MATHEMATICA, Tome 50 (73), N° 1, 2008, pp. 39–49

SOME SUFFICIENT CONDITIONS FOR UNIVALENCE AND SUBORDINATION RESULTS OF CERTAIN ANALYTIC AND UNIVALENT FUNCTIONS

B.A. FRASIN, M. DARUS and S. SIREGAR

Abstract. Very recently, Frasin and Darus [2] introduced the class $\mathcal{B}(\alpha)$ of analytic functions and gave some properties for this class. The aim of this paper is to obtain some sufficient conditions for univalence and subordination results for functions of the class $\mathcal{B}(\alpha)$.

MSC 2000. 30C45.
Key words. Analytic functions, univalent functions, starlike functions and convex functions, subordination.

1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. A function f belonging to $\mathcal{S}$ is said to be starlike of order α if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathcal{U}) \tag{2}$$

for some α $(0 \leq \alpha < 1)$. We denote by $\mathcal{S}^*_\alpha$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of order α in $\mathcal{U}$. Also, a function f belonging to $\mathcal{S}$ is said to be convex of order α if it satisfies

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathcal{U}) \tag{3}$$

for some α $(0 \leq \alpha < 1)$. We denote by $\mathcal{K}_\alpha$ the subclass of $\mathcal{A}$ consisting of functions which are convex of order α in $\mathcal{U}$.

A function f belonging to $\mathcal{S}$ is said to be close-to-convex of order α if there exists a function g belonging to $\mathcal{S}^*_\alpha$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > \alpha \qquad (z \in \mathcal{U}) \tag{4}$$

for some α $(0 \leq \alpha < 1)$. We denote by $\mathcal{C}_\alpha$ the subclass of $\mathcal{A}$ consisting of functions which are close-to-convex of order α in $\mathcal{U}$. Let the functions f and

g be analytic in $\mathcal{U}$. Then we say that the function f is subordinate to g in $\mathcal{U}$ if there exists an analytic function w in $\mathcal{U}$ with $w(0) = 0$ and $|w| < 1$ $(z \in \mathcal{U})$ such that $f(z) = g(w(z))$. We denote this subordination by $f(z) \prec g(z)$ or, shortly, $f \prec g$.

A function $f \in \mathcal{A}$ is said to be a member of the class $\mathcal{B}(\alpha)$ if and only if

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 - \alpha. \tag{5}$$

Note that the condition (5) implies

$$\operatorname{Re}\left\{ \frac{z^2 f'(z)}{f^2(z)} \right\} > \alpha. \tag{6}$$

for some α $(0 \le \alpha < 1)$ and for all $z \in \mathcal{U}$.

Frasin and Darus [2] have defined the class $\mathcal{B}(\alpha)$ and investigated some interesting properties for this class. In this paper we shall give new additional results for functions of the class $\mathcal{B}(\alpha)$.

2. SOME PROPERTIES OF THE CLASS $\mathcal{B}(\alpha)$

In order to prove our main results, we recall the following lemmas:

Lemma 1. ([3]) *Let w be analytic in $\mathcal{U}$ and such that $w(0) = 0$. If the map $z \in \mathcal{U} \mapsto |w(z)| \in \mathbb{R}$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathcal{U}$, then we have*

$$z_0 w'(z_0) = k w(z_0), \tag{7}$$

where $k \ge 1$ is a real number.

Lemma 2. ([8]) *Let $f \in \mathcal{A}$ satisfy the condition*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 \qquad (z \in \mathcal{U}). \tag{8}$$

Then f is univalent in $\mathcal{U}$.

Lemma 3. ([6]) *Let p be an analytic function in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \ne 0$ $(z \in \mathcal{U})$. If there exists a point $z_0 \in \mathcal{U}$ such that*

$$|\arg p(z)| < \frac{\pi}{2}\eta \qquad \text{for } |z| < |z_0| \tag{9}$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\eta$$

with $0 < \eta \le 1$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = \mathrm{i}k\eta,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1, \qquad \textit{when } \arg p(z_0) = \frac{\pi}{2}\eta,$$

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \leq -1, \qquad \textit{when } \arg p(z_0) = -\frac{\pi}{2}\eta,$$

and

$$p(z_0)^{\frac{1}{\eta}} = \pm a\mathrm{i}, \quad (a > 0).$$

LEMMA 4. ([7]) *If $f \in \mathcal{A}$ satisfies the condition*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha - \beta \qquad (z \in \mathcal{U}) \tag{10}$$

for $\alpha \geq 0$, $0 < \beta \leq 1/2(1-\gamma)$, and $\gamma = \alpha/(1+\beta)$, then f belongs to the class $\mathcal{C}_\rho$, where $\rho = (1+\beta)/[(1+\beta)(1+2\beta) - 2\alpha\beta]$. Therefore f is close-to-convex of order ρ in $\mathcal{U}$.

Applying Lemma 1, we prove

THEOREM 1. *Let $f \in \mathcal{A}$. If*

$$\left|\frac{z^2 f'(z)}{f^2(z)} + \frac{zf''(z)}{f'(z)} - \frac{2zf'(z)}{f(z)} + 1\right| < \frac{1-\alpha}{2\alpha} \qquad (z \in \mathcal{U}), \tag{11}$$

where $\frac{1}{2} \leq \alpha < 1$, then $f \in \mathcal{B}(\alpha)$.

Proof. We define $w(z)$ by

$$\frac{z^2 f'(z)}{f^2(z)} = \frac{1 + (1-2\alpha)w(z)}{1 - w(z)} \quad (w(z) \neq 1), \tag{12}$$

and note that w is regular in $\mathcal{U}$ and $w(0) = 0$. By logarithmic differentiation we get from (12) that

$$\frac{zf''(z)}{f'(z)} - \frac{2zf'(z)}{f(z)} + 2 = \frac{(1-2\alpha)zw'(z)}{1 + (1-2\alpha)w(z)} + \frac{zw'(z)}{1 - w(z)}. \tag{13}$$

It follows from (12) and (13) that

$$\frac{z^2 f'(z)}{f^2(z)} + \frac{zf''(z)}{f'(z)} - \frac{2zf'(z)}{f(z)} + 2 =$$
$$= \frac{1 + (1-2\alpha)w(z)}{1 - w(z)} + \frac{(1-2\alpha)zw'(z)}{1 + (1-2\alpha)w(z)} + \frac{zw'(z)}{1 - w(z)},$$

or, equivalently,

$$\begin{aligned} &\frac{z^2 f'(z)}{f^2(z)} + \frac{zf''(z)}{f'(z)} - \frac{2zf'(z)}{f(z)} + 1 = \\ &= \frac{2(1-\alpha)w(z)}{1 - w(z)}\left(1 + \frac{zw'(z)}{[1 + (1-2\alpha)w(z)]w(z)}\right). \end{aligned} \tag{14}$$

Suppose there exists $z_o \in \mathcal{U}$ such that

$$\max_{|z|<|z_o|} |w(z)| = |w(z_o)| = 1 \quad (w(z_o) \neq -1).$$

Then, by Lemma 1, we have

$$z_o w'(z) = k w(z_o),$$

where $k \geq 1$ is a real number. From (14) we get

$$\begin{aligned} &\left| \frac{z_o^2 f'(z_o)}{f^2(z_o)} + \frac{z_o f''(z_o)}{f'(z_o)} - \frac{2z_o f'(z_o)}{f(z_o)} + 1 \right| = \\ &\qquad = \left| \frac{2(1-\alpha)w(z_o)}{1-w(z_o)} \left(1 + \frac{z_o w'(z_o)}{[1+(1-2\alpha)w(z_o)]w(z_o)} \right) \right| \geq \\ &\qquad \geq \left| \frac{2(1-\alpha)w(z_o)}{1-w(z_o)} \right| \left| \frac{z_o w'(z_o)}{[1+(1-2\alpha)w(z_o)]w(z_o)} \right| \geq \\ &\qquad \geq \frac{(1-\alpha)k}{2\alpha} \geq \\ &\qquad \geq \frac{1-\alpha}{2\alpha}, \end{aligned}$$

which contradicts our assumption (11). Therefore $|w(z)| < 1$ holds for all $z \in \mathcal{U}$. We finally conclude that $f \in \mathcal{B}(\alpha)$. □

Putting $\alpha = \frac{1}{2}$ in Theorem 1, we get

Corollary 2.1. Let $f \in \mathcal{A}$. If

$$\left| \frac{z^2 f'(z)}{f^2(z)} + \frac{z f''(z)}{f'(z)} - \frac{2z f'(z)}{f(z)} + 1 \right| < \frac{1}{2} \qquad (z \in \mathcal{U}). \tag{15}$$

then $f \in \mathcal{B}(\frac{1}{2})$.

Next, we prove

Theorem 2. *Let $f \in \mathcal{B}(\alpha)$ for some $0 \leq \alpha \leq \frac{1}{2}$ such that $f \in \mathcal{K}_\beta$ for some $0 \leq \beta < 1$. Then the following inequality holds for every $z \in \mathcal{U}$*

$$\operatorname{Re}\left\{ \frac{z f'(z)}{f(z)} \right\} \geq \frac{(\beta+1) - 4(1-\alpha)(\beta+1)|z| + (\beta+1)(1-\alpha)|z|^2}{2(1-|z|)(1-(1-2\alpha)|z|)}. \tag{16}$$

Proof. Since $f \in \mathcal{B}(\alpha)$, we can write

$$\frac{z^2 f'(z)}{f^2(z)} = \frac{1+(1-2\alpha)w(z)}{1-w(z)} \tag{17}$$

for some analytic map w in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathcal{U})$. Applying the Schwarz Lemma, (17) can be written as

$$\frac{z^2 f'(z)}{f^2(z)} = \frac{1+(1-2\alpha)z\Phi(z)}{1-\Phi(z)} \qquad (z \in \mathcal{U}), \tag{18}$$

where Φ is analytic in $\mathcal{U}$ and satisfies $|\Phi(z)| \leq 1$ for $z \in \mathcal{U}$. Differentiating both sides of (18) logarithmically, we obtain

$$\frac{zf'(z)}{f(z)} = \frac{zf''(z)}{2f'(z)} + 1 - \frac{(1-\alpha)(z\Phi'(z) + z\Phi(z))}{(1 - z\Phi(z))(1 + (1-2\alpha)z\Phi(z))}, \tag{19}$$

or, equivalently,

$$\frac{zf'(z)}{f(z)} = \frac{1}{2}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} + \frac{1}{2} - \frac{(1-\alpha)(z\Phi'(z) + z\Phi(z))}{(1 - z\Phi(z))(1 + (1-2\alpha)z\Phi(z))}. \tag{20}$$

From [5, p. 168] we see that

$$\left|\Phi'(z)\right| \leq \frac{1 - |\Phi(z)|^2}{1 - |z|^2} \qquad (z \in \mathcal{U}). \tag{21}$$

Therefore, from (20) and (21), it follows that

$$\begin{aligned}
&\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \\
&\geq \frac{1}{2}\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} + \frac{1}{2} - \frac{(1-\alpha)(|z\Phi'(z)| + |z\Phi(z)|)}{(1 - |z\Phi(z)|)(1 - (1-2\alpha)\,|z\Phi(z)|)} \geq \\
&\geq \frac{\beta+1}{2} - \frac{(1-\alpha)\,|z|\,(|z| + |\Phi(z)|)}{(1-|z|^2)(1 - (1-2\alpha)\,|z\Phi(z)|)} \geq \\
&\geq \frac{\beta+1}{2} - \frac{(1-\alpha)\,|z|}{(1-|z|)(1-(1-2\alpha)\,|z|)} = \\
&= \frac{(\beta+1)(1-|z|)(1-(1-2\alpha)\,|z|) - 2(1-\alpha)\,|z|}{2(1-|z|)(1-(1-2\alpha)\,|z|)} = \\
&= \frac{(\beta+1) - 4(1-\alpha)(\beta+1)\,|z| + (\beta+1)(1-\alpha)\,|z|^2}{2(1-|z|)(1-(1-2\alpha)\,|z|)}
\end{aligned}$$

which completes the proof of Theorem 2. □

COROLLARY 1. *Let $0 \leq \alpha \leq \frac{1}{2}$ and suppose that $f \in \mathcal{B}(\alpha)$ is a convex function. Then, for every $z \in \mathcal{U}$,*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \frac{1 - 4(1-\alpha)\,|z| + (1-\alpha)\,|z|^2}{2(1-|z|)(1-(1-2\alpha)\,|z|)}. \tag{22}$$

Next, we prove

THEOREM 3. *Let $f \in \mathcal{A}$ and suppose that $z^2 f'(z)/f^2(z) \neq \delta$ in $\mathcal{U}$. If*

$$\left|\arg\left\{\frac{z^2 f'(z)}{f^2(z)}\left(\frac{zf''(z)}{f'(z)} + 2\frac{zf'(z)}{f(z)}\right)\right\}\right| < \frac{\pi\xi}{2} \quad (0 < \xi \leq 1), \tag{23}$$

then

$$\arg\left|\left(\frac{z^2 f'(z)}{f^2(z)} - \delta\right)\right| < \frac{\pi\eta}{2} \quad (0 \leq \delta \leq 1), \tag{24}$$

where η $(0 < \eta \leq 1)$ *is the solution of the equation*

$$\xi = \eta + \frac{2}{\pi}\tan^{-1}\left(\frac{\eta - 2\delta(1-\delta)\,|a|\sin\frac{\pi}{2}\eta}{2 + 2\delta(1-\delta)\,|a|\cos\frac{\pi}{2}\eta}\right). \tag{25}$$

Proof. Put

$$p(z) = \frac{1}{1-\delta}\left(\frac{z^2 f'(z)}{f^2(z)} - \delta\right). \tag{26}$$

Then p is analytic $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathcal{U}$. By logarithmic differentiations of both sides of (26), we get

$$\frac{zf''(z)}{f'(z)} + 2\frac{zf'(z)}{f(z)} = \frac{(1-\delta)zp'(z)}{\delta + (1-\delta)p(z)} + 2. \tag{27}$$

Therefore we obtain

$$\frac{z^2 f'(z)}{f^2(z)}\left(\frac{zf''(z)}{f'(z)} + 2\frac{zf'(z)}{f(z)}\right) = (1-\delta)p(z)\left(\frac{zp'(z)}{p(z)} + \frac{2\delta(1-\delta)}{p(z)} + 2\right). \tag{28}$$

Suppose there exists a point $z_0 \in \mathcal{U}$ such that

$$|\arg p(z)| < \frac{\pi}{2}\eta, \quad \text{for } |z| < |z_0|,$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\eta.$$

Then, applying Lemma 3, we can write that

$$\frac{z_0 p'(z_0)}{p(z_0)} = \mathrm{i}k\eta,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1, \qquad \text{when } \arg p(z_0) = \frac{\pi}{2}\eta,$$

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \leq -1, \qquad \text{when } \arg p(z_0) = -\frac{\pi}{2}\eta,$$

and

$$p(z_0)^{\frac{1}{\eta}} = \pm a\mathrm{i} \quad (a > 0).$$

Suppose first that $p(z_0)^{\frac{1}{\eta}} = a\mathrm{i}$ $(a > 0)$. Then we obtain

$$\begin{aligned}
&\arg\left\{\frac{z_0^2 f'(z_0)}{f^2(z_0)}\left(\frac{z_0 f''(z_0)}{f'(z_0)} + 2\frac{z_0 f'(z_0)}{f(z_0)}\right)\right\} = \\
&= \arg\left\{(1-\delta)p(z_0)\left(\frac{z_0 p'(z_0)}{p(z_0)} + \frac{2\delta(1-\delta)}{p(z_0)} + 2\right)\right\} = \\
&= \arg p(z_0) + \arg\left(\frac{z_0 p'(z_0)}{p(z_0)} + \frac{2\delta(1-\delta)}{p(z_0)} + 2\right) = \\
&= \arg p(z_0) + \arg\left(\mathrm{i}\eta k + 2\delta(1-\delta)(\mathrm{i}a)^{-\eta} + 2\right) = \\
&= \frac{\pi\eta}{2} + \tan^{-1}\left(\frac{\eta k - 2\delta(1-\delta)\,|a|\sin\frac{\pi}{2}\eta}{2 + 2\delta(1-\delta)\,|a|\cos\frac{\pi}{2}\eta}\right) \geq \\
&\geq \frac{\pi\eta}{2} + \tan^{-1}\left(\frac{\eta - 2\delta(1-\delta)\,|a|\sin\frac{\pi}{2}\eta}{2 + 2\delta(1-\delta)\,|a|\cos\frac{\pi}{2}\eta}\right) = \\
&= \frac{\pi}{2}\xi,
\end{aligned}$$

where ξ is given by (25). This contradicts assumption (24) of our theorem.

Next suppose that $p(z_0)^{\frac{1}{\eta}} = -a\mathrm{i}$ $(a > 0)$. Applying the same method as above, we obtain

$$\begin{aligned}
&\arg\left\{\frac{z_0^2 f'(z_0)}{f^2(z_0)}\left(\frac{z_0 f''(z_0)}{f'(z_0)} + 2\frac{z_0 f'(z_0)}{f(z_0)}\right)\right\} \leq \\
&\leq -\frac{\pi\eta}{2} - \tan^{-1}\left(\frac{\eta - 2\delta(1-\delta)\,|a|\sin\frac{\pi}{2}\eta}{2 + 2\delta(1-\delta)\,|a|\cos\frac{\pi}{2}\eta}\right) = \\
&= -\frac{\pi}{2}\xi,
\end{aligned}$$

where ξ is given by (25). This contradicts assumption (24). This finishes the proof of Theorem 3. □

Putting $\delta = 0$ in Theorem 3, we get

Corollary 2. *Let $f \in \mathcal{A}$ and suppose that $z^2 f'(z)/f^2(z) \neq 0$ in $\mathcal{U}$. If*

$$\left|\arg\left\{\frac{z^2 f'(z)}{f^2(z)}\left(\frac{z f''(z)}{f'(z)} + 2\frac{z f'(z)}{f(z)}\right)\right\}\right| < \frac{\pi\xi}{2} \quad (0 < \xi \leq 1), \tag{29}$$

then $f \in \mathcal{B}(\eta)$, where $\eta (0 < \eta \leq 1)$ is the solution of the equation

$$\xi = \eta + \frac{2}{\pi}\tan^{-1}\frac{\eta}{2}. \tag{30}$$

Applying Lemma 4, we next prove

Theorem 4. *Let the function f be in the class $\mathcal{B}(\alpha)$. If $f \in \mathcal{S}^*_\alpha$ and*

$$\left|w'(z)\right| \leq \frac{\alpha(\beta + \alpha - 3)}{1-\alpha}, \tag{31}$$

where w is analytic in $\mathcal{U}$ with $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathcal{U})$, $\alpha > 0$, $0 < \beta \leq 1/2(1-\gamma)$, and $\gamma = \alpha/(1+\beta)$, then f belongs to the class $\mathcal{C}_\delta$, where $\delta = (1+\beta)/[(1+\beta)(1+2\beta) - 2\alpha\beta]$. Therefore f is close-to-convex of order δ in $\mathcal{U}$.

Proof. Let $f \in \mathcal{B}(\alpha)$, then

$$\frac{z^2 f'(z)}{f^2(z)} = 1 + (1-\alpha)w(z) \qquad (z \in \mathcal{U}), \tag{32}$$

where w is analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathcal{U})$. By logarithmic differentiation we get from (32) that

$$1 + z\frac{f''(z)}{f'(z)} = \frac{(1-\alpha)zw'(z)}{1+(1-\alpha)w(z)} + 2\left(\frac{zf'(z)}{f(z)} - 1\right) + 1. \tag{33}$$

From (33), we obtain

$$\begin{aligned} \left|1 + z\frac{f''(z)}{f'(z)}\right| &\leq \left|\frac{(1-\alpha)zw'(z)}{1+(1-\alpha)w(z)}\right| + 2\left|\frac{zf'(z)}{f(z)} - 1\right| + 1 \\ &\leq \frac{(1-\alpha)}{\alpha}\left|w'(z)\right| + 2(1-\alpha) + 1 \\ &\leq \beta - \alpha \end{aligned}$$

and so

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha - \beta \qquad (z \in \mathcal{U}). \tag{34}$$

Lemma 4 yields that $f \in \mathcal{C}_\delta$, where $\delta = (1+\beta)/[(1+\beta)(1+2\beta) - 2\alpha\beta]$. □

Now, we prove

THEOREM 5. *Let $f \in \mathcal{B}(\alpha)$. If $f \in \mathcal{K}_\alpha$, then*

$$\left|zw'(z)\right| < \begin{cases} 6(1-\alpha), & \text{if } 0 \leq \alpha \leq \frac{1}{2} \\ 6\alpha, & \text{if } \frac{1}{2} \leq \alpha < 1, \end{cases} \tag{35}$$

where w is analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$.

Proof. We define $w(z)$ by

$$\frac{z^2 f'(z)}{f^2(z)} = \frac{1 + (1-2\alpha)w(z)}{1 - w(z)}. \tag{36}$$

Then w is analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$. By logarithmic differentiation we get from (36) that

$$\frac{zf''(z)}{f'(z)} - \frac{2zf'(z)}{f(z)} + 2 = \frac{2(1-\alpha)zw'(z)}{(1-w(z))(1+(1-2\alpha)w(z))}, \tag{37}$$

hence

$$\left|\frac{2(1-\alpha)zw'(z)}{(1-w(z))(1+(1-2\alpha)w(z))}\right| \leq \left|\frac{zf''(z)}{f'(z)}\right| + 2\left|\frac{zf'(z)}{f(z)} - 1\right|. \tag{38}$$

Since $f \in \mathcal{K}_\alpha \subset \mathcal{S}^*_\alpha$, relation (38) implies

$$\left|\frac{2(1-\alpha)zw'(z)}{(1-w(z))(1+(1-2\alpha)w(z))}\right| \leq 1-\alpha+2(1-\alpha), \tag{39}$$

or, equivalently,

$$\begin{aligned} \left|zw'(z)\right| &\leq \frac{3}{2}\left|(1-w(z))\right|\left|1+(1-2\alpha)w(z)\right| \\ &\leq 3\left|1+(1-2\alpha)w(z)\right| \\ &\leq \begin{cases} 6(1-\alpha), & \text{if } 0 \leq \alpha \leq \frac{1}{2} \\ 6\alpha, & \text{if } \frac{1}{2} \leq \alpha < 1. \end{cases} \end{aligned}$$

□

3. SUBORDINATION RESULTS

In order to prove our subordination results, we shall make use of the following results given in [1].

LEMMA 5. *Let p and h be analytic functions in $\mathcal{U}$ such that $p(0) = h(0) = 1$. Assume that h is convex and univalent in $\mathcal{U}$ satisfying the condition* $\operatorname{Re}\{\beta h(z)+\gamma\} > 0$ *for complex numbers β, γ and for all $z \in \mathcal{U}$. If p, h, β and γ satisfy the Briot-Bouquet differential equation*

$$p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} = h(z), \tag{40}$$

then $p(z) \prec h(z)$ $(z \in \mathcal{U})$.

LEMMA 6. *Under the hypothesis of Lemma 5, if the Briot-Bouquet differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z)+\gamma} = h(z) \quad (q(0) = 1) \tag{41}$$

has a univalent solution q, then $p(z) \prec q(z) \prec h(z)$. Furthermore, q is the best dominant.

We prove first the following subordination result.

THEOREM 6. *Let h be a convex and univalent function in $\mathcal{U}$ such that $h(0) = 1$ and* $\operatorname{Re}\{h(z)\} > 0$ *for $z \in \mathcal{U}$. If $f \in \mathcal{A}$ satisfies*

$$\frac{z^2 f'(z)}{f^2(z)} - z^2\left(\frac{z}{f(z)}\right)'' \prec h(z) \qquad (z \in \mathcal{U}), \tag{42}$$

then

$$\frac{z^2 f'(z)}{f^2(z)} \prec h(z) \qquad (z \in \mathcal{U}). \tag{43}$$

Proof. Define the function p by

$$\frac{z^2 f'(z)}{f^2(z)} = p(z) \qquad (z \in \mathcal{U}). \tag{44}$$

Then p is analytic in $\mathcal{U}$ with $p(0) = 1$. Differentiating both sides in (44), we obtain

$$-z^2 \left(\frac{z}{f(z)}\right)'' = zp'(z). \tag{45}$$

From (44) and (45) we get

$$\frac{z^2 f'(z)}{f^2(z)} - z^2 \left(\frac{z}{f(z)}\right)'' = p(z) + zp'(z). \tag{46}$$

Taking $\beta = 0$ and $\gamma = 1$ in Lemma 5, we finish the proof of Theorem 6. □

Putting $h(z) = [1 + (1 - 2\alpha)z]/(1 - z)$ in Theorem 6, we obtain

Corollary 3. *If $f \in \mathcal{A}$ satisfies*

$$\frac{z^2 f'(z)}{f^2(z)} - z^2 \left(\frac{z}{f(z)}\right)'' \prec \frac{1 + (1 - 2\alpha)z}{1 - z} \qquad (z \in \mathcal{U}), \tag{47}$$

then $f \in \mathcal{B}(\alpha)$.

By making use of Corollary 5 and Lemma 2, we have

Corollary 4. *If $f \in \mathcal{A}$ satisfies*

$$\frac{z^2 f'(z)}{f^2(z)} - z^2 \left(\frac{z}{f(z)}\right)'' \prec \frac{1 + z}{1 - z} \qquad (z \in \mathcal{U}), \tag{48}$$

then f is univalent in $\mathcal{U}$.

By replacing $p(z)$ by $z^2 f'(z)/f^2(z)$ and taking $\beta = 0$ and $\gamma = 1$ in Lemma 5 and Lemma 6, we can easily obtain

Theorem 7. *Under the hypothesis of Theorem 6, if the Briot-Bouquet differential equation*

$$q(z) + zq'(z) = h(z) \qquad (q(0) = 1)$$

has a univalent solution, then

$$\frac{z^2 f'(z)}{f^2(z)} \prec q(z) \prec h(z). \tag{49}$$

Furthermore, q is the best dominant.

Acknowledgements. The work presented here was partially supported by SAGA: STGL-012-2006, Academy of Sciences, Malaysia. The authors would like to thank the referee for some suggestions to improve the paper.

REFERENCES

[1] Eenigenburg, P.J., Miller, S.S., Mocanu, P.T. and Reade, M.O., *On a Briot-Bouquet differential subordination,* Rev. Roumaine Math. Pures Appl., **29** (1984), 567–573.

[2] Frasin, B.A. and Darus, M., *On certain analytic univalent functions*, Internat. J. Math. and Math. Sci., **25 (5)** (2001), 305–310.

[3] Jack, I.S., *Functions starlike and convex of order* α, J. London. Math. Soc., **3** (1971), 469–474.

[4] Miller, S.S. and Mocanu, P.T., *Second order differential inequalities in the complex plane*, J. Math. Ana. Appl., **65** (1978), 289–305.

[5] Z. Nehari, *Conformal Mapping*, McGraw-Hill, New York, 1952.

[6] Nunokawa, M., *On some angular estimates of analytic functions*, Math. Japonica, **41** (1995), 447–452.

[7] Owa, S., *The order of close-to-convexity for certain univalent functions,* J. Math. Ana. Appl., **138** (1989), 393–396.

[8] Ozaki, S. and Nunokawa, M., *The Schwarzian derivative and univalent functions*, Proc. Amer. Math. Soc., **33 (2)** (1972), 392–394.

Received June 16, 2005

B.A. Frasin
Al al-Bayt University
Department of Mathematics
Mafraq, Jordan
E-mail: bafrasin@yahoo.com

M. Darus and S. Siregar
School of Mathematical Sciences
Faculty of Science and Technology
University Kebangsaan Malaysia
Bangi 43600, Malaysia
E-mail: maslina@pkrisc.cc.ukm.my