{12}$$

where $\Delta^i_{\frac{1}{n}}f(0)$ is the finite difference of order i, with the step $\frac{1}{n}$ and the starting point 0 of the function f, that is,

$$\Delta^i_{\frac{1}{n}}f(0) = \sum_{\upsilon=0}^{i}(-1)^{\upsilon}\binom{i}{\upsilon}f\left(\frac{i-\upsilon}{n}\right). \tag{13}$$

b) *The operators $P_n^{\langle\alpha\rangle}$ can be represented by*

$$P_n^{\langle\alpha\rangle}(f;x) = \frac{1}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}B_n(f;t)\mathrm{d}t, \tag{14}$$

for any $x \in]0,1[$, any $n \in \mathbb{N}$ and $\alpha > 0$, where

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\mathrm{d}t, \qquad a,b \in \mathbb{R}_+ \tag{15}$$

is the Beta function or Euler function of first kind, which has the property

$$B(a+1,b) = \frac{a}{a+b}B(a,b). \tag{16}$$

3. MAIN RESULTS

In order to establish a general formula concerning calculation of the test functions by Bernstein-Stancu operators, we need to prove the following.

LEMMA 3. *Let α be a non-negative parameter, which may depend only on the natural number n. If $x \in]0,1[$, then*

$$B\left(\frac{x}{\alpha}+j-i, \frac{1-x}{\alpha}\right) = \frac{x^{[j-i,-\alpha]}}{1^{[j-i,-\alpha]}} \cdot B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right), \tag{17}$$

where $x^{[j-i,-\alpha]} = x(x+\alpha)\cdot\ldots\cdot(x+\overline{j-i-1}\alpha)$ and $1^{[j-i,-\alpha]} = (1+\alpha)\cdot\ldots\cdot(1+\overline{j-i-1}\alpha)$.

Proof. The relation (17) can be proved taking the mathematical induction into account. Assume that (17) holds. Using the property (16) of the Beta function, it follows that

$$B\left(\frac{x}{\alpha}+j-i+1, \frac{1-x}{\alpha}\right) = \frac{\frac{x}{\alpha}+j-i}{\frac{x}{\alpha}+j-i+\frac{1}{\alpha}-\frac{x}{\alpha}} B\left(\frac{x}{\alpha}+j-i, \frac{1-x}{\alpha}\right)$$

$$= \frac{(x+\overline{j-i}\alpha)x^{[j-i,-\alpha]}}{(1+\overline{j-i}\alpha)1^{[j-i,-\alpha]}} B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right) = \frac{x^{[j-i+1,-\alpha]}}{1^{[j-i+1,-\alpha]}} B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right). \qquad \square$$

THEOREM 4. *For any $j,n \in \mathbb{N}$, any $x \in]0,1[$ and $\alpha > 0$, the following*

$$P_n^{\langle\alpha\rangle}(e_j;x) = \frac{1}{n^j}\sum_{i=0}^{j-1} S(j,j-i)(n)_{j-i}\frac{x^{[j-i,-\alpha]}}{1^{[j-i,-\alpha]}} \tag{18}$$

holds.

Proof. Using the representation (14) for the Bernstein-Stancu operators, it follows that

$$P_n^{\langle\alpha\rangle}(e_j;x) = \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)}\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}B_n(e_j;t)\mathrm{d}t,$$

then by (11), (15) and (17), one obtains

$$\begin{aligned}P_n^{\langle\alpha\rangle}(e_j;x) &= \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)}\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\cdot\frac{1}{n^j}\sum_{i=0}^{j-1}S(j,j-i)(n)_{j-i}t^{j-i}\mathrm{d}t\\
&= \frac{1}{n^j}\sum_{i=0}^{j-1}S(j,j-i)(n)_{j-i}\frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)}\int_0^1 t^{\frac{x}{\alpha}+j-i-1}(1-t)^{\frac{1-x}{\alpha}-1}\mathrm{d}t\\
&= \frac{1}{n^j}\sum_{i=0}^{j-1}S(j,j-i)(n)_{j-i}\frac{B\left(\frac{x}{\alpha}+j-i, \frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)}\\
&= \frac{1}{n^j}\sum_{i=0}^{j-1}S(j,j-i)(n)_{j-i}\frac{x^{[j-i,-\alpha]}}{1^{[j-i,-\alpha]}}. \qquad \square\end{aligned}$$

Corollary 5. *The degree of the polynomial* $P_n^{\langle\alpha\rangle} e_j$ *is*

$$d_{n,j} := \begin{cases} j, & if \quad j \le n \\ n, & if \quad j > n, \end{cases} \tag{19}$$

for any $n, j \in \mathbb{N}$ *and* $\alpha > 0$.

Proof. It follows immediately from Theorem 4. □

Remark. The formula (18) can also be proved by using the representation (12) of Bernstein-Stancu operators, such that, taking (13), (6) into account and the fact that $\Delta_h^i e_j = 0$, if $i > j$, it follows that

$$P_n^{\langle\alpha\rangle}(e_j; x) = \sum_{i=1}^{j} \binom{n}{i} \frac{x^{[i,-\alpha]}}{1^{[i,-\alpha]}} \sum_{\upsilon=0}^{i} (-1)^{\upsilon} \binom{i}{\upsilon} \left(\frac{i-\upsilon}{n}\right)^j$$

$$= \frac{1}{n^j} \sum_{i=1}^{j} \binom{n}{i} \cdot i! S(j,i) \frac{x^{[i,-\alpha]}}{1^{[i,-\alpha]}} = \frac{1}{n^j} \sum_{i=1}^{j} S(j,i)(n)_i \frac{x^{[i,-\alpha]}}{1^{[i,-\alpha]}}$$

$$= \frac{1}{n^j} \sum_{i=1}^{j} S(j, j+1-i)(n)_{j+1-i} \frac{x^{[j+1-i,-\alpha]}}{1^{[j+1-i,-\alpha]}} = \frac{1}{n^j} \sum_{i=0}^{j-1} S(j, j-i)(n)_{j-i} \frac{x^{[j-i,-\alpha]}}{1^{[j-i,-\alpha]}}.$$

Application 6. In what follows, for $j \in \{1, 2, 3, 4\}$ we present the first four cases concerning calculation of the test functions by Bernstein-Stancu operators.

Case 1: $j = 1$. $P_n^{\langle\alpha\rangle}(e_1; x) = \frac{1}{n} S(1,1)(n)_1 \frac{x^{[1,-\alpha]}}{1^{[1,-\alpha]}} = \frac{1}{n} nx = x$.

Case 2: $j = 2$.

$$P_n^{\langle\alpha\rangle}(e_2; x) = \frac{1}{n^2} \sum_{i=0}^{1} S(2, 2-i)(n)_{2-i} \frac{x^{[2-i,-\alpha]}}{1^{[2-i,-\alpha]}}$$

$$= \frac{1}{n^2} \left(S(2,2)(n)_2 \frac{x^{[2,-\alpha]}}{1^{[2,-\alpha]}} + S(2,1)(n)_1 \frac{x^{[1,-\alpha]}}{1^{[1,-\alpha]}} \right)$$

$$= \frac{1}{n^2} \left(\frac{n(n-1)x(x+\alpha)}{1+\alpha} + nx \right) = \frac{1}{1+\alpha} \left(\frac{x(1-x)}{n} + x(x+\alpha) \right).$$

Case 3: $j = 3$.

$$P_n^{\langle\alpha\rangle}(e_3; x) = \frac{1}{n^3} \sum_{i=0}^{2} S(3, 3-i)(n)_{3-i} \frac{x^{[3-i,-\alpha]}}{1^{[3-i,-\alpha]}}$$

$$= \frac{1}{n^3} \left(S(3,3)(n)_3 \frac{x^{[3,-\alpha]}}{1^{[3,-\alpha]}} + S(3,2)(n)_2 \frac{x^{[2,-\alpha]}}{1^{[2,-\alpha]}} + S(3,1)(n)_1 \frac{x^{[1,-\alpha]}}{1^{[1,-\alpha]}} \right)$$

$$= \frac{(n-1)(n-2)x(x+\alpha)(x+2\alpha)}{n^2(1+\alpha)(1+2\alpha)} + \frac{3(n-1)x(x+\alpha)}{n^2(1+\alpha)} + \frac{x}{n^2},$$

where $S(3,2) = 2 \cdot S(2,2) + S(2,1) = 3$.

Case 4: $j=4$.

$$P_n^{\langle\alpha\rangle}(e_4;x)=\frac{1}{n^4}\sum_{i=0}^{3}S(4,4-i)(n)_{4-i}\frac{x^{[4-i,-\alpha]}}{1^{[4-i,-\alpha]}}$$
$$=\frac{1}{n^4}\left(S(4,4)(n)_4\frac{x^{[4,-\alpha]}}{1^{[4,-\alpha]}}+S(4,3)(n)_3\frac{x^{[3,-\alpha]}}{1^{[3,-\alpha]}}+S(4,2)(n)_2\frac{x^{[2,-\alpha]}}{1^{[2,-\alpha]}}\right.$$
$$\left.+S(4,1)(n)_1\frac{x^{[1,-\alpha]}}{1^{[1,-\alpha]}}\right)=\frac{1}{n^4}\left(\frac{(n)_4x(x+\alpha)(x+2\alpha)(x+3\alpha)}{(1+\alpha)(1+2\alpha)(1+3\alpha)}\right.$$
$$\left.+\frac{6(n)_3x(x+\alpha)(x+2\alpha)}{(1+\alpha)(1+2\alpha)}+\frac{7(n)_2x(x+\alpha)}{1+\alpha}+(n)_1x\right),$$

where $S(4,2)=2\cdot S(3,2)+S(3,1)=7$ and $S(4,3)=3\cdot S(3,3)+S(3,2)=6$.

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Received October 20, 2010
Accepted March 4, 2011

North University of Baia Mare
Department of Mathematics
and Computer Science
Str. Victoriei 76
430122 Baia Mare, Romania
E-mail: danmiclausrz@yahoo.com