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# A NOTE ON APPROXIMATION PROPERTIES OF DERIVATIVES OF SCHOENBERG SPLINES

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**Abstract.** We analyze approximation properties of derivatives of variation-diminishing Schoenberg splines with emphasis on the case of purely equidistant knots. New direct inequalities regarding simultaneous approximation up to the second derivative are obtained in terms of the classical second order modulus of smoothness. For adequate polynomial degree and sufficiently smooth functions these *quantitative* estimates imply a *simultaneous approximation order* which is *quadratic* with respect to mesh size. These results remain valid if we drop the general requirement of data given outside the basic interval. Numerical tests verify our theoretical error bounds.

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**Key words.** Schoenberg operator, splines, derivatives, degree of approximation, simultaneous approximation, modulus of smoothness, Hestenes extension.

## 1. INTRODUCTION

Given integers $d \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$, $m \in \mathbb{N} := \{1, 2, 3, \dots\}$, and a strictly increasing sequence of real numbers $(t_i)_{i=1}^{m-1}$, which naturally partitions the real axis $\mathbb{R}$ into $m$ right-open intervals $\{T_i\}_{i=0}^{m-1}$, a $(d-1)$-times continuously differentiable function $s \in C^{d-1}(\mathbb{R})$ is a *spline* of *degree* at most $d$ with *minimal defect* at its *breakpoints* $(t_i)$ if on each *segment* $T_i$ it is a polynomial of degree at most $d$. These splines form an $(m+d)$-dimensional linear space $\Pi_{d,(t_i)}$ which contains the set $\Pi_d$ of polynomials with maximum degree $d$. Extending the sequence of breakpoints to a sequence of *knots*

$$\mathbf{t} := (t_{-d+1} \leq \cdots \leq t_0 < \cdots < t_m \leq \cdots \leq t_{m+d-1}), \tag{1}$$

it is possible to recursively define a basis $\{N_{i,d,\mathbf{t}}\}_{i=-d}^{m-1}$ of $\Pi_{d,(t_i)}$ by

$$N_{i,d,\mathbf{t}}(x) := \begin{cases} \frac{t_{i+d+1}-x}{t_{i+d+1}-t_{i+1}} N_{i+1,d-1,\mathbf{t}}(x) & , i = -d, \\ \frac{x-t_i}{t_{i+d}-t_i} N_{i,d-1,\mathbf{t}}(x) \\ \qquad + \frac{t_{i+d+1}-x}{t_{i+d+1}-t_{i+1}} N_{i+1,d-1,\mathbf{t}}(x) & , -d+1 \leq i \leq m-2, \\ \frac{x-t_i}{t_{i+d}-t_i} N_{i,d-1,\mathbf{t}}(x) & , i = m-1, \end{cases} \tag{2}$$

We are obliged to Prof. Ioan Raşa for clarifying some details about higher-order convexity and to Prof. Valery A. Zheludev for sending us a copy of his survey [34] which also includes an extensive bibliography of English *and* Russian language literature on approximation by local splines.

for $d \geq 1$ and all $x \in \mathbb{R}$, where $N_{i,0,\mathbf{t}}$ is the characteristic function of the $i$-th segment, i.e.,

$$N_{i,0,\mathbf{t}}(x) := \begin{cases} 1 & , \ x \in T_i, \\ 0 & , \ x \in \mathbb{R} \setminus T_i. \end{cases} \tag{3}$$

The *B(asis)-splines* $\{N_{i,d,\mathbf{t}}\}$ are *normalized* to sum to one identically. Hence, any spline $s \in \Pi_{d,(t_i)}$ has a unique representation

$$s = \sum_{i=-d}^{m-1} a_i N_{i,d,\mathbf{t}} \tag{4}$$

as affine combination of points $a_i \in \mathbb{R}$, $-d \leq i \leq m-1$.

Details on the theoretical and historical background of splines and the excellent approximation properties of spline functions are presented in the sterling books by de Boor [6], Nürnberger [25], and Schumaker [28]. As a consequence of their aesthetic visual appearance and the availability of efficient evaluation, manipulation, and rendering routines, splines have also been widely used in Computer Aided Geometric Design (CAGD) [1, 2, 5, 8, 14], particularly in the modeling of parametric curves and surfaces.

Lyche and Schumaker [21] construct explicit *local* spline operators for real-valued functions $f$ defined on intervals by fixing the coefficients $a_i$ in (4) with the aid of linear functionals $\lambda_i$, i.e.

$$a_i = \lambda_i(f), \tag{5}$$

such that smooth functions $f$ are approximated with a reasonable order of accuracy. This class of approximation schemes includes Schoenberg's classical *variation-diminishing* method [27], specified by

$$\mathcal{S}_{d,\mathbf{t}} : \mathbb{R}^{[\bar{t}_0, \bar{t}_m]} \ni f \mapsto \sum_{i=-d}^{m-1} f\left(\xi_{i,d,\mathbf{t}}\right) N_{i,d,\mathbf{t}} \in \Pi_{d,(t_i)}[t_0, t_m] \tag{6}$$

for $d \geq 1$, where *Greville's abscissae* $\{\xi_{i,d,\mathbf{t}}\}_{i=-d}^{m-1}$ are given by

$$\xi_{i,d,\mathbf{t}} := \frac{1}{d} \sum_{j=1}^{d} t_{i+j} \tag{7}$$

and $[\bar{t}_0, \bar{t}_m] := [\xi_{-d,d,\mathbf{t}}, \xi_{m-1,d,\mathbf{t}}] \supseteq [t_0, t_m]$. We point out that the *domain* $[\bar{t}_0, \bar{t}_m]$ coincides with the *basic interval* $[t_0, t_m]$ if and only if $t_{-d+1} = t_0$ and $t_m = t_{m+d-1}$. This traditional setting is also called the *clamped* case, since it implies

$$\mathcal{S}_{d,\mathbf{t}}(f; t_0) = f(t_0), \qquad \mathrm{D}\mathcal{S}_{d,\mathbf{t}}(f; t_0) = \frac{f(\xi_{-d+1,d,\mathbf{t}}) - f(t_0)}{\xi_{-d+1,d,\mathbf{t}} - t_0}, \tag{8}$$

$$\mathcal{S}_{d,\mathbf{t}}(f; t_m) = f(t_m), \qquad \mathrm{D}\mathcal{S}_{d,\mathbf{t}}(f; t_m) = \frac{f(t_m) - f(\xi_{m-2,d,\mathbf{t}})}{t_m - \xi_{m-2,d,\mathbf{t}}}. \tag{9}$$

Two prominent instances of clamped Schoenberg splines constitute

(a) *piecewise linear interpolants* for $d = 1$,
(b) *Bernstein polynomials* [3, 19] with respect to $[t_0, t_m]$ for $m = 1$.

Schoenberg's approach unifies these effective techniques and, moreover, enables control of the trade-off between accuracy and smoothness of the approximant in terms of polynomial degree $d$ and knots $\mathbf{t}$. This relation can be deduced from direct estimates of the approximation error $\|\mathcal{S}_{d,\mathbf{t}} f - f\|_{[0,1]}$ involving the upper bound

$$\mathcal{S}_{d,\mathbf{t}} \left((e_1 - x)^2; x\right) \leq \min \left\{ \frac{\bar{t}_m - \bar{t}_0}{2d}, \frac{d+1}{12} \|\mathbf{t}\|^2 \right\} \tag{10}$$

of the operator's second moment established by Marsden [23] for $d \geq 2$ and $x \in [t_0, t_m]$. Here and below, $\|\mathbf{t}\| := \max_{-d+1 \leq i \leq m+d-2} \{t_{i+1} - t_i\}$ denotes the *mesh size* of the sequence $\mathbf{t}$, and $e_r$, $r \in \mathbb{N}_0$, is the monomial function mapping every real number to its $r$-th power.

For later reference, we recall some basic and well known properties of Schoenberg's operator.

Remark 1 (Marsden [22], Schoenberg [27]).

(i) $\mathcal{S}_{d,\mathbf{t}} : \mathbb{R}^{[\bar{t}_0, \bar{t}_m]} \to \mathbb{R}^{[t_0, t_m]}$ is *discretely defined*, *linear* and *positive.*
(ii) $\mathcal{S}_{d,\mathbf{t}} L = L$ for all linear polynomials $L \in \Pi_1$.
(iii) For $d \geq 2$, the first derivative of $\mathcal{S}_{d,\mathbf{t}} f$, $f \in \mathbb{R}^{[\bar{t}_0, \bar{t}_m]}$, exists and is given by

$$\mathrm{D}\mathcal{S}_{d,\mathbf{t}} f = \sum_{i=-d+1}^{m-1} \frac{f(\xi_{i,d,\mathbf{t}}) - f(\xi_{i-1,d,\mathbf{t}})}{\xi_{i,d,\mathbf{t}} - \xi_{i-1,d,\mathbf{t}}} N_{i,d-1,\mathbf{t}}. \tag{11}$$

In the sequel, we restrict our view to knot sequences $\mathbf{t}$ which are *uniform* with respect to the basic interval $[t_0, t_m] = [0, 1]$, that is, $t_i = \frac{i}{m}$, $0 \leq i \leq m$. If, moreover, $t_{-d+1} = t_0$ and $t_m = t_{m+d-1}$, we refer to $\mathcal{S}_{d,\mathbf{t}}$ as *clamped uniform* Schoenberg operator and alternatively use the notation $\mathcal{S}_{d,m}$. In this particular case, the following classical Voronovskaya-type result conveys a saturation rate which is quadratic with respect to mesh size $h := \|\mathbf{t}\| = \frac{1}{m}$ for the class $C^2[0, 1]$.

Theorem 1 (Marsden [22], Schoenberg [27]). *Let $d \geq 2$, $f \in C^2[0, 1]$, and $x \in (0, 1)$. Then it holds*

$$\frac{m^2}{d+1} \left[\mathcal{S}_{d,m}(f; x) - f(x)\right] \xrightarrow{\frac{m}{d} \to \infty} \frac{1}{24} f''(x). \tag{12}$$

Furthermore, one can state

Theorem 2 (Marsden [22]). *Let $r \in \{0, 1\}$, $d \geq r + 1$, and $f \in C^r[0, 1]$. Then it holds*

$$\|\mathrm{D}^r \mathcal{S}_{d,m} f - \mathrm{D}^r f\|_{[0,1]} \xrightarrow{m+d \to \infty} 0. \tag{13}$$

Thus, for degree $d \geq 2$, clamped uniform Schoenberg splines simultaneously approximate continuously differentiable real functions and their first derivatives with arbitrary precision as $m + d$, the number of data points, tends to infinity.

However, while Bernstein polynomials approximate *all* derivatives of smooth functions simultaneously and uniformly [18], and preserve convexity of *any* order [16], Schoenberg splines generally do *not* possess these desirable features.

EXAMPLE 1 (cf. Beutel et al. [4], Marsden [22]). For degree $d = 3$, segments $m \geq 3$, and points $x \in \left[0, \frac{1}{m}\right]$ we obtain

$$\mathcal{S}_{3,m}(e_2; x) = \frac{x}{m}\left(-\frac{1}{18}m^2x^2 + mx + \frac{1}{3}\right) \tag{14}$$

by carrying out some elementary computations involving the B-spline recurrence (2), (3). It follows that

$$\mathrm{D}\mathcal{S}_{3,m}(e_2; x) = \frac{1}{m}\left(-\frac{1}{6}m^2x^2 + 2mx + \frac{1}{3}\right), \tag{15}$$

$$\mathrm{D}^2\mathcal{S}_{3,m}(e_2; x) = 2 - \frac{m}{3}\,x. \tag{16}$$

Thus, we have

$$\left|\mathrm{D}\mathcal{S}_{3,m}\left(e_2; \frac{1}{m}\right) - \mathrm{D}e_2\left(\frac{1}{m}\right)\right| = \frac{1}{6m}, \tag{17}$$

$$\left|\mathrm{D}^2\mathcal{S}_{3,m}\left(e_2; \frac{1}{m}\right) - \mathrm{D}^2e_2\left(\frac{1}{m}\right)\right| = \frac{1}{3}, \tag{18}$$

and

$$\mathrm{D}^3\mathcal{S}_{3,m}(e_2; 0) = -\frac{m}{3} < 0 = \mathrm{D}^3e_2(0). \tag{19}$$

Equations (17) and (18) show that we generally cannot expect the uniform approximation error with respect to the first and the second derivative to behave better than $O(h)$ and $O(1)$, respectively. (19) disproves (local) convexity preservation of order 3.

An alternative to the clamped uniform approach is to consider *purely equidistant* knots $t_i = \frac{i}{m}$, $-d+1 \leq i \leq m+d-1$. In this case, adopting a notion commonly used in CAGD for qualifying corresponding parametric spline curves [5], we refer to $\mathcal{Q}_{d,m} := \mathcal{S}_{d,\mathbf{t}}$ as *floating uniform* Schoenberg operator. Instead of $N_{i,d,\mathbf{t}}$ we also write $N_{i,d,m}$, and – like in the clamped uniform setting – we put $h := \|\mathbf{t}\| = \frac{1}{m}$. Greville's abscissae specialize to $\xi_{i,d,m} := \xi_{i,d,\mathbf{t}} = \left(i + \frac{d+1}{2}\right)h$, $-d \leq i \leq m-1$. Consequently, $\mathcal{Q}_{d,m}$ is defined for functions given on the domain $[\bar{0}, \bar{1}] = [-\frac{d-1}{2}h, 1 + \frac{d-1}{2}h]$ which, for $d \geq 2$, constitutes a proper superset of the basic interval $[0, 1]$. For $d = 1$, both floating and clamped uniform Schoenberg approximation collapse to piecewise linear interpolation, i.e., $\mathcal{Q}_{1,m} = \mathcal{S}_{1,m}$. However, higher-degree floating uniform Schoenberg splines generally do *not* share endpoint interpolation properties with their clamped counterparts.

Besides these putative disadvantages, the choice of purely equidistant knots entails certain benefits over the traditional construction involving coalescing boundary knots. Indeed, the following statement shows that, except for trivial cases, the optimal error of simultaneous approximation asymptotically behaves like $O(h^2)$.

THEOREM 3 (cf. Zheludev [33], Theorem 2). *Let* $r \in \mathbb{N}_0$, $d \geq r+2$, $f \in C^{r+2}[\bar{0}, \bar{1}]$, *and* $x \in [0,1]$. *Then it holds*

$$\frac{m^2}{d+1}\left[\mathrm{D}^r \mathcal{Q}_{d,m}(f;x) - \mathrm{D}^r f(x)\right] \xrightarrow[\text{uniformly}]{m\to\infty} \mathrm{D}^r\left[\frac{1}{24}f''(x)\right]. \tag{20}$$

It is the purpose of this paper to complement this result with quantitative direct estimates in terms of the classical second order modulus of smoothness.

## 2. PRELIMINARIES AND AUXILIARY RESULTS

DEFINITION 1. Let $a, b \in \mathbb{R}$, $a \leq b$, and $f \in \mathbb{R}^{[a,b]}$. For $r \in \mathbb{N}_0$ and $\delta \in \mathbb{R}_{\geq 0}$ the $r$-th *modulus of smoothness* of $f$ with respect to $[a,b]$ is specified by

$$\omega_{r,[a,b]}(f;\delta) := \sup_{\substack{0\leq\epsilon\leq\delta \\ a\leq x\leq b-r\epsilon}} \left|\Delta_\epsilon^r f(x)\right|, \tag{21}$$

where $\Delta_\epsilon^r f(x)$ denotes the $r$-th *forward difference* of $f(x)$ with *step size* $\epsilon$, i.e., $\Delta_\epsilon^0 f(x) = f(x)$ and $\Delta_\epsilon^r f(x) = \Delta_\epsilon^{r-1} f(x+\epsilon) - \Delta_\epsilon^{r-1} f(x)$ for $r \geq 1$.

In regard to our further reasoning, we summarize some fundamental characteristics of these moduli.

REMARK 2 (cf. Schumaker [28], p. 55f, and references specified therein). Let $I \subset \mathbb{R}$ be a compact interval, $f \in \mathbb{R}^I$, $r, s \in \mathbb{N}_0$, and $\delta \in \mathbb{R}_{\geq 0}$. Then we have:

(i) $\mathbb{R}^I \ni f \mapsto \omega_{r,I}(f;\delta) \in \mathbb{R}$ is a *seminorm*.
(ii) $\mathbb{R}_{\geq 0} \ni \delta \mapsto \omega_{r,I}(f;\delta) \in \mathbb{R}$ is *non-decreasing*.
(iii) $\omega_{r+s,I}(f;\delta) \leq 2^s \omega_{r,I}(f;\delta)$.
(iv) $\omega_{r+s,I}(f;\delta) \leq \delta^s \omega_{r,I}(f^{(s)};\delta)$, if $f \in C^s(i)$.

DEFINITION 2. Let $I \subset \mathbb{R}$, $J \subseteq I$ be compact intervals, $F \subseteq \mathbb{R}^I$, and $r \in \mathbb{N}_0$.

(i) A function $f \in \mathbb{R}^I$ is said to be *r-convex* if its $r$-th *divided differences* $\mathbf{\Delta}^r[\theta_0, \ldots, \theta_r]f$ are non-negative for all distinct nodes $\theta_i \in I$, $0 \leq i \leq r$.
(ii) We denote the set of all $r$-convex functions $f \in \mathbb{R}^I$ by $K^r(i)$.
(iii) An operator $\mathcal{A} : F \to \mathbb{R}^J$ is called *r-convex* if $\mathcal{A}(F \cap K^r(i)) \subseteq K^r(J)$.

REMARK 3.

(i) 0-, 1-, and 2-convex functions are also referred to as *non-negative*, *non-decreasing*, and *convex (from below)*, respectively.

(ii) Let $I \subset \mathbb{R}$ be a compact interval, $r \in \mathbb{N}_0$, and $f \in C^r(i)$. In this case, the mean value theorem stated by Schwarz [29] guarantees existence of a point $\xi \in I$ such that $r!\boldsymbol{\Delta}^r[\theta_0, \dots, \theta_r]f = \mathrm{D}^r f(\xi)$ for all distinct nodes $\theta_i \in I$, $0 \leq i \leq r$. Conversely, Hopf proves in his thesis [13, p. 16] that $r! \lim_{\theta_0,\dots,\theta_r \to \theta} \boldsymbol{\Delta}^r[\theta_0, \dots, \theta_r]f = \mathrm{D}^r f(\theta)$ for all $\theta \in I$ (cf. [7], [31, p. 18f]). As a consequence, any function $f \in C^r(i)$ is $r$-convex if and only if $\mathrm{D}^r f(x) \geq 0$ for all $x \in I$.

(iii) 0-convex operators are usually called *positive*.

(iv) Knoop and Pottinger [16] introduce the notion of *almost* $r$-convex operators generalizing the classical term used by Lupaş [20].

The subsequent quantitative Korovkin-type statement improves earlier results established by Knoop and Pottinger [16], and Gonska [10]. It is the key ingredient for the inequalities of Section 3.

Theorem 4 (Kacsó [15]). *Let $I \subset \mathbb{R}$ and $J \subseteq I$ be compact intervals, and let $r \in \mathbb{N}_0$. If $\mathcal{L} : C^r(i) \to C^r(J)$ is a linear and (almost) $r$-convex operator with $\mathcal{L}(\Pi_{r-1}(i)) \subseteq \Pi_{r-1}(J)$, then we have*

$$(22)\qquad \begin{aligned} |\mathrm{D}^r\mathcal{L}(f;x) - \mathrm{D}^r f(x)| \leq{}& |\alpha_{\mathcal{L},r}(x) - 1|\, |\mathrm{D}^r f(x)| \\ &+ \tfrac{\beta_{\mathcal{L},r}(x)}{\delta}\, \omega_{1,I}\left(\mathrm{D}^r f; \delta\right) \\ &+ \left[\alpha_{\mathcal{L},r}(x) + \tfrac{\gamma_{\mathcal{L},r}(x)}{2\delta^2}\right] \omega_{2,I}\left(\mathrm{D}^r f; \delta\right) \end{aligned}$$

*for all $f \in C^r(i)$, $x \in J$, and $\delta \in \left(0, \frac{\mathrm{length}\,(i)}{2}\right]$, where*

$$(23)\qquad \alpha_{\mathcal{L},r}(x) := \mathrm{D}^r\mathcal{L}\left(\tfrac{1}{r!}\, e_r; x\right),$$

$$(24)\qquad \beta_{\mathcal{L},r}(x) := \left|\mathrm{D}^r\mathcal{L}\left(\tfrac{1}{(r+1)!}\, e_{r+1} - \tfrac{1}{r!}\, x\, e_r; x\right)\right|,$$

$$(25)\qquad \gamma_{\mathcal{L},r}(x) := \mathrm{D}^r\mathcal{L}\left(\tfrac{2}{(r+2)!}\, e_{r+2} - \tfrac{2}{(r+1)!}\, x\, e_{r+1} + \tfrac{1}{r!}\, x^2\, e_r; x\right).$$

*Here, $\mathrm{D}^r\mathcal{L}$ operates on the function in a variable $t$, independent of $x$.*

Remark 4.

(i) The case $r = 0$ is a remarkable result due to Păltănea [26]. There it is stated that the upper bound on $\delta$ can be eliminated for operators $\mathcal{L}$ which preserve linear polynomials.

(ii) From the proof given by Kacsó [15] it is obvious that, likewise, the restriction on $\delta$ is *not* necessary in the general case if $\alpha_{\mathcal{L},r} = 1$ and $\beta_{\mathcal{L},r} = 0$, identically.

Our next objective is to justify the applicability of Theorem 4 to floating uniform Schoenberg splines.

Lemma 1. *For $r \in \mathbb{N}_0$, $r \le d-1$, the $r$-th derivative of $\mathcal{Q}_{d,m} f$, $f \in \mathbb{R}^{[\bar{0},\bar{1}]}$, exists and is given by*

$$\mathrm{D}^r \mathcal{Q}_{d,m} f = \sum_{i=-d+r}^{m-1} r!\, \mathbf{\Delta}_h^r f(\xi_{i-r,d,m}) N_{i,d-r,m}, \tag{26}$$

*where* $\mathbf{\Delta}_h^r f(\xi_{i-r,d,m}) := \mathbf{\Delta}^r[\xi_{i-r,d,m}, \ldots, \xi_{i,d,m}] f$.

*Proof.* Considering $r = 0$, it suffices to observe that

$$f(\xi_{i,d,m}) = 0!\, \mathbf{\Delta}_h^0 f(\xi_{i-0,d,m}). \tag{27}$$

Let $r_0 \in \mathbb{N}_0$, $r_0 \le d-2$. Assuming (26) for $r = r_0$, verifying

$$\frac{\mathbf{\Delta}_h^{r_0} f(\xi_{i-r_0,d,m}) - \mathbf{\Delta}_h^{r_0} f(\xi_{i-r_0-1,d,m})}{\xi_{i,d-r_0,m} - \xi_{i-1,d-r_0,m}} = (r_0+1)\, \mathbf{\Delta}_h^{r_0+1} f(\xi_{i-(r_0+1),d,m}), \tag{28}$$

and utilizing (11), we conclude

$$\begin{aligned} \mathrm{D}^{r_0+1} \mathcal{Q}_{d,m} f &= \mathrm{D} \sum_{i=-d+r_0}^{m-1} r_0!\, \mathbf{\Delta}_h^{r_0} f(\xi_{i-r_0,d,m}) N_{i,d-r_0,m} \\ &= \sum_{i=-d+r_0+1}^{m-1} (r_0+1)!\, \mathbf{\Delta}_h^{r_0+1} f(\xi_{i-(r_0+1),d,m}) N_{i,d-(r_0+1),m}. \end{aligned} \tag{29}$$

This completes the proof. □

Taking into account the positivity of Schoenberg's operator, we immediately obtain

Corollary 1. *$\mathcal{Q}_{d,m}$ is $r$-convex for all $r \in \mathbb{N}_0$, $r \le d-1$.*

In view of the operator's linear precision, indeed, all requirements of Theorem 4 in regard to $\mathcal{L} = \mathcal{Q}_{d,m}$ are satisfied for $r \in \{0,1,2\}$ and $d \ge r+1$. It remains to compute or estimate the quantities $\alpha_{\mathcal{Q}_{d,m},r}(x)$, $\beta_{\mathcal{Q}_{d,m},r}(x)$, $\gamma_{\mathcal{Q}_{d,m},r}(x)$ for $x \in [0,1]$.

It is a classical, but perhaps not commonly known fact that the moments of floating uniform Schoenberg operators of sufficiently high degree are constant.

Proposition 1 (Marsden and Riemenschneider [24], Zheludev [33]). *Let $x \in [0,1]$. Then it holds*

$$\mathcal{Q}_{d,m}\left((e_1-x)^0; x\right) = 1, \tag{30}$$

$$\mathcal{Q}_{d,m}\left((e_1-x)^1; x\right) = 0, \tag{31}$$

$$\mathcal{Q}_{d,m}\left((e_1-x)^2; x\right) = \tfrac{d+1}{12}\, h^2, \qquad d \ge 2, \tag{32}$$

$$\mathcal{Q}_{d,m}\left((e_1-x)^3; x\right) = 0, \qquad d \ge 3, \tag{33}$$

$$\mathcal{Q}_{d,m}\left((e_1-x)^4; x\right) = \tfrac{(d+1)(5d+3)}{240}\, h^4, \qquad d \ge 4. \tag{34}$$

Simple calculations lead to

Corollary 2. *For* $x \in [0,1]$, *we have*

(35) $$\mathcal{Q}_{d,m}(e_0;x) = 1,$$

(36) $$\mathcal{Q}_{d,m}(e_1;x) = x,$$

(37) $$\mathcal{Q}_{d,m}(e_2;x) = x^2 + \tfrac{d+1}{12}h^2, \qquad d \geq 2,$$

(38) $$\mathcal{Q}_{d,m}(e_3;x) = x^3 + \tfrac{d+1}{4}h^2 x, \qquad d \geq 3,$$

(39) $$\mathcal{Q}_{d,m}(e_4;x) = x^4 + \tfrac{d+1}{2}h^2 x^2 + \tfrac{(d+1)(5d+3)}{240}h^4, \qquad d \geq 4.$$

Corollary 3. *Let* $r \in \{0,1,2\}$, $d \geq r+1$, *and* $x \in [0,1]$. *Then it holds*

(40) $$\alpha_{\mathcal{Q}_{d,m},r}(x) = 1,$$

(41) $$\beta_{\mathcal{Q}_{d,m},r}(x) = 0.$$

*Moreover, if* $d \geq r+2$, *we have*

(42) $$\gamma_{\mathcal{Q}_{d,m},r}(x) = \tfrac{d+1}{12}h^2.$$

It is also possible to find explicit representations of the monomials' images under lower degree operators. After elementary but tedious computations involving the B-spline recurrence (2), (3) one actually arrives at

Proposition 2. *For* $x \in [ih,(i+1)h]$, $0 \leq i \leq m-1$, *we have*

(43) $$\mathcal{Q}_{1,m}(e_2;x) = (2i+1)hx - (i^2+i)h^2,$$

(44) $$\mathcal{Q}_{2,m}(e_3;x) = \left(3i+\tfrac{3}{2}\right)hx^2 - \left(3i^2+3i-\tfrac{1}{4}\right)h^2 x + \left(i^3+\tfrac{3}{2}i^2+\tfrac{1}{2}i\right)h^3,$$

(45) $$\mathcal{Q}_{3,m}(e_4;x) = (4i+2)\,hx^3 - \left(6i^2+6i-1\right)h^2x^2 + \left(4i^3+6i^2+2i\right)h^3 x - \left(i^4+2i^3+i^2-\tfrac{1}{3}\right)h^4.$$

Corollary 4. *Let* $r \in \{0,1,2\}$ *and* $d = r+1$. *Then for* $x \in [ih,(i+1)h]$, $0 \leq i \leq m-1$, *we have*

(46) $$\gamma_{\mathcal{Q}_{d,m},r}(x) = -\left[x - \left(i+\tfrac{1}{2}\right)h\right]^2 + \tfrac{d+2}{12}h^2 \leq \tfrac{d+2}{12}h^2.$$

*The given upper bound is sharp.*

*Proof.* For $r = 0$, we get

(47) $$\begin{aligned}\gamma_{\mathcal{Q}_{1,m},0}(x) &= \mathcal{Q}_{1,m}\left(\tfrac{2}{2!}\,e_2 - \tfrac{2}{1!}\,x\,e_1 + \tfrac{1}{0!}\,x^2\,e_0;x\right)\\ &= (2i+1)hx - (i^2+i)h^2 - 2x^2 + x^2\\ &= -\left[x-\left(i+\tfrac{1}{2}\right)h\right]^2 + \tfrac{3}{12}h^2\\ &\leq \tfrac{3}{12}h^2.\end{aligned}$$

Similarly, we obtain

(48) $$\gamma_{\mathcal{Q}_{2,m},1}(x) = -\left[x-\left(i+\tfrac{1}{2}\right)h\right]^2 + \tfrac{4}{12}h^2 \leq \tfrac{4}{12}h^2,$$

(49) $$\gamma_{\mathcal{Q}_{3,m},2}(x) = -\left[x-\left(i+\tfrac{1}{2}\right)h\right]^2 + \tfrac{5}{12}h^2 \leq \tfrac{5}{12}h^2.$$

The upper bounds are sharp for $x = \left(i+\tfrac{1}{2}\right)h$. □

## 3. MAIN RESULTS

The apparatus established in the previous section permits us to formulate

THEOREM 5. *Let $r \in \{0,1,2\}$, $f \in C^r[\bar{0},\bar{1}]$, and $\delta \in \mathbb{R}_{>0}$. Then we have*

$$\|\mathrm{D}^r \mathcal{Q}_{d,m} f - \mathrm{D}^r f\|_{[0,1]} \le \begin{cases} \left(1 + \frac{d+2}{24}\frac{h^2}{\delta^2}\right) \omega_{2,[\bar{0},\bar{1}]}(\mathrm{D}^r f;\delta) & , d = r+1, \\ \left(1 + \frac{d+1}{24}\frac{h^2}{\delta^2}\right) \omega_{2,[\bar{0},\bar{1}]}(\mathrm{D}^r f;\delta) & , d \ge r+2. \end{cases} \tag{50}$$

There are several ways to eliminate the free parameter $\delta$ in (50). Putting $\delta := \frac{h}{2}$ leads to

COROLLARY 5. *Let $r \in \{0,1,2\}$ and $f \in C^r[\bar{0},\bar{1}]$. Then we have*

$$\|\mathrm{D}^r \mathcal{Q}_{d,m} f - \mathrm{D}^r f\|_{[0,1]} \le \begin{cases} \left(1 + \frac{d+2}{6}\right) \omega_{2,[\bar{0},\bar{1}]}\left(\mathrm{D}^r f;\frac{h}{2}\right) & , d = r+1, \\ \left(1 + \frac{d+1}{6}\right) \omega_{2,[\bar{0},\bar{1}]}\left(\mathrm{D}^r f;\frac{h}{2}\right) & , d \ge r+2. \end{cases} \tag{51}$$

Depending on the situation, other choices of $\delta$ might be more adequate. A different approach gives rise to

COROLLARY 6. *Let $r \in \{0,1,2\}$, $d \ge r+1$, and $f \in C^r[\bar{0},\bar{1}]$. Then it holds*

$$\|\mathrm{D}^r \mathcal{Q}_{d,m} f - \mathrm{D}^r f\|_{[0,1]} \le \omega_{2,[\bar{0},\bar{1}]}\left(\mathrm{D}^r f;\frac{1+(d-1)h}{2}\right). \tag{52}$$

*Proof.* We observe that

$$\omega_{2,[\bar{0},\bar{1}]}(\mathrm{D}^r f;\delta) \le \omega_{2,[\bar{0},\bar{1}]}\left(\mathrm{D}^r f;\frac{1+(d-1)h}{2}\right). \tag{53}$$

Letting $\delta \to \infty$ in (50) proves the assertion. □

For sufficiently smooth functions we obtain normwise estimates.

COROLLARY 7. *Let $r \in \{0,1,2\}$ and $f \in C^{r+2}[\bar{0},\bar{1}]$. Then we have*

$$\|\mathrm{D}^r \mathcal{Q}_{d,m} f - \mathrm{D}^r f\|_{[0,1]} \le \begin{cases} \frac{d+2}{24} h^2 \left\|\mathrm{D}^{r+2} f\right\|_{[\bar{0},\bar{1}]} & , d = r+1, \\ \frac{d+1}{24} h^2 \left\|\mathrm{D}^{r+2} f\right\|_{[\bar{0},\bar{1}]} & , d \ge r+2. \end{cases} \tag{54}$$

*Proof.* Taking into account that

$$\omega_{2,[\bar{0},\bar{1}]}(\mathrm{D}^r f;\delta) \le \delta^2 \left\|\mathrm{D}^{r+2} f\right\|_{[\bar{0},\bar{1}]} \tag{55}$$

for all $f \in C^{r+2}[\bar{0},\bar{1}]$ and letting $\delta \to 0$ in (50) completes the proof. □

Optimality of constants will be discussed elsewhere. Here, we only give

EXAMPLE 2 (Piecewise Linear Interpolation). Let $d = 1$ and $r = 0$.
(i) For $f \in C[0,1]$ estimate (52) takes on the form

$$\|\mathcal{Q}_{1,m} f - f\|_{[0,1]} \le \omega_{2,[0,1]}\left(f;\tfrac{1}{2}\right). \tag{56}$$

The constant 1 in front of the second order modulus of smoothness *cannot* be replaced by any other constant strictly less than 1 (cf. Whitney [32], p. 69, for $m = 1$).

(ii) For $f \in C^2[0,1]$ inequality (54) reads

$$\|\mathcal{Q}_{1,m} f - f\|_{[0,1]} \leq \tfrac{1}{8} h^2 \left\|\mathrm{D}^2 f\right\|_{[0,1]} . \tag{57}$$

This estimate is also given in [6, p. 31f]. There, piecewise linear interpolation is also shown to be *nearly optimal* in the sense that the error of approximation can, *at best*, be halved by going over to a *best possible* approximation to $f$ from $\Pi_{1,\left(\frac{i}{m}\right)}[0,1]$.

## 4. HESTENES EXTENSION

So far, we have basically neglected the fact that floating uniform Schoenberg splines generally depend on data outside the basic interval. We address this problem with the aid of the subsequent *extension* operator introduced by Hestenes [12] as generalization of a *reflection principle* considered by Lichtenstein [17].

Definition 3 (Hestenes [12]). For $r \in \mathbb{N}_0$, $f \in \mathbb{R}^{[0,1]}$, and $x \in [-1,2]$ let

$$\mathcal{H}_r(f;x) := \begin{cases} \sum_{j=0}^r \eta_{j,r} f\left(-\frac{x}{2^j}\right) & , x < 0, \\ f(x) & , 0 \leq x \leq 1, \\ \sum_{j=0}^r \eta_{j,r} f\left(1 + \frac{1-x}{2^j}\right) & , 1 < x, \end{cases} \tag{58}$$

where the coefficients $\eta_{j,r}$ are uniquely determined as solution of the *Vandermonde* system

$$\sum_{j=0}^{r} \eta_{j,r} \left(-\frac{1}{2^j}\right)^i = 1, \quad 0 \leq i \leq r. \tag{59}$$

Fundamental properties of Hestenes' extension operator include the following.

Remark 5 (cf. Hestenes [12] and Sperling [30], p. 147f).

(i) $\mathcal{H}_r : \mathbb{R}^{[0,1]} \to \mathbb{R}^{[-1,2]}$ is a *pointwise discretely defined* and *linear* operator.
(ii) $\mathcal{H}_r P = P$ for all polynomials $P \in \Pi_r$.
(iii) $\mathcal{H}_r\left(C^s[0,1]\right) \subseteq C^s[-1,2]$ for all $s \in \mathbb{N}_0$, $s \leq r$.

The key observation of this section is

Proposition 3 (Global Smoothness Preservation). *Let $I$ be a compact interval with $[0,1] \subseteq I \subseteq [-1,2]$, $r \in \mathbb{N}_0$, $f \in C^r[0,1]$, and $\delta \in \left(0, \frac{1}{2}\right]$. Then we have*

$$\omega_{2,I}\left(\mathrm{D}^r \mathcal{H}_{r+2} f; \delta\right) \leq C_r\, \omega_{2,[0,1]}\left(\mathrm{D}^r f; \delta\right) \tag{60}$$

*for some constant $C_r \in \mathbb{R}_{\geq 0}$ which is independent of $f$ and $\delta$.*

*Proof.* For arbitrary $g \in C^{r+2}[0,1]$ we have

$$\begin{aligned}
&\omega_{2,I}\left(\mathrm{D}^r\mathcal{H}_{r+2}f;\delta\right)\\
&\quad\le \omega_{2,I}\left(\mathrm{D}^r\mathcal{H}_{r+2}(f-g);\delta\right)+\omega_{2,I}\left(\mathrm{D}^r\mathcal{H}_{r+2}g;\delta\right)\\
&\quad\le 4\left\|\mathrm{D}^r\mathcal{H}_{r+2}(f-g)\right\|_I+\delta^2\left\|\mathrm{D}^{r+2}\mathcal{H}_{r+2}g\right\|_I\\
&\quad\le \max\left\{1,\sum_{j=0}^{r}|\eta_{j,r+2}|\right\}\left(4\left\|\mathrm{D}^r(f-g)\right\|_{[0,1]}+\delta^2\left\|\mathrm{D}^{r+2}g\right\|_{[0,1]}\right).
\end{aligned}$$

Following Gonska and Kovacheva [11], it is possible to choose $g \in C^{r+2}[0,1]$ such that

$$\left\|\mathrm{D}^r(f-g)\right\|_{[0,1]} \le \frac{3}{4}\,\omega_{2,[0,1]}\left(\mathrm{D}^r f;\delta\right), \tag{61}$$

$$\left\|\mathrm{D}^{r+2}g\right\|_{[0,1]} \le \frac{3}{2}\,\delta^{-2}\,\omega_{2,[0,1]}\left(\mathrm{D}^r f;\delta\right). \tag{62}$$

It follows that

$$\omega_{2,I}\left(\mathrm{D}^r\mathcal{H}_{r+2}f;\delta\right) \le \frac{9}{2}\max\left\{1,\sum_{j=0}^{r}|\eta_{j,r+2}|\right\}\omega_{2,[0,1]}\left(\mathrm{D}^r f;\delta\right). \tag{63}$$

Putting $C_r := \frac{9}{2}\max\left\{1,\sum_{j=0}^{r}|\eta_{j,r+2}|\right\}$ clearly proves the assertion. □

COROLLARY 8. *Let $I$ be a compact interval with $[0,1] \subseteq I \subseteq [-1,2]$, $r \in \mathbb{N}_0$, and $\mathcal{A} : C^r(i) \to C^r[0,1]$ an arbitrary operator. If the estimate*

$$\left\|\mathrm{D}^r\mathcal{A}f-\mathrm{D}^r f\right\|_{[0,1]} \le \Gamma_r(\delta)\,\omega_{2,I}\left(\mathrm{D}^r f;\delta\right) \tag{64}$$

*is correct for all $f \in C^r(i)$ and certain quantities $\delta \in \left(0,\frac{1}{2}\right]$, $\Gamma_r(\delta) \in \mathbb{R}_{\ge 0}$ which do* not *depend on $f$, then the inequality*

$$\left\|\mathrm{D}^r\mathcal{A}\mathcal{H}_{r+2}g-\mathrm{D}^r g\right\|_{[0,1]} \le C_r\,\Gamma_r(\delta)\,\omega_{2,[0,1]}\left(\mathrm{D}^r g;\delta\right) \tag{65}$$

*holds for all $g \in C^r[0,1]$, where $C_r$ is given as in Proposition 3.*

*Proof.* Under the given prerequisites, the claim follows from

$$\begin{aligned}
\left\|\mathrm{D}^r\mathcal{A}\mathcal{H}_{r+2}g-\mathrm{D}^r g\right\|_{[0,1]} &= \left\|\mathrm{D}^r\mathcal{A}\mathcal{H}_{r+2}g-\mathrm{D}^r\mathcal{H}_{r+2}g\right\|_{[0,1]}\\
&\le \Gamma_r(\delta)\,\omega_{2,I}\left(\mathrm{D}^r\mathcal{H}_{r+2}g;\delta\right)\\
&\le C_r\,\Gamma_r(\delta)\,\omega_{2,[0,1]}\left(\mathrm{D}^r g;\delta\right).
\end{aligned}$$

□

This leads to

THEOREM 6. *Let $r \in \{0,1,2\}$, $f \in C^r[0,1]$, and $\delta \in \left(0,\frac{1}{2}\right]$. If, moreover, $m \ge \frac{d-1}{2}$, then $[\bar{0},\bar{1}] \subseteq [-1,2]$ and we have*

$$\begin{aligned}
&\left\|\mathrm{D}^r\mathcal{Q}_{d,m}\mathcal{H}_{r+2}f-\mathrm{D}^r f\right\|_{[0,1]}\\
&\qquad\le C_r\begin{cases}\left(1+\frac{d+2}{24}\frac{h^2}{\delta^2}\right)\omega_{2,[0,1]}(\mathrm{D}^r f;\delta) & ,\ d=r+1,\\ \left(1+\frac{d+1}{24}\frac{h^2}{\delta^2}\right)\omega_{2,[0,1]}(\mathrm{D}^r f;\delta) & ,\ d\ge r+2,\end{cases}
\end{aligned} \tag{66}$$

*where* $C_r$ *is given as in Proposition* 3.

Thus, the order of simultaneous approximation by floating uniform Schoenberg splines can be retained if there is only discrete data available in the basic interval $[0,1]$.

Explicit upper bounds on the constants $C_r$ can be found in Sperling [30], p. 162. Due to their huge size, however, they seem to be of theoretical interest only.

## 5. NUMERICAL RESULTS

We finish our analysis with some numerical tests. Given $m+3$ data points at the adequate Greville abscissae, we compare the accuracy of Bernstein polynomials with those of cubic clamped and floating uniform Schoenberg splines up to the second derivative. The operators $\mathcal{Q}_{3,m}\mathcal{H}_2$ and $\mathcal{Q}_{3,m}\mathcal{H}_3$ require 4 and 6 additional data points, respectively. All computations are performed in double precision floating-point arithmetic (cf. Goldberg [9]).

*Table 1* – Uniform approximation

| Size | $\|\mathcal{L}f-f\|_{[0,1]}$, where $f=[e_1(1-e_1)]^2$ and $\mathcal{L}$ is | | | |
|---|---|---|---|---|
| $m$ | $\mathcal{S}_{m+2,1}$ | $\mathcal{S}_{3,m}$ | $\mathcal{Q}_{3,m}$ | $\mathcal{Q}_{3,m}\mathcal{H}_2$ |
| 1 | $2{,}55\cdot10^{-2}$ | $2{,}55\cdot10^{-2}$ | $6{,}67\cdot10^{-1}$ | $1{,}15\cdot10^{-1}$ |
| 10 | $9{,}19\cdot10^{-3}$ | $1{,}63\cdot10^{-3}$ | $3{,}37\cdot10^{-3}$ | $2{,}37\cdot10^{-3}$ |
| 100 | $1{,}21\cdot10^{-3}$ | $2{,}99\cdot10^{-5}$ | $3{,}33\cdot10^{-5}$ | $3{,}23\cdot10^{-5}$ |
| 1000 | $1{,}25\cdot10^{-4}$ | $3{,}29\cdot10^{-7}$ | $3{,}33\cdot10^{-7}$ | $3{,}32\cdot10^{-7}$ |
| 10000 | – | $3{,}33\cdot10^{-9}$ | $3{,}33\cdot10^{-9}$ | $3{,}33\cdot10^{-9}$ |

*Table 2* – Uniform approximation of the first derivative

| Size | $\|\mathrm{D}\mathcal{L}f-\mathrm{D}f\|_{[0,1]}$, where $f=[e_1(1-e_1)]^2$ and $\mathcal{L}$ is | | | |
|---|---|---|---|---|
| $m$ | $\mathcal{S}_{m+2,1}$ | $\mathcal{S}_{3,m}$ | $\mathcal{Q}_{3,m}$ | $\mathcal{Q}_{3,m}\mathcal{H}_2$ |
| 1 | $1{,}48\cdot10^{-1}$ | $1{,}48\cdot10^{-1}$ | $2{,}00\cdot10^{-0}$ | $3{,}44\cdot10^{-1}$ |
| 10 | $7{,}00\cdot10^{-2}$ | $3{,}11\cdot10^{-2}$ | $2{,}00\cdot10^{-2}$ | $1{,}61\cdot10^{-2}$ |
| 100 | $9{,}61\cdot10^{-3}$ | $3{,}31\cdot10^{-3}$ | $2{,}00\cdot10^{-4}$ | $1{,}96\cdot10^{-4}$ |
| 1000 | $9{,}96\cdot10^{-4}$ | $3{,}33\cdot10^{-4}$ | $2{,}00\cdot10^{-6}$ | $2{,}00\cdot10^{-6}$ |
| 10000 | – | $3{,}33\cdot10^{-5}$ | $2{,}00\cdot10^{-8}$ | $2{,}00\cdot10^{-8}$ |

While Bernstein polynomials always show a linear rate of convergence, we observe that *clamped* uniform Schoenberg splines lose one degree of approximation with each further derivative. Clearly, *floating* uniform Schoenberg splines outperform both competitors. The quadratic order of approximation is even retained, if we exclusively sample data from inside the basic interval $[0,1]$, as

*Table 3* – Uniform approximation of the second derivative

| Size | $\lVert D^2\mathcal{L}f - D^2 f\rVert_{[0,1]}$, where $f = [e_1(1-e_1)]^2$ and $\mathcal{L}$ is | | | | |
|---|---|---|---|---|---|
| $m$ | $\mathcal{S}_{m+2,1}$ | $\mathcal{S}_{3,m}$ | $\mathcal{Q}_{3,m}$ | $\mathcal{Q}_{3,m}\mathcal{H}_2$ | $\mathcal{Q}_{3,m}\mathcal{H}_3$ |
| 1 | $2{,}30\cdot 10^{-0}$ | $2{,}30\cdot 10^{-0}$ | $5{,}00\cdot 10^{-0}$ | $2{,}69\cdot 10^{-0}$ | $2{,}22\cdot 10^{-0}$ |
| 10 | $9{,}94\cdot 10^{-1}$ | $5{,}04\cdot 10^{-1}$ | $5{,}00\cdot 10^{-2}$ | $7{,}02\cdot 10^{-1}$ | $5{,}00\cdot 10^{-2}$ |
| 100 | $1{,}35\cdot 10^{-1}$ | $3{,}24\cdot 10^{-1}$ | $5{,}00\cdot 10^{-4}$ | $7{,}45\cdot 10^{-2}$ | $5{,}00\cdot 10^{-4}$ |
| 1000 | $1{,}39\cdot 10^{-2}$ | $3{,}32\cdot 10^{-1}$ | $5{,}00\cdot 10^{-6}$ | $7{,}50\cdot 10^{-3}$ | $5{,}00\cdot 10^{-6}$ |
| 10000 | – | $3{,}33\cdot 10^{-1}$ | $5{,}00\cdot 10^{-8}$ | $7{,}50\cdot 10^{-4}$ | $5{,}17\cdot 10^{-8}$ |

long as the extension is constructed to be smooth enough. For comparison, we have depicted the columns for $\mathcal{Q}_{3,m}\mathcal{H}_2$ and $\mathcal{Q}_{3,m}\mathcal{H}_3$, with the smoother images coming from $\mathcal{H}_3$. Apparently the choice of $\mathcal{H}_3$ suffices to guarantee quadratic convergence in the second derivative for the special function $f = [e_1(1-e_1)]^2$. For a twice continuously differentiable function this can be derived from the first inequality in Theorem 6 for $d = 3$ only for $\mathcal{H}_4 = \mathcal{H}_{2+2}$. It is therefore desirable to further investigate how pessimistic the inequalities of Theorem 6 are in the general case.

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