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A NOTE ON ANNIHILATORS AND INJECTIVITY

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Abstract. It is proved that every two-sided ideal of a ring A is generated by a central idempotent if and only if every two-sided ideal of A is the left and right annihilator of an element of A and the intersection of the Jacobson radical, the left singular ideal and the right singular ideal of A is zero. The following generalization of injective modules, distinct from p-injective modules, is studied: a left A-module M is said to satisfy property (*) if, for any left submodule N of M isomorphic to a complement left submodule C of M, every left A-monomorphism of N into C extends to a left A-homomorphism of M into C.

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Throughout, A denotes an associative ring with identity and A-modules are unital. J, Z, Y will stand respectively for the Jacobson radical, the left singular ideal and the right singular ideal of A. An ideal of A will always mean a two-sided ideal of A. Of course, J, Z, Y are ideals of A. For any left A-module M, $Z(M) = \{y \in M \mid l(y)$ is an essential left ideal of $A\}$ is the left singular submodule of M. The singular submodule $Z(R)$ of a right A-module R is similarly defined. Thus $Z = Z({}_AA)$ and $Y = Z(A_A)$. ${}_AM$ is called singular (respectively non-singular) if $Z(M) = M$ (respectively $Z(M) = 0$). A is called a left non-singular ring if $Z = 0$. As usual, a submodule N of M is called a complement (or closed) submodule of M if N has no proper essential extension in M [7]. For results on non-singular rings and modules, consult Goodearl's classic [7]. The concept of non-singular rings is fundamental in the development of ring theory after the structure theory of N. Jacobson (cf. [6, p. 180]).

Following [6], write "A is VNR" if A is a von Neumann regular ring. A is VNR if and only if every left (right) A-module is p-injective ([1], [2], [12], [16], [18]) if and only if every left (right) A-module is YJ-injective [23, Theorem 9].

A left A-module M is called (a) p-injective if, for every principal left ideal P of A, any left A-homomorphism of P into M extends to one of A into M (cf. [6, p. 122], [15, p. 340], [18]); (b) YJ-injective if, for every $0 \neq a \in A$, there exist a positive integer n such that $a^n \neq 0$ and any left A-homomorphism of Aa^n into M extends to one of A into M ([4], [16], [19], [20], [22], [23]). P-injectivity and YJ-injectivity are similarly defined on the right side.

A is called a left p-injective (respectively YJ-injective) ring if ${}_AA$ is p-injective (respectively YJ-injective). YJ-injectivity is called GP-injectivity in [3], [10], [11], [13]. It may be noted that A is left YJ-injective if and only if, for every $0 \neq a \in A$, there exist a positive integer n such that a^nA is a non-zero right annihilator [19, Lemma 3]. A left (right) ideal of A is called reduced if it contains no non-zero nilpotent element. A is called fully (respectively (a) fully left; (b) fully right) idempotent if every ideal (respectively (a) left ideal; (b) right ideal) of A is idempotent.

Recall that A is a biregular ring if, for every $a \in A$, AaA is generated by a central idempotent. This motivates the next result.

THEOREM 1. *The following conditions are equivalent:*

(1) *Every ideal of A is generated by a central idempotent;*
(2) *Every ideal of A is the left and right annihilator of an element of A and $J \cap Z \cap Y = 0$.*

Proof. Assume (1). Since J cannot contain a non-zero idempotent, $J = 0$. Now for any ideal T of A, $T = Ae$, where e is a central idempotent in A. Since $T = l(1-e) = r(1-e)$, assertion (1) implies (2).

Assume (2). Suppose there exists a non-zero ideal T of A such that $T^2 = 0$. If $0 \neq t \in T$, then $l(AtA)$ is an essential right ideal of A and $r(AtA)$ is an essential left ideal of A. By hypothesis, $AtA = l(b) = r(b)$, $b \in A$. Then $AtA = l(AbA) = r(AbA)$ and $r(AtA) = r(l(AbA)) = AbA$, which implies that AbA is an essential left ideal of A. Similarly, AbA is an essential right ideal of A. Now $AbA = l(c) = r(c)$, $c \in A$, which implies that $AbA = l(AcA) = r(AcA)$, whence $AcA \subseteq Z \cap Y$. Suppose that $AcA \neq 0$. Since AbA is essential in ${}_AA$, $N = AcA \cap AbA$ is a non-zero left ideal of A and $N^2 \subseteq AbAcA = 0$, which yields $N \subseteq J$, whence $N \subseteq AcA \cap J \subseteq Z \cap Y \cap J = 0$, which is a contradiction. We have proved that A is a semi-prime ring.

Now for any $z \in Z$, $AzA = l(AuA) = r(AuA)$, $u \in A$. Since A is semi-prime, $r(AzA) = l(AzA) = l(r(AuA)) = AuA$. Then $AzA + AuA = AzA + r(AzA)$ is an essential left ideal of A and $AzA + r(AzA) = l(w) = r(w)$, $w \in A$. Since $AzAw = 0$ and $l(AzA)w = r(AzA)w = 0$, we have $w \in r(l(AzA)) = AzA$. Therefore $(AwA)^2 \subseteq (AzA)(AwA) = 0$ and since A is semi-prime, $w = 0$. Now $AzA + r(AzA) = A$ and since $AzA \cap r(AzA) = 0$ (because A is semi-prime), we have $A = AzA \oplus r(AzA)$. But Z cannot contain a non-zero idempotent and hence $z = 0$, which yields $Z = 0$. For any ideal T of A, $T = l(d)$, $d \in A$, and if ${}_AE$ is an essential extension of ${}_AT$, for any $y \in E$, there exists an essential left ideal L of A such that $Ly \subseteq T$. Then $Lyd = 0$ implies that $yd \in Z = 0$, whence $y \in l(d) = T$. Therefore T is a complement left ideal of A. Now $T \cap r(T) = 0$ (A being semi-prime), and if K is a complement left ideal of A such that $S = (T \oplus r(T)) \oplus K$ is an essential left ideal of A, then $TK \subseteq T \cap K = 0$ implies that $K \subseteq r(T)$, whence $K = 0$ and $S = T \oplus r(T)$ is an essential left ideal of A. But S is a complement left ideal of A as above and hence $A = S = T \oplus r(T)$. Therefore T is generated by a central idempotent (in as much as A is semi-prime). Thus (2) implies (1). □

Corollary 1. *If every ideal of A is the left and right annihilator of an element of A and $J \cap Z \cap Y = 0$, then A is biregular.*

Applying [21, Theorem 1.6] to Theorem 1, we get

Corollary 2. *If every complement left ideal of A is an ideal of A and every ideal of A is the left and right annihilator of an element of A with $J \cap Z \cap Y = 0$, then A is a reduced fully left idempotent left Goldie ring.*

Remark. In Theorem 1, the condition $J \cap Z \cap Y = 0$ is not superfluous in (2) (otherwise, any principal left and right ideal quasi-Frobenius ring would be semi-simple Artinian!).

We now turn to generalizations of injectivity. As usual, a left A-module M is called continuous if (a) every complement left submodule of M is a direct summand of M and (b) every left submodule of M which is isomorphic to a direct summand of M is a direct summand of M. A is called a left continuous ring (in the sense of Y. Utumi [14]) if ${}_AA$ is continuous. If A is left continuous, then A/J is left continuous regular and $Z = J$ [14, Lemma 4.1].

Here we introduce an effective generalization of injective modules, distinct from p-injective modules.

Definition 1. A left A-module M satisfies property (*) if, for any left submodule N of M which is isomorphic to a complement left submodule C of M, every left A-monomorphism of N into C extends to a left A-homomorphism of M into C.

Since any simple left A-module satisfies property (*), then a module satisfying property (*) needs not be p-injective (otherwise, any ring would be fully left and right idempotent (cf. [1, p. 121] and [15, p. 340]). The converse is not true either (otherwise, any VNR ring would be continuous (cf. Theorem 3 below)).

Theorem 2. *Let M be a left A-module satisfying property (*), $E = \mathrm{End}({}_AM)$ and $J(E)$ the Jacobson radical of E. Then $J(E) = \{f \in E \mid \ker f$ is essential in ${}_AM\}$ and $E/J(E)$ is VNR.*

Proof. Set $T = \{f \in E \mid \ker f \text{ is essential in } {}_AM\}$. Then it is well-known that T is an ideal of E. We show that $T \subseteq J(E)$. Let $f \in T$, $b \in E$. With $u = 1 - bf$, since $\ker f \cap \ker u = 0$, we have $\ker u = 0$. If $v : uM \to M$ is the inverse isomorphism of $M \to uM$, then v extends to an endomorphism h of ${}_AM$. For any $m \in M$, $hu(m) = h(um) = v(um) = vu(m) = m$, which proves that hu is the identity map on M. Therefore $f \in J(E)$ and hence $T \subseteq J(E)$. Now if $0 \neq \bar{g} \in E/J(E)$, $g \in E$, $g \notin J(E)$ implies that $g \notin T$. Let C be a non-zero complement left submodule of M such that $L = \ker g \oplus C$ is an essential submodule of M. The restriction r of g to C is an isomorphism of C onto

$r(C)$. Let $z\colon r(C) \to C$ denote the inverse isomorphism of r. By hypothesis, z extends to a left A-homomorphism $t\colon M \to C$. If $j\colon C \to M$ is the inclusion map, for every $c \in C$, $jt \in E$, $jtg(c) = jtr(c) = jzr(c) = j(c) = c$, which yields $L \subseteq \ker(gjtg - g)$, whence $\bar{g}(\overline{jt})\bar{g} = \bar{g} \in E/J(E)$, proving that $E/J(E)$ is a VNR ring. It remains to show that $J(E) \subseteq T$. If we suppose that there exists $w \in J(E)$ such that $w \notin T$, the preceding argument shows that there exist $d \in E$ such that $wdw - w \in T$. Since $dw \in J(E)$, there exists $s \in E$ such that $(1 - dw)s = 1$. Then $w = w(1 - dw)s = (w - wdw)s \in T$ (in as much as T is an ideal of E), which is a contradiction. Finally, we have $J(E) = T = \{f \in E \mid \ker f \text{ is essential in } {}_AM\}$. □

Lemma 1. *Any continuous left A-module satisfies property* (*).

Proof. If ${}_AM$ is continuous and N a submodule isomorphic to a complement left submodule C of M, then both N and C are direct summands of M. In that case, any left A-monomorphism of N into C extends to a left A-homomorphism of M into C. Therefore M satisfies property (*). □

Combining Lemma 1 with Theorem 2, we get

Proposition 1. *Let M be a continuous left A-module and $E = End({}_AM)$. Then the Jacobson radical of E is $J(E) = \{f \in E \mid \ker f$ is essential in ${}_AM\}$ and $E/J(E)$ is a VNR ring.*

Lemma 2. *Let M be a left A-module satisfying property* (*). *Then every complement left submodule of M is a direct summand of M.*

Proof. Let C be a complement left submodule of M. If $i : C \to C$ is the identity map on C, then i extends to a left A-homomorphism of M into C. If $j : C \to M$ is the inclusion map, then there exist a left A-homomorphism $h : M \to C$ such that $hj = i$. This proves that C is a direct summand of M. □

We say that "A satisfies property (*)" if ${}_AA$ satisfies property (*).

Theorem 3. *The following conditions are equivalent:*

(1) *A is left continuous regular;*
(2) *A is a left p-injective left non-singular ring satisfying property* (*).

Proof. (1) implies (2) by Lemma 1.

Assume (2). Then any left ideal of A isomorphic to a direct summand of ${}_AA$ is a direct summand of ${}_AA$ (cf. [20, p. 439]). By Lemma 2, A is left continuous. Since $Z = 0$, A is VNR by [14, Lemma 4.1]. Thus (2) implies (1). □

As usual, for a left submodule N of a left A-module M, $\mathrm{Cl}_M(N) = \{y \in M \mid Ly \subseteq N$ for some essential left ideal L of $A\}$ is the closure of N in M. A theorem of I. Kaplansky asserts that a commutative ring A is VNR if and only if every simple A-module is injective. This has motivated a large number of papers on generalizations of those rings in the non-commutative case. Rings whose simple singular right modules are YJ-injective are studied in [4], [10], [11].

Proposition 2. *Let A be a semi-prime ring whose simple singular right modules are YJ-injective. For any homomorphic image Q of a left A-module satisfying property* (*)*, $Z(Q)$ is a direct summand of Q.*

Proof. Let $g : M \to Q$ be an epimorphism of left A-modules M, Q with M satisfying property (*). Then $M/\ker g \cong Q$. Since every simple singular right A-module is YJ-injective and A is semi-prime, $Z = 0$ [22, Proposition 2]. Since g is an epimorphism, $g^{-1}(Z(Q)) = \mathrm{Cl}_M(\ker g)$. Since $Z = 0$, $\mathrm{Cl}_M(\ker g)$ is a complement submodule of M by [17, Theorem 4]. By Lemma 2, $\mathrm{Cl}_M(\ker g)$ is a direct summand of M. Therefore $M = g^{-1}(Z(Q)) \oplus N$ for some submodule N of M. This yields $Q = g(M) = Z(Q) \oplus g(N)$, where $g(N) \cong N$. □

Proposition 3. *The following conditions are equivalent:*

(1) *A is a left Noetherian ring whose p-injective left modules are injective;*
(2) *Every p-injective left A-module is injective;*
(3) *Every p-injective left A-module satisfies property* (*).

Proof. Clearly, (1) implies (2) while (2) implies (3).

Assume (3). Let M be a p-injective left A-module and ${}_AE$ the injective hull of ${}_AM$. Set $Q = {}_AM \oplus_A E$. Then ${}_AQ$ is p-injective, which therefore satisfies property (*). Let $u\colon M \to E$ be the inclusion map and $j\colon E \to Q$ the natural injection. Then $ju\colon M \to Q$, and since ${}_AQ$ satisfies property (*), the identity map $i\colon M \to M$ extends to a left A-homomorphism $h\colon Q \to M$. Therefore $hju = i$. Since $u\colon M \to E$ is the inclusion map and $hj\colon E \to M$ a map such that $(hj)u$ is the identity map on M, ${}_AM$ is a direct summand of ${}_AE$, which yields $M = E$ injective. If S is a direct sum of injective left A-modules, since a direct sum of p-injective left A-modules is p-injective, then S is p-injective and hence injective. This proves that A is left Noetherian [5, Theorem 20.1]. Thus (3) implies (1). □

YJ-injectivity effectively generalizes p-injectivity, even for rings [3]. Since a left and right YJ-injective left Noetherian ring is quasi-Frobenius, we get

Corollary 3. *If A is a left and right YJ-injective ring whose p-injective left modules satisfy property* (*)*, then A is quasi-Frobenius.*

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