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THE SECOND HANKEL DETERMINANT $H_2(n)$ FOR ODD STARLIKE AND CONVEX FUNCTIONS

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Abstract. For odd starlike and convex functions f defined on the open unit disk $\mathbb{U}$, the upper bounds of the functional $|a_n a_{n+2} - a_{n+1}^2|$, defined by using the second Hankel determinant $H_2(n)$ due to J. W. Noonan and D. K. Thomas (see [4]), are studied. Furthermore, applying the second Hankel determinant $H_2(n)$, a new operator $\mathcal{H}$ is introduced and the properties of new functions $\mathcal{H}f$ are discussed.

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Key words. Hankel determinant, odd analytic function, odd starlike function, odd convex function.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic on the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Furthermore, let $\mathcal{P}$ denote the class of functions p of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k, \tag{2}$$

which are analytic on $\mathbb{U}$ and satisfy

$$\operatorname{Re} p(z) > 0 \qquad (z \in \mathbb{U}).$$

Every element $p \in \mathcal{P}$ is called a *Carathéodory function* (cf. [1]).

If $f \in \mathcal{A}$ satisfies the following inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}) \tag{3}$$

for some α $(0 \leq \alpha < 1)$, then f is said to be *starlike of order* α *in* $\mathbb{U}$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of all functions f which are starlike of order α in $\mathbb{U}$.

Similarly, we say that f is a member of the class $\mathcal{K}(\alpha)$ of *convex functions of order* α *in* $\mathbb{U}$ if $f \in \mathcal{A}$ satisfies the following inequality

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathbb{U}) \tag{4}$$

for some α $(0 \leq \alpha < 1)$.

Also, let $\mathcal{A}_{\text{odd}} \subset \mathcal{A}$ be the class of odd functions f normalized by

$$f(z) = z + \sum_{m=1}^{\infty} a_{2m+1} z^{2m+1}, \tag{5}$$

which are analytic on $\mathbb{U}$. Moreover, we define the following subclasses of $\mathcal{A}_{\text{odd}}$

$$\mathcal{S}^*_{\text{odd}}(\alpha) = \mathcal{A}_{\text{odd}} \cap \mathcal{S}^*(\alpha), \quad \mathcal{K}_{\text{odd}}(\alpha) = \mathcal{A}_{\text{odd}} \cap \mathcal{K}(\alpha).$$

A function $f \in \mathcal{S}^*_{\text{odd}}(\alpha)$ is called an *odd starlike function of order* α, while an element $f \in \mathcal{K}_{\text{odd}}(\alpha)$ is a *convex function of order* α.

For simplicity we write

$$\mathcal{S}^*_{\text{odd}} = \mathcal{S}^*_{\text{odd}}(0) \quad \text{and} \quad \mathcal{K}_{\text{odd}} = \mathcal{K}_{\text{odd}}(0).$$

Remark 1. Let $f \in \mathcal{A}_{\text{odd}}$. Then

$$f(z) \in \mathcal{K}_{\text{odd}}(\alpha) \quad \text{if and only if} \quad zf'(z) \in \mathcal{S}^*_{\text{odd}}(\alpha)$$

and

$$f(z) \in \mathcal{S}^*_{\text{odd}}(\alpha) \quad \text{if and only if} \quad \int_0^z \frac{f(\zeta)}{\zeta} \mathrm{d}\zeta \in \mathcal{K}_{\text{odd}}(\alpha).$$

Example 2. The function f defined by

$$f(z) = \frac{z}{(1-z^2)^{1-\alpha}}$$

belongs to $\mathcal{S}^*_{\text{odd}}(\alpha)$, while the function g given by

$$g(z) = z \, {}_2F_1\left(\frac{1}{2}, 1-\alpha; \frac{3}{2}; z^2\right),$$

where ${}_2F_1(a, b; c; z)$ represents the hypergeometric function, lies in $\mathcal{K}_{\text{odd}}(\alpha)$.

In [4], Noonan and Thomas introduced the q–th Hankel determinant as

$$H_q(n) = \det \begin{pmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{pmatrix} \qquad (n, q \in \mathbb{N} = \{1, 2, 3, \cdots\}).$$

This determinant has been discussed by several authors. For example, it is known that the Fekete and Szegö functional $|a_3 - a_2^2|$ is equal to $|H_2(1)|$ (see, [2]), and that the functional $|a_2 a_4 - a_3^2|$ is equivalent to $|H_2(2)|$.

Janteng, Halim, and Darus showed in [3] the following theorems.

Theorem 3. *Let $f \in \mathcal{S}^*$. Then*

$$|a_2 a_4 - a_3^2| \le 1.$$

Equality is attained for functions of the following form

$$f(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + 4z^4 + \cdots$$

and

$$f(z) = \frac{z}{1-z^2} = z + z^3 + z^5 + z^7 + \cdots .$$

THEOREM 4. *Let* $f \in \mathcal{K}$. *Then*

$$|a_2 a_4 - a_3^2| \leq \frac{1}{8}.$$

The present paper is motivated by these results and the purpose of this investigation is to find upper bounds of the functional $|a_n a_{n+2} - a_{n+1}^2| = |H_2(n)|$, given by the second Hankel determinant, for functions f in the class $\mathcal{S}^*_{\text{odd}}(\alpha)$ and $\mathcal{K}_{\text{odd}}(\alpha)$, respectively.

2. PROPERTIES OF THE CLASSES $\mathcal{S}^*_{\text{ODD}}(\alpha)$ AND $\mathcal{K}_{\text{ODD}}(\alpha)$

In this section, we derive upper bounds of $|a_{2m+1}|$ for functions f in $\mathcal{S}^*_{\text{odd}}(\alpha)$ and $\mathcal{K}_{\text{odd}}(\alpha)$. We apply the following lemmas to obtain our results.

LEMMA 5. *The equality*

$$1 + \sum_{l=1}^{m} \frac{\prod_{j=1}^{l}(j-\alpha)}{l!} = \frac{\prod_{j=2}^{m+1}(j-\alpha)}{m!}$$

holds for any m $(m = 1, 2, 3, \ldots)$.

Proof. For the case $m = 1$, noting that $1 + \prod_{j=1}^{1}(j-\alpha) = \prod_{j=2}^{2}(j-\alpha) = 2-\alpha$, the assertion of the lemma holds true. Next, we suppose that the equality

$$1 + \sum_{l=1}^{M} \frac{\prod_{j=1}^{l}(j-\alpha)}{l!} = \frac{\prod_{j=2}^{M+1}(j-\alpha)}{M!}$$

is valid for some M $(M \geq 1)$. Then

$$\begin{aligned}
1 + \sum_{l=1}^{M+1} \frac{\prod_{j=1}^{l}(j-\alpha)}{l!} &= 1 + \sum_{l=1}^{M} \frac{\prod_{j=1}^{l}(j-\alpha)}{l!} + \frac{\prod_{j=1}^{M+1}(j-\alpha)}{(M+1)!} \\
&= \frac{\prod_{j=2}^{M+1}(j-\alpha)}{M!} + \frac{\prod_{j=1}^{M+1}(j-\alpha)}{(M+1)!} \\
&= \frac{\prod_{j=2}^{M+1}(j-\alpha)}{M!} \left(1 + \frac{1-\alpha}{M+1}\right) = \frac{\prod_{j=2}^{M+2}(j-\alpha)}{(M+1)!}.
\end{aligned}$$

The statement follows now by mathematical induction. □

The following result is fundamental for Carathéodory functions.

LEMMA 6. (cf. [1], [5]) *If a function p, defined by $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$, belongs to $\mathcal{P}$, then $|c_k| \leq 2$ for each k $(k = 1, 2, 3, \dots)$. Equality holds for*

$$p(z) = \frac{1+z}{1-z} = 1 + \sum_{k=1}^{\infty} 2z^k.$$

From this lemma, we deduce immediately the following result.

LEMMA 7. *If an even function $p(z) = 1 + \sum_{k=1}^{\infty} c_{2k} z^{2k}$ satisfies*

$$\operatorname{Re}\, p(z) > \alpha \qquad (z \in \mathbb{U})$$

for some α $(0 \leq \alpha < 1)$, then $|c_{2k}| \leq 2(1-\alpha)$ for each k $(k = 1, 2, 3, \dots)$, with equality for

$$p(z) = \frac{1 + (1-2\alpha)z^2}{1-z^2} = 1 + \sum_{k=1}^{\infty} 2(1-\alpha) z^{2k}.$$

Proof. Put $q(z) = \dfrac{p(z)-\alpha}{1-\alpha}$. Then $q(z) = 1 + \sum_{k=1}^{\infty} \dfrac{c_{2k}}{1-\alpha} z^{2k}$, hence $q \in \mathcal{P}$. Thus, it follows from Lemma 6 that

$$\left| \frac{c_{2k}}{1-\alpha} \right| \leq 2 \qquad (k = 1, 2, 3, \dots)$$

or, equivalently,

$$|c_{2k}| \leq 2(1-\alpha) \qquad (k = 1, 2, 3, \dots).$$

□

From these, we derive now the following important preliminary results.

THEOREM 8. *Let $f \in \mathcal{S}^*_{\mathrm{odd}}(\alpha)$. Then*

$$|a_{2m+1}| \leq \frac{\prod_{j=1}^{m} (j-\alpha)}{m!} \qquad (m = 1, 2, 3, \dots),$$

with equality for

$$f(z) = \frac{z}{(1-z^2)^{1-\alpha}} = z + \sum_{m=1}^{\infty} \frac{\prod_{j=1}^{m} (j-\alpha)}{m!} z^{2m+1}.$$

Proof. Since $f \in \mathcal{S}^*_{\mathrm{odd}}(\alpha)$, there is a function p of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} c_{2k} z^{2k}$$

satisfying $\operatorname{Re} p(z) > \alpha$ $(z \in \mathbb{U})$ and such that

$$f'(z) = \frac{f(z)}{z} p(z). \tag{6}$$

Equation (6) is equivalent to

$$1 + \sum_{m=1}^{\infty} (2m+1) a_{2m+1} z^{2m} = 1 + \sum_{m=1}^{\infty} \left(\sum_{l=0}^{m} a_{2l+1} c_{2(m-l)} \right) z^{2m}, \tag{7}$$

where $a_1 = c_0 = 1$. Equalizing the coefficient of z^{2m} on both sides of the above equality for each m, and applying Lemma 7, we obtain the following inequality

$$\begin{aligned} |a_{2m+1}| = \frac{1}{2m} \left| \sum_{l=0}^{m-1} a_{2l+1}\, c_{2(m-l)} \right| &\leq \frac{1}{2m} \sum_{l=0}^{m-1} |a_{2l+1}| \cdot |c_{2(m-l)}| \\ &\leq \frac{1-\alpha}{m} \sum_{l=0}^{m-1} |a_{2l+1}|. \end{aligned}$$

Since $a_1 = 1$, we get that $|a_3| \leq (1-\alpha)|a_1| = 1-\alpha$,

$$|a_5| \leq \frac{1-\alpha}{2} (|a_1| + |a_3|) \leq \frac{1-\alpha}{2} (1 + (1-\alpha)) = \frac{(1-\alpha)(2-\alpha)}{2},$$

and

$$\begin{aligned} |a_7| &\leq \frac{1-\alpha}{3} (|a_1| + |a_3| + |a_5|) \\ &\leq \frac{1-\alpha}{3} \left(1 + (1-\alpha) + \frac{(1-\alpha)(2-\alpha)}{2} \right) = \frac{(1-\alpha)(2-\alpha)(3-\alpha)}{6}. \end{aligned}$$

Therefore, we expect that $|a_{2m+1}| \leq \dfrac{\prod_{j=1}^{m} (j-\alpha)}{m!}$ $(m = 1, 2, 3, \dots)$. Actually, supposing $|a_{2m+1}| \leq \dfrac{\prod_{j=1}^{m} (j-\alpha)}{m!}$ $(m = 1, 2, 3, \dots, M)$ and using Lemma 5, we derive

$$\begin{aligned} |a_{2(M+1)+1}| &\leq \frac{1-\alpha}{M+1} \sum_{l=0}^{M} |a_{2l+1}| \\ &\leq \frac{1-\alpha}{M+1} \left\{ 1 + \sum_{l=1}^{M} \frac{\prod_{j=1}^{l} (j-\alpha)}{l!} \right\} \\ &= \frac{1-\alpha}{M+1} \frac{\prod_{j=2}^{M+1} (j-\alpha)}{M!} = \frac{\prod_{j=1}^{M+1} (j-\alpha)}{(M+1)!}. \end{aligned}$$

The inequality to be proved follows now by mathematical induction. Equality is attained for $f \in \mathcal{S}^*_{\mathrm{odd}}(\alpha)$ given by

$$\frac{zf'(z)}{f(z)} = \frac{1+(1-2\alpha)z^2}{1-z^2}.$$

□

Taking $\alpha = 0$ in Theorem 8, we get the following result.

COROLLARY 9. *Let* $f \in \mathcal{S}^*_{\mathrm{odd}}$. *Then*

$$|a_{2m+1}| \leq 1 \qquad (m = 1, 2, 3, \dots),$$

with equality for

$$f(z) = \frac{z}{1-z^2} = z + \sum_{m=1}^{\infty} z^{2m+1}.$$

We can obtain similarly upper bounds of $|a_{2m+1}|$ for odd convex functions f.

THEOREM 10. *Let* $f \in \mathcal{K}_{\mathrm{odd}}(\alpha)$. *Then*

$$|a_{2m+1}| \leq \frac{\prod_{j=1}^{m}(j-\alpha)}{(2m+1)\, m!} \qquad (m = 1, 2, 3, \dots),$$

with equality for

$$f(z) = z\,{}_2F_1\left(\frac{1}{2}, 1-\alpha; \frac{3}{2}; z^2\right) = z + \sum_{m=1}^{\infty} \frac{\prod_{j=1}^{m}(j-\alpha)}{(2m+1)\, m!} z^{2m+1}.$$

Proof. By Remark 1, it is clear that if $f \in \mathcal{K}_{\mathrm{odd}}(\alpha)$, then

$$(2m+1)|a_{2m+1}| \leq \frac{\prod_{j=1}^{m}(j-\alpha)}{m!}$$

or, equivalently,

$$|a_{2m+1}| \leq \frac{\prod_{j=1}^{m}(j-\alpha)}{(2m+1)\, m!}.$$

□

For $\alpha = 0$ in Theorem 10 we obtain the next result.

COROLLARY 11. *Let* $f \in \mathcal{K}_{\mathrm{odd}}$. *Then*

$$|a_{2m+1}| \leq \frac{1}{2m+1} \qquad (m = 1, 2, 3, \dots),$$

with equality for

$$f(z)=\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)=z+\sum_{m=1}^{\infty}\frac{1}{2m+1}z^{2m+1}.$$

3. MAIN RESULTS

Applying Theorem 8 and Theorem 10, we get upper bounds for the second Hankel determinant $|H_2(n)|=|a_na_{n+2}-a_{n+1}^2|$ for functions in $\mathcal{S}^*_{\mathrm{odd}}(\alpha)$ and $\mathcal{K}_{\mathrm{odd}}(\alpha)$.

THEOREM 12. *Let $f\in\mathcal{S}^*_{\mathrm{odd}}(\alpha)$. Then*

$$|H_2(n)|=|a_na_{n+2}-a_{n+1}^2|\leq\begin{cases}1-\alpha & (n=1),\\[2ex] \dfrac{\prod_{j=1}^{m}(j-\alpha)^2}{(m!)^2} & (n=2m),\\[2ex] \dfrac{\left(\prod_{j=1}^{m}(j-\alpha)^2\right)(m+1-\alpha)}{m!\,(m+1)!} & (n=2m+1),\end{cases}$$

where $m=1,2,3,\dots$, with equality for

$$f(z)=\frac{z}{(1-z^2)^{1-\alpha}}=z+\sum_{m=1}^{\infty}\frac{\prod_{j=1}^{m}(j-\alpha)}{m!}z^{2m+1}.$$

Proof. Since $f\in\mathcal{S}^*_{\mathrm{odd}}(\alpha)$, it follows that

$$|a_na_{n+2}-a_{n+1}^2|=\begin{cases}|a_1a_3-a_2^2|=|a_1|\cdot|a_3| & (n=1),\\[1ex] |a_{2m}a_{2(m+1)}-a_{2m+1}^2|=|a_{2m+1}|^2 & (n=2m),\\[1ex] |a_{2m+1}a_{2m+3}-a_{2(m+1)}^2|=|a_{2m+1}|\cdot|a_{2m+3}| & (n=2m+1),\end{cases}$$

where $m=1,2,3,\dots$. By Theorem 8 we obtain the asserted inequalities. □

When $\alpha=0$ we get the following particular result.

COROLLARY 13. *Let $f\in\mathcal{S}^*_{\mathrm{odd}}$. Then*

$$|H_2(n)|=|a_na_{n+2}-a_{n+1}^2|\leq 1\qquad(n=1,2,3,\dots),$$

with equality for

$$f(z)=\frac{z}{1-z^2}=z+\sum_{m=1}^{\infty}z^{2m+1}.$$

We also derive the following results for odd convex functions f by applying Theorem 10.

THEOREM 14. *Let* $f \in \mathcal{K}_{\text{odd}}(\alpha)$. *Then*

$$|H_2(n)| = |a_n a_{n+2} - a_{n+1}^2| \leq \begin{cases} \dfrac{1-\alpha}{3} & (n=1), \\[2ex] \dfrac{\prod\limits_{j=1}^{m} (j-\alpha)^2}{(2m+1)^2 (m!)^2} & (n=2m), \\[2ex] \dfrac{\left(\prod\limits_{j=1}^{m} (j-\alpha)^2\right)(m+1-\alpha)}{(2m+1)(2m+3)m!\,(m+1)!} & (n=2m+1), \end{cases}$$

where $m = 1, 2, 3, \ldots,$ *with equality for*

$$f(z) = z\,{}_2F_1\left(\frac{1}{2}, 1-\alpha; \frac{3}{2}; z^2\right) = z + \sum_{m=1}^{\infty} \frac{\prod\limits_{j=1}^{m} (j-\alpha)}{(2m+1)\, m!} z^{2m+1}.$$

Setting $\alpha = 0$, we get the following particular result.

COROLLARY 15. *Let* $f \in \mathcal{K}_{\text{odd}}$. *Then*

$$|H_2(n)| = |a_n a_{n+2} - a_{n+1}^2| \leq \begin{cases} \dfrac{1}{4m^2-1} & (n=2m-1), \\ \dfrac{1}{(2m+1)^2} & (n=2m), \end{cases}$$

with equality for

$$f(z) = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right) = z + \sum_{m=1}^{\infty} \frac{1}{2m+1} z^{2m+1}.$$

4. APPLICATIONS AND OPEN PROBLEMS

We consider now a new operator related to the second Hankel determinant $H_2(n)$.

DEFINITION 16. For $f \in \mathcal{A}$ with $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ define

$$\mathcal{H}f(z) = z + \sum_{n=2}^{\infty} \left(a_n^2 - a_{n-1}a_{n+1}\right) z^n = z - \sum_{n=2}^{\infty} H_2(n-1) z^n.$$

Note that the above operator $\mathcal{H}$, applied to a function $f \in \mathcal{A}$, can be written as

$$\mathcal{H}f(z) = (f * f)(z) - \left(zf * \frac{f}{z}\right)(z),$$

where $*$ means the convolution (or Hadamard) product of two functions.

We recall now the following result due to Robertson [6].

Lemma 17. *Let $f \in \mathcal{K}(\alpha)$. Then*

$$|a_n| \le \frac{\prod_{j=2}^{n}(j-2\alpha)}{n!} \qquad (n = 2, 3, 4, \dots).$$

In particular, for $\alpha = 0$, if $f \in \mathcal{K}$, then

$$|a_n| \le 1 \qquad (n = 2, 3, 4, \dots).$$

Using the operator $\mathcal{H}$ given by Definition 16 and taking into account Corollary 13, we can conjecture that the new function $\mathcal{H}f$ may be in the class $\mathcal{K}$ if $f \in \mathcal{S}^*_{\mathrm{odd}}$. But this is not true, as it is shown by the following counter-example.

Remark 18. Let $f(z) = z + \frac{1}{3}z^3 \in \mathcal{A}_{\mathrm{odd}}$. A simple computation gives us

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) = \mathrm{Re}\left(\frac{1+z^2}{1+\frac{1}{3}z^2}\right) > 0 \quad (z \in \mathbb{U}).$$

Therefore, $f \in \mathcal{S}^*_{\mathrm{odd}}$. On the other hand, we see that

$$g(z) = \mathcal{H}f(z) = z - \frac{1}{3}z^2 + \frac{1}{9}z^3 \notin \mathcal{K},$$

because for the point $z_0 = \dfrac{231 + 33\sqrt{95}\mathrm{i}}{400} \in \mathbb{U}$ $\left(|z_0| = \dfrac{99}{100} < 1\right)$ we have

$$\mathrm{Re}\left(1 + \frac{z_0 g''(z_0)}{g'(z_0)}\right) = -\frac{994883}{31204889} < 0.$$

Inspired by the above result, we formulate an interesting problem below.

Problem 1. Find the class $\mathcal{M}$ of functions satisfying the property that if $f \in \mathcal{S}^*_{\mathrm{odd}}$, then the new function $\mathcal{H}f \in \mathcal{M}$.

Moreover, we can also formulate the following generalized problem.

Problem 2. Find the class $\mathcal{N}(\alpha)$ of functions satisfying the property that if $f \in \mathcal{S}^*(\alpha)$, then the new function $\mathcal{H}f \in \mathcal{N}(\alpha)$.

REFERENCES

[1] Duren, P.L., *Univalent functions*, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.
[2] Fekete, M. and Szegö, G., *Eine Bemerkung über ungerade schlichte Funktionen*, J. London Math. Soc., **8** (1933), 85–89.
[3] Janteng, A., Halim, S.A. and Darus, M., *Hankel determinant for starlike and convex functions*, Int. J. Math. Anal., **1** (2007), 619–625.
[4] Noonan, J.W. and Thomas, D.K., *On the second Hankel determinant of areally mean p-valent functions*, Trans. Amer. Math. Soc., **223** (2) (1976), 337–346.
[5] Pommerenke, Ch., *Univalent functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.
[6] Robertson, M.S., *On the theory of univalent functions*, Ann. of Math., **37** (1936), 374–408.

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