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ON UNIVALENCE CRITERIA FOR MEROMORPHIC FUNCTIONS

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Abstract. The object of this paper is to obtain some conditions for the univalence of meromorphic functions defined on the punctured unit disk $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

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1. INTRODUCTION

Let A denote the class of analytic functions g defined on the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ which satisfy the usual normalization condition:

$$g(0) = g'(0) - 1 = 0.$$

Also denote by S the subclass of A consisting of those functions g that are analytic and univalent on U, and by I the interval $[0, \infty)$. Further let $\widetilde{\sum}$ be the class consisting of the functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k, \tag{1}$$

i.e., which are analytic on the punctured unit disk $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U \setminus \{0\}$, and which have a simple pole at the origin.

For a function $f \in \widetilde{\sum}$, the differential operators D^n (see [1]) are defined by

$$\mathrm{D}^0 f(z) = f(z), \quad \mathrm{D}^1 f(z) = zf'(z) + \frac{2}{z},$$
$$\mathrm{D}^2 f(z) = z(\mathrm{D}^1 f(z))' + \frac{2}{z} = zf'(z) + z^2 f''(z),$$
$$\mathrm{D}^n f(z) = z(\mathrm{D}^{n-1} f(z))' + \frac{2}{z}, \text{ for } n \in \mathbb{N} = \{1, 2, \cdots\}.$$

In this paper we obtain some univalence conditions for meromorphic functions of the form (1).

2. PRELIMINARIES

In proving our results, we will need the following theorem due to Ch. Pommerenke (see [2, 3]).

THEOREM 1. *Let $L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$, $a_1(t) \neq 0$ be analytic on U_r for all $t \in I$, locally absolutely continuous on I, and locally uniform with respect to U_r. Suppose that for almost all $t \in I$*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \quad \forall z \in U_r,$$

where $p(z,t)$ is analytic on U and satisfies the condition $\Re\,(p(z,t)) > 0$ for all $z \in U$, $t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and if $\{L(z,t)/a_1(t)\}$ forms a normal family in U_r, then, for each $t \in I$, the function $L(\cdot,t)$ has an analytic and univalent extension to the whole open unit disk U.

3. MAIN RESULTS

Making use of the Theorem 1, we can now prove our main results.

THEOREM 2. *Let $s = \alpha + i\beta$ be a complex number such that $\alpha > 0$ and let $f \in \widetilde{\sum}$. If the following inequalities*

$$\left|\mathrm{i}\beta + \alpha\left(1 + \frac{2f(z) + zf'(z)}{z[\mathrm{D}^n f(z)]'}\right)\right| < |s| \tag{2}$$

and

$$\left|\mathrm{i}\beta + |z|^{2/\alpha}\,\alpha\left(1 + \frac{2f(z) + zf'(z)}{z[\mathrm{D}^n f(z)]'}\right)\right. \\ \left. -(1-|z|^{2/\alpha})\alpha\left[(1-s)\frac{zf'(z)}{f(z)} + s\left(1 + \frac{z[\mathrm{D}^n f(z)]''}{[\mathrm{D}^n f(z)]'}\right) + 1\right]\right| \leq |s| \tag{3}$$

are satisfied for all $z \in U^*$, then the function f is univalent on U^*.

Proof. We prove that there exists a real number $r \in (0,1]$ such that the function $L : U_r \times I \to \mathbb{C}$, defined formally by

$$L(z,t) = \mathrm{e}^{-2st}z^2 f(\mathrm{e}^{-st}z)\left\{1 - (\mathrm{e}^{2t}-1)\mathrm{e}^{-st}z\frac{[\mathrm{D}^n f(\mathrm{e}^{-st}z)]'}{f(\mathrm{e}^{-st}z)}\right\}^s,$$

is analytic on U_r for all $t \in I$.

Let us consider the function ϕ_1 given by

$$\phi_1(z,t) = \mathrm{e}^{-st}z\frac{[\mathrm{D}^n f(\mathrm{e}^{-st}z)]'}{f(\mathrm{e}^{-st}z)}.$$

For all $t \in I$, the function $\phi_1(\cdot,t)$ is analytic on U and $\phi_1(0,t) = -1$. Thus there exists a disc U_{r_1}, $0 < r_1 < 1$, such that $\phi_1(z,t) \neq 0$ for all $t \in I$ and $z \in U_{r_1}$. It can be easily shown that the function

$$\phi_2(z,t) = \left\{1 + (\mathrm{e}^{2t}-1)\phi_1(z,t)\right\}^s$$

is analytic on U_{r_1} and that $\phi_2(0,t) = \mathrm{e}^{2st}$ for all $t \in I$. From these considerations it follows that the function

$$L(z,t) = \mathrm{e}^{-2st}z^2 f(\mathrm{e}^{-st}z)\phi_2(z,t)$$

is analytic on U_{r_1} for all $t \in I$ and that it is of the form

$$L(z,t) = \mathrm{e}^{st} z + \cdots .$$

Since $\alpha > 0$, we have that

$$\lim_{t\to\infty} |a_1(t)| = \infty.$$

Moreover, $a_1(t) \neq 0$ for all $t \in I$.

Thus $\{L(z,t)/a_1(t)\}_{t\in I}$ forms a normal family of analytic functions on U_{r_2}, $0 < r_2 < r_1$. From the analyticity of $\frac{\partial L(z,t)}{\partial t}$, we obtain that for all fixed numbers $T > 0$ and r_3, $0 < r_3 < r_2$, there exists a constant $K > 0$ (that depends on T and r_3) such that

$$\left|\frac{\partial L(z,t)}{\partial t}\right| < K, \ \forall z \in U_{r_3}, \ t \in [0,T] .$$

Therefore, the function L is locally absolutely continuous on I and locally uniform with respect to U_{r_3}.

The function p, defined by

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z} \Big/ \frac{\partial L(z,t)}{\partial t},$$

is analytic on a disk U_r, $0 < r < r_3$, for all $t \in I$. In order to prove that the function p has an analytic extension to U and that $\Re(p(z,t)) > 0$ for all $t \in I$, we will show that the function w, given by

$$w(z,t) = \frac{p(z,t)-1}{p(z,t)+1}, \tag{4}$$

has an analytic extension to U and that $|w(z,t)| < 1$, for all $z \in U$ and $t \in I$. From equality (4) we have

$$w(z,t) = \frac{(1+s)\psi(z,t)+2}{(1-s)\psi(z,t)-2}, \tag{5}$$

where

(6)

$$\psi(z,t) = \frac{1}{s}\mathrm{e}^{-2t}\left(\frac{\mathrm{e}^{st}}{z}\frac{2f(\mathrm{e}^{-st}z)}{[\mathrm{D}^n f(\mathrm{e}^{-st}z)]'} + \frac{f'(\mathrm{e}^{-st}z)}{[\mathrm{D}^n f(\mathrm{e}^{-st}z)]'} + 1\right)$$
$$+ (1-\mathrm{e}^{-2t})\left(\left(1-\frac{1}{s}\right)\frac{\mathrm{e}^{-st}zf'(\mathrm{e}^{-st}z)}{f(\mathrm{e}^{-st}z)} - \frac{\mathrm{e}^{-st}z[\mathrm{D}^n f(\mathrm{e}^{-st}z)]''}{[\mathrm{D}^n f(\mathrm{e}^{-st}z)]'} - \left(\frac{2}{s}+1\right)\right)$$

for $z \in U$ and $t \in I$. The inequality $|w(z,t)| < 1$, for all $z \in U$ and $t \in I$, where $w(z,t)$ is defined by (5), is equivalent to

$$\left|\psi(z,t) + \frac{1}{\alpha}\right| < \frac{1}{\alpha}, \ \alpha = \Re(s), \ \forall z \in U, \ t \in I.$$

Define now

$$\Phi(z,t) = \psi(z,t) + \frac{1}{\alpha}, \ \forall z \in U, \ t \in I.$$

From (2) and (6) we have

$$|\Phi(z,0)| = \left| \mathrm{i}\beta + \alpha\left(\frac{2f(z)+zf'(z)}{z[\mathrm{D}^n f(z)]'} + 1\right)\right| < |s|\,. \tag{7}$$

Inequality (2) yields

$$|w(z,0)| < 1 \text{ for all } z \in U.$$

Let $t > 0$. Since for every $z \in \overline{U} = \{z \in \mathbb{C} : \ |z| \le 1\}$ and every $t > 0$

$$\left|\mathrm{e}^{-st} z\right| \le \left|\mathrm{e}^{-\alpha t}\right| = \mathrm{e}^{-\alpha t} < 1,$$

it follows that Φ is an analytic function on $\bar{U}$. Making use of the maximum modulus principle, we obtain that for each arbitrarily fixed $t > 0$ there exists $\theta = \theta(t) \in \mathbb{R}$ such that

$$|\Phi(z,t)| < \max_{|z|=1} |\Phi(z,t)| = \left|\Phi(\mathrm{e}^{\mathrm{i}\theta}, t)\right|, \text{ for all } z \in U \text{ and } t \in I.$$

Let $u = \mathrm{e}^{-st}\mathrm{e}^{\mathrm{i}\theta}$. Then $|u| = \mathrm{e}^{-\alpha t}$ and $\mathrm{e}^{-2t} = |u|^{2/\alpha}$. Thus, by (6), we get

$$\begin{aligned}\left|\Phi(\mathrm{e}^{\mathrm{i}\theta}, t)\right| = \frac{1}{\alpha\,|s|}\Bigg| &\mathrm{i}\beta + \alpha\,|u|^{2/\alpha}\left(\frac{2f(u)+uf'(u)}{u[\mathrm{D}^n f(u)]'} + 1\right) \\ &-(1-|u|^{2/\alpha})\alpha\left[(1-s)\frac{uf'(u)}{f(u)} + s\left(1 + \frac{u[\mathrm{D}^n f(u)]''}{[\mathrm{D}^n f(u)]'}\right) + 1\right]\Bigg|.\end{aligned}$$

Since $u \in U$, inequality (3) implies

$$\left|\Phi(\mathrm{e}^{\mathrm{i}\theta}, t)\right| \le \frac{1}{\alpha}. \tag{8}$$

From (7) and (8) we conclude that

$$|\Phi(z,t)| = \left|\psi(z,t) + \frac{1}{\alpha}\right| < \frac{1}{\alpha} \text{ for all } z \in U \text{ and } t \in I.$$

Therefore $|w(z,t)| < 1$ for all $z \in U$ and $t \in I$.

Since all conditions of Theorem 1 are satisfied, we obtain that the function $L(\cdot, t)$ has an analytic and univalent extension $\mathcal{L}$ to the whole open unit disk U, for every $t \in I$. For $t = 0$ we have $\mathcal{L}(z, 0) = f(z)$, for $z \in U^*$, and therefore the function f is analytic and univalent on U^*. □

THEOREM 3. *Let $s = \alpha + \mathrm{i}\beta$ be a complex number such that $\alpha \ge 1$ and let $f \in \widetilde{\sum}$. If the following inequalities*

$$\left| \mathrm{i}\beta + \alpha\left(\frac{2f(z)+zf'(z)}{z[\mathrm{D}^n f(z)]'} + 1\right)\right| < |s| \tag{9}$$

and

$$\begin{aligned}\Bigg| &\mathrm{i}\beta + |z|^2\,\alpha\left(\frac{2f(z)+zf'(z)}{z[\mathrm{D}^n f(z)]'} + 1\right) \\ &-(1-|z|^2)\alpha\left[(1-s)\frac{zf'(z)}{f(z)} + s\left(1 + \frac{z[\mathrm{D}^n f(z)]''}{[\mathrm{D}^n f(z)]'}\right) + 1\right]\Bigg| \le |s| \end{aligned}\tag{10}$$

are satisfied for all $z \in U^*$, then the function f is univalent on U^*.

Proof. Define for $\lambda \in [0,1]$ the function

$$\varphi(z,\lambda) = \lambda k(z) + (1-\lambda)\, l(z), \text{ for } z \in U,$$

where

$$k(z) = \mathrm{i}\beta + \alpha\left(\frac{2f(z)+zf'(z)}{z[\mathrm{D}^n f(z)]'} + 1\right),$$

and

$$l(z) = \left\{-\alpha\left[(1-s)\frac{zf'(z)}{f(z)} + s\left(1+\frac{z[\mathrm{D}^n f(z)]''}{[\mathrm{D}^n f(z)]'}\right)+1\right]+\mathrm{i}\beta\right\}.$$

For $z \in U$ and $\lambda \in [0,1]$, the point $\phi(z,\lambda)$ lies on the segment with endpoints $k(z)$ and $l(z)$. The function $\phi(\cdot,\lambda)$ is analytic on U for all $\lambda \in [0,1]$. From (9) and (10) we get for all $z \in U$

$$|\varphi(z,1)| = |k(z)| < |s|, \tag{11}$$

and

$$\left|\varphi(z,|z|^2)\right| \le |s|. \tag{12}$$

If λ increases from $\lambda_1 = |z|^2$ to $\lambda_2 = 1$, then, for fixed $z \in U$, the point $\phi(z,\lambda)$ moves on the segment whose endpoints are $\varphi(z,|z|^2)$ and $\phi(z,1)$. Because $\alpha \ge 1$, the relations (11) and (12) yield that

$$\left|\varphi\left(z,|z|^{2/\alpha}\right)\right| \le |s|, \text{ for } z \in U. \tag{13}$$

Now observe that the inequality (13) is just condition (3) from Theorem 2. Hence it follows from Theorem 2 that the function f is analytic and univalent on U^*. □

If we take $n = 0$, respectively, $n = 1$ in Theorem 3, then we obtain the following results.

Corollary 4. *Let $s = \alpha + \mathrm{i}\beta$ be a complex number such that $\alpha \ge 1$ and let $f \in \widetilde{\sum}$. If the following inequalities*

$$\left|\mathrm{i}\beta + 2\alpha\left(\frac{f(z)}{zf'(z)}+1\right)\right| < |s|$$

and

$$\left|\mathrm{i}\beta + 2|z|^2\alpha\left(\frac{f(z)}{zf'(z)}+1\right) -(1-|z|^2)\alpha\left[(1-s)\frac{zf'(z)}{f(z)} + s\left(1+\frac{zf''(z)}{f'(z)}\right)+1\right]\right| \le |s|$$

are satisfied for all $z \in U^*$, then the function f is univalent on U^*.

Corollary 5. *Let $s = \alpha + \mathrm{i}\beta$ be a complex number such that $\alpha \geq 1$ and let $f \in \widetilde{\sum}$. If the following inequalities*

$$\left| \mathrm{i}\beta + \alpha \left(\frac{z^3 f''(z) + 2z^2 f'(z) + 2zf(z) - 2}{z^3 f''(z) + z^2 f'(z) - 2} \right) \right| < |s|$$

and

$$\left| \mathrm{i}\beta + |z|^2 \alpha \left(\frac{z^3 f''(z) + 2z^2 f'(z) + 2zf(z) - 2}{z^3 f''(z) + z^2 f'(z) - 2} \right) \right.$$
$$-(1 - |z|^{2/\alpha})\alpha$$
$$\left. \left[(1-s)\frac{zf'(z)}{f(z)} + s\left(\frac{z^4 f'''(z) + 3z^3 f''(z) + z^2 f'(z) + 2}{z^3 f''(z) + z^2 f'(z) - 2} \right) + 1 \right] \right| \leq |s|$$

are satisfied for all $z \in U^*$, then the function f is univalent on U^*.

For $s = 1$ and $n = 0$ in Theorem 3 we get the following result.

Corollary 6. *Let $f \in \widetilde{\sum}$. If the following inequalities*

$$\left| \frac{f(z)}{zf'(z)} + 1 \right| < \frac{1}{2} \tag{14}$$

and

$$\left| 2|z|^2 \left(\frac{f(z)}{zf'(z)} + 1 \right) - (1 - |z|^2) \left(2 + \frac{zf''(z)}{f'(z)} \right) \right| \leq 1 \tag{15}$$

are satisfied for all $z \in U^*$, then the function f is univalent on U^*.

Example 7. Let $n \geq 1$. Then the function

$$f(z) = \frac{1}{z} + \frac{1}{5n} z^n \tag{16}$$

is analytic and univalent on U^*.

Proof. From equality (16) we have

$$\frac{f(z)}{zf'(z)} + 1 = \frac{\frac{1}{5}(\frac{n+1}{n}) z^{n+1}}{-1 + \frac{1}{5} z^{n+1}} \tag{17}$$

and

$$2 + \frac{zf''(z)}{f'(z)} = \frac{\frac{n+1}{5} z^{n+1}}{-1 + \frac{1}{5} z^{n+1}}. \tag{18}$$

Taking into account (17) and (18), we obtain

$$\left| \frac{f(z)}{zf'(z)} + 1 \right| = \left| \frac{\frac{1}{5}(\frac{n+1}{n}) z^{n+1}}{1 - \frac{1}{5} z^{n+1}} \right| \leq \frac{\frac{1}{5}(\frac{n+1}{n}) |z|^{n+1}}{1 - \frac{1}{5} |z|^{n+1}} < \frac{1}{4} \left(1 + \frac{1}{n} \right) \leq \frac{1}{2}$$

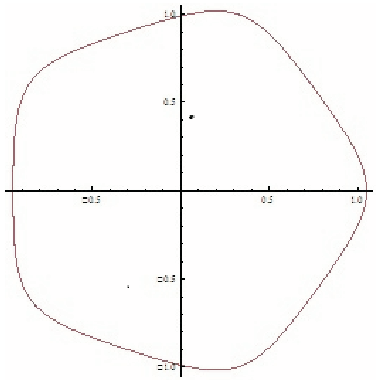

Fig. 3.1 – The graph of the function $f(z) = \frac{1}{z} + \frac{1}{5n} z^n$.

and

$$\begin{aligned} & \left| 2\,|z|^2 \left(\frac{f(z)}{zf'(z)} + 1 \right) - (1 - |z|^2) \left(2 + \frac{zf''(z)}{f'(z)} \right) \right| \\ = & \left| 2\,|z|^2 \left(\frac{\frac{1}{5}(\frac{n+1}{n}) z^{n+1}}{-1 + \frac{1}{5} z^{n+1}} \right) - (1 - |z|^2) \left(\frac{\frac{n+1}{5} z^{n+1}}{-1 + \frac{1}{5} z^{n+1}} \right) \right| \\ = & \left| \left(\frac{\frac{n+1}{5} z^{n+1}}{-1 + \frac{1}{5} z^{n+1}} \right) \left(\frac{n+2}{n} |z|^2 - 1 \right) \right| \\ < & \frac{1}{2} \left(1 + \frac{1}{n} \right) \\ \leq & 1. \end{aligned}$$

These inequalities show that the conditions (14) and (15) of Corollary 6 are satisfied. It follows that the function f defined by (16) is analytic and univalent on U^*. □

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